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# INEQUALITIES FOR THE DUAL RELATIVE OPERATOR ENTROPY

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**Abstract.** In this paper, we introduce the concept of *dual relative entropy* defined by

$$D(A|B) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}$$

for positive invertible operators $A$ and $B$ and establish various upper and lower bounds for the error operator in approximating the $D(A|B)$ by

$$\frac{m\ln m}{M-m}(MA-B)+\frac{M\ln M}{M-m}(B-mA)$$

under the natural assumption $mA \leq B \leq MA$ for some $m$, $M$ with $0<m<M$. Applications for the operator entropy are also given. Some trace inequalities are derived as well.

Kamei and Fujii [8], [9] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators $A$ and $B$, by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}, \tag{1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [16].

In general, we can define for positive operators $A$, $B$

$$S(A|B) := s-\lim_{\varepsilon\to 0+} S(A+\varepsilon I|B)$$

if it exists, here $I$ is the identity operator.

For the entropy function $\eta(t) = -t\ln t$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A = S(A|I) \geq 0$$

for positive contraction $A$. This shows that the relative operator entropy (1) is a relative version of the operator entropy.

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Following [10, p. 149-p. 155], we recall some important properties of relative operator entropy for $A$ and $B$ positive invertible operators:
(i) We have the equalities

$$S(A|B) = -A^{1/2}\left(\ln A^{1/2}B^{-1}A^{1/2}\right)A^{1/2} = B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2}; \tag{2}$$

(ii) We have the inequalities

$$S(A|B) \le A(\ln\|B\| - \ln A) \text{ and } S(A|B) \le B - A;$$

(iii) For any $C$, $D$ positive invertible operators we have that

$$S(A+B|C+D) \ge S(A|C) + S(B|D);$$

(iv) If $B \le C$ then

$$S(A|B) \le S(A|C);$$

(v) If $B_n \downarrow B$ then

$$S(A|B_n) \downarrow S(A|B);$$

(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B);$$

(vii) For every operator $T$ we have

$$T^*S(A|B)T \le S(T^*AT|T^*BT).$$

The relative operator entropy is jointly concave, namely, for any positive invertible operators $A$, $B$, $C$, $D$ we have

$$S(tA + (1-t)B|tC + (1-t)D) \ge tS(A|C) + (1-t)S(B|D)$$

for any $t \in [0,1]$.

For other results on the relative operator entropy see [6], [12], [13], [15] and [17].

In the recent paper [5] we have obtained amongst other the following result in approximating the relative operator entropy $S(A|B)$ by some simpler quantity:

**Theorem 1.** *Let $A$, $B$ be two positive invertible operators such that the condition*

$$mA \le B \le MA, \tag{3}$$

*for some $m$, $M$ with $0 < m < M$, is valid, then we have*

$$\begin{aligned}
&\frac{1}{2M^2}(B - mA)A^{-1}(MA - B) \qquad (4)\\
&\le S(A|B) - \frac{\ln m}{M-m}(MA - B) - \frac{\ln M}{M-m}(B - mA)\\
&\le \frac{1}{2m^2}(B - mA)A^{-1}(MA - B).
\end{aligned}$$

In particular, we have the following result for the operator entropy:

**Corollary 1.** *Assume that* $pI \leq C \leq PI$ *for some constants* $p$, $P$ *with* $0 < p < P$. *Then we have for operator entropy* $\eta(C) = -C \ln C$ *that*

$$\begin{aligned} &\frac{p}{2P}(IP - C)C^{-1}(C - Ip) \\ &\leq \eta(C) + \frac{P\ln P}{P-p}(C - pI) + \frac{p\ln p}{P-p}(PI - C) \\ &\leq \frac{P}{2p}(IP - C)C^{-1}(C - Ip). \end{aligned} \tag{5}$$

Taking into account the above, we can introduce the concept of *dual relative entropy* defined by

$$D(A|B) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}$$

for positive invertible operators $A$ and $B$.

Observe that, if we replace in (2) $B$ with $A$, then we get

$$\begin{aligned} S(B|A) &= A^{1/2}\eta\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \\ &= A^{1/2}\left(-A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}, \end{aligned}$$

therefore we have

$$A^{1/2}\left(A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} = -S(B|A)$$

for positive invertible operators $A$ and $B$, which shows that the dual relative entropy has the following representation in terms of the relative entropy:

$$D(A|B) = -S(B|A) \tag{6}$$

for positive invertible operators $A$ and $B$. It is also well know that, in general $S(A|B)$ is not equal to $S(B|A)$.

Motivated by the above results, we establish in this paper some error bounds in approximation of the dual relative entropy $D(A|B)$ with the simpler quantity

$$\frac{m\ln m}{M-m}(MA - B) + \frac{M\ln M}{M-m}(B - mA) \tag{7}$$

under the natural assumptions (3) for the operators $A$ and $B$, namely $mA \leq B \leq MA$, for some $m$, $M$ with $0 < m < M$. For this purpose, we use some scalar inequalities for convex functions from [2], [3] and [4]. Applications for the operator entropy $\eta(C) = -C\ln C = S(C|I)$ under the natural assumption $pI \leq C \leq PI$ for some constants $p$, $P$ with $0 < p < P$, are also provided.

## 1. Absolute Value Upper and Lower Bounds

With the assumption that the operators $A$ and $B$ satisfy the condition $mA \leq B \leq MA$, for some $m$, $M$ with $0 < m < M$, define the *error operator*

$$E_{m,M}(A,B) := \frac{m \ln m}{M-m}(MA - B) + \frac{M \ln M}{M-m}(B - mA) - D(A|B), \quad (8)$$

which represent the error in approximating the dual relative operator entropy by the operator from (7)

The next result provided some upper and lower bounds for the error operator $E_{m,M}(A,B)$.

**Theorem 2.** *Let $A$, $B$ be two positive invertible operators such that the condition (3) is valid, then we have*

$$\begin{aligned} &2\left(\frac{1}{2}A - \frac{1}{M-m}A^{1/2}\left|A^{-1/2}\left(B - \frac{m+M}{2}A\right)A^{-1/2}\right|A^{1/2}\right)K(m,M) \\ &\leq E_{m,M}(A,B) \\ &\leq 2\left(\frac{1}{2}A + \frac{1}{M-m}A^{1/2}\left|A^{-1/2}\left(B - \frac{m+M}{2}A\right)A^{-1/2}\right|A^{1/2}\right)K(m,M), \end{aligned} \quad (9)$$

*where*

$$\begin{aligned} K(m,M) &:= \left[\frac{m \ln m + M \ln M}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right] \\ &= \ln\left(\frac{G(m^m, M^M)}{[A(m,M)]^{A(m,M)}}\right) \end{aligned}$$

*and $G(a,b) := \sqrt{ab}$ is the geometric mean while $A(a,b) := \frac{a+b}{2}$ is the arithmetic mean of positive numbers $a$, $b$.*

*Proof.* Recall the following result obtained by the author in 2006 [2] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} &n \min_{j \in \{1,2,\ldots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right] \\ &\leq \frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j) - \Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right) \\ &\leq n \max_{j \in \{1,2,\ldots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right], \end{aligned} \quad (10)$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset $C$ of the linear space $X$, $\{x_j\}_{j\in\{1,2,\ldots,n\}}$ are vectors in $C$ and $\{p_j\}_{j\in\{1,2,\ldots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$. For $n = 2$, we deduce from (10) that

$$\begin{aligned} &2r\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \\ &\leq \nu\Phi(x)+(1-\nu)\Phi(y)-\Phi[\nu x+(1-\nu)y] \\ &\leq 2R\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{11}$$

for any $x$, $y \in \mathbb{R}$ and $\nu \in [0,1]$, where $r = \min\{\nu, 1-\nu\}$ and $R = \max\{\nu, 1-\nu\}$.

Now, if we take in (11) the convex function $\Phi(t) = t\ln t$, $t > 0$, then we get

$$\begin{aligned} &2r\left[\frac{x\ln x+y\ln y}{2}-\left(\frac{x+y}{2}\right)\ln\left(\frac{x+y}{2}\right)\right] \\ &\leq \nu x\ln x+(1-\nu)y\ln y-[\nu x+(1-\nu)y]\ln[\nu x+(1-\nu)y] \\ &\leq 2R\left[\frac{x\ln x+y\ln y}{2}-\left(\frac{x+y}{2}\right)\ln\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{12}$$

for any $x$, $y > 0$ and $\nu \in [0,1]$.

This is an inequality of interest in itself as well.

Now, if we take in (12) $x = m$, $y = M$ and $\nu = \frac{M-u}{M-m} \in [0,1]$ with $u \in [m, M]$ then we get

$$\begin{aligned} &2\min\left\{\frac{M-u}{M-m},\frac{u-m}{M-m}\right\} \\ &\times\left[\frac{m\ln m+M\ln M}{2}-\left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right] \\ &\leq \frac{M-u}{M-m}m\ln m+\frac{u-m}{M-m}M\ln M-u\ln u \\ &\leq 2\max\left\{\frac{M-u}{M-m},\frac{u-m}{M-m}\right\} \\ &\times\left[\frac{m\ln m+M\ln M}{2}-\left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]. \end{aligned} \tag{13}$$

Since

$$\min\left\{\frac{M-u}{M-m},\frac{u-m}{M-m}\right\}=\frac{1}{2}-\left|\frac{u-\frac{m+M}{2}}{M-m}\right|$$

and

$$\max\left\{\frac{M-u}{M-m},\frac{u-m}{M-m}\right\}=\frac{1}{2}+\left|\frac{u-\frac{m+M}{2}}{M-m}\right|,$$

then from (13) we have

$$2\left(\frac{1}{2}-\frac{1}{M-m}\left|u-\frac{m+M}{2}\right|\right)K\left(m,M\right) \tag{14}$$
$$\leq \frac{M-u}{M-m}m\ln m+\frac{u-m}{M-m}M\ln M-u\ln u$$
$$\leq 2\left(\frac{1}{2}+\frac{1}{M-m}\left|u-\frac{m+M}{2}\right|\right)K\left(m,M\right)$$

for any $u\in[m,M]$.

Using the continuous functional calculus we have from (14) that

$$2\left(\frac{1}{2}I-\frac{1}{M-m}\left|X-\frac{m+M}{2}I\right|\right)K\left(m,M\right) \tag{15}$$
$$\leq m\ln m\frac{MI-X}{M-m}+M\ln M\frac{X-mI}{M-m}-X\ln X$$
$$\leq 2\left(\frac{1}{2}I+\frac{1}{M-m}\left|X-\frac{m+M}{2}I\right|\right)K\left(m,M\right)$$

for any selfadjoint operator $X$ with the property that $mI\leq X\leq MI$.

Multiplying both sides of (3) by $A^{-1/2}$ we get

$$mI\leq A^{-1/2}BA^{-1/2}\leq MI$$

and by replacing $X$ by $A^{-1/2}BA^{-1/2}$ in (15) we obtain

$$2\left(\frac{1}{2}I-\frac{1}{M-m}\left|A^{-1/2}BA^{-1/2}-\frac{m+M}{2}I\right|\right)K\left(m,M\right) \tag{16}$$
$$\leq m\ln m\frac{MI-A^{-1/2}BA^{-1/2}}{M-m}+M\ln M\frac{A^{-1/2}BA^{-1/2}-mI}{M-m}$$
$$-A^{-1/2}BA^{-1/2}\ln(A^{-1/2}BA^{-1/2})$$
$$\leq 2\left(\frac{1}{2}I+\frac{1}{M-m}\left|A^{-1/2}BA^{-1/2}-\frac{m+M}{2}I\right|\right)K\left(m,M\right).$$

Multiplying both sides of (16) by $A^{1/2}$ we get the desired result (9). □

**Remark 1.** *One can observe that the inequalities (10) are a simple consequence of Theorem 1, p.717 from* [14]. *Similar scalar inequalities as those in the proof of the theorem were obtained in* [1] *and* [11].

**Remark 2.** *If $A$ and $B$ commute, then*

$$A^{1/2}\left|A^{-1/2}\left(B-\frac{m+M}{2}A\right)A^{-1/2}\right|A^{1/2}=\left|B-\frac{m+M}{2}A\right|,$$
$$S\left(B|A\right)=B\left(\ln A-\ln B\right)$$

*and by (9) we have*

$$(0 \leq) \, 2\left(\frac{1}{2}A - \frac{1}{M-m}\left|B - \frac{m+M}{2}A\right|\right) K(m,M) \tag{17}$$
$$\leq \frac{m\ln m}{M-m}(MA - B) + \frac{M\ln M}{M-m}(B - mA) - B(\ln B - \ln A)$$
$$\leq 2\left(\frac{1}{2}A + \frac{1}{M-m}\left|B - \frac{m+M}{2}A\right|\right) K(m,M).$$

The above result can be applied for the operator entropy

$$\eta(C) = -C\ln C = S(C|I)$$

as follows:

**Corollary 2.** *Assume that $pI \leq C \leq PI$ for some $p$, $P$ with $0 < p < P$. Then we have that*

$$(0 \leq) \, 2\left(\frac{1}{2}I - \frac{1}{P-p}\left|C - \frac{p+P}{2}I\right|\right) K(p,P) \tag{18}$$
$$\leq \frac{p\ln p}{P-p}(PI - C) + \frac{P\ln P}{P-p}(C - pI) + \eta(C)$$
$$\leq 2\left(\frac{1}{2}I + \frac{1}{P-p}\left|C - \frac{p+P}{2}I\right|\right) K(p,P).$$

## 2. An Upper Bound in Terms of Logarithm

We have the following inequality of interest for convex functions, see for instance [3]:

**Lemma 1.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval $I$, $a, b \in \mathring{I}$, the interior of $I$ and $\nu \in [0,1]$. Then*

$$0 \leq (1-\nu) f(a) + \nu f(b) - f((1-\nu)a + \nu b)$$
$$\leq \nu(1-\nu)(b-a)\left[f'_-(b) - f'_+(a)\right]. \tag{19}$$

*In particular, we have*

$$0 \leq \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \leq \frac{1}{4}(b-a)\left[f'_-(b) - f'_+(a)\right]. \tag{20}$$

*The constant $\frac{1}{4}$ is best possible in both inequalities from (20).*

We can state the following result:

**Theorem 3.** *Let $A$, $B$ be two positive invertible operators such that the condition (3) is valid, then we have*

$$(0 \leq) E_{m,M}(A,B) \leq \frac{\ln M - \ln m}{M-m}(B-mA)A^{-1}(MA-B) \leq \frac{1}{4}(M-m)(\ln M - \ln m)A. \tag{21}$$

*Proof.* If we consider the convex function $f(t) = t\ln t$, $t>0$, then $f'(t) = \ln t + 1$ and by (19) we have

$$0 \leq (1-\nu)a\ln a + \nu b\ln b - ((1-\nu)a+\nu b)\ln((1-\nu)a+\nu b) \leq \nu(1-\nu)(b-a)(\ln b - \ln a) \tag{22}$$

for any $a,b>0$ and and $\nu \in [0,1]$.

On applying the inequality (22) on the interval $[m,M]$ and for $\nu = \frac{x-m}{M-m} \in [0,1]$ with $x\in[m,M]$ then we get

$$0 \leq m\ln m\frac{M-x}{M-m} + M\ln M\frac{x-m}{M-m} - x\ln x \leq \frac{(x-m)(M-x)}{M-m}(\ln M - \ln m) \leq \frac{1}{4}(M-m)(\ln M - \ln m). \tag{23}$$

Using the continuous functional calculus we have from (23) that

$$0 \leq m\ln m\frac{MI-X}{M-m} + M\ln M\frac{X-mI}{M-m} - X\ln X \leq (\ln M - \ln m)\frac{(X-mI)(M-XI)}{M-m} \leq \frac{1}{4}(M-m)(\ln M-\ln m)I \tag{24}$$

for any selfadjoint operator $X$ with the property that $mI \leq X \leq MI$.

By replacing $X$ by $A^{-1/2}BA^{-1/2}$ in (15) we get

$$\begin{aligned} 0 &\leq m\ln m\frac{MI - A^{-1/2}BA^{-1/2}}{M-m} + M\ln M\frac{A^{-1/2}BA^{-1/2}-mI}{M-m} \\ &- A^{-1/2}BA^{-1/2}\ln(A^{-1/2}BA^{-1/2}) \\ &\leq (\ln M - \ln m)\frac{\left(A^{-1/2}BA^{-1/2}-mI\right)\left(MI - A^{-1/2}BA^{-1/2}\right)}{M-m} \\ &\leq \frac{1}{4}(M-m)(\ln M - \ln m)I. \end{aligned} \tag{25}$$

Multiplying both sides of (25) by $A^{1/2}$ we get the desired result (21). □

**Corollary 3.** *Assume that $pI \leq C \leq PI$ for some $p$, $P$ with $0<p<P$. Then we have that*

$$(0 \leq) \frac{p\ln p}{P-p}(PI-C) + \frac{P\ln P}{P-p}(C-pI) + \eta(C) \tag{26}$$

$$\leq \frac{\ln P - \ln p}{P - p} (C - pI) (PI - C) \leq \frac{1}{4} (P - p) (\ln P - \ln p).$$

## 3. Further Lower and Upper Bounds

We have the following result, see for instance [4]:

**Lemma 2.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of $I$. If there exists the constants $d$, $D$ such that*

$$d \leq f'' (t) \leq D \text{ for any } t \in \mathring{I}, \tag{27}$$

*then*

$$\frac{1}{2}\nu (1 - \nu) d (b - a)^2 \leq (1 - \nu) f (a) + \nu f (b) - f ((1 - \nu) a + \nu b) \tag{28}$$
$$\leq \frac{1}{2}\nu (1 - \nu) D (b - a)^2$$

*for any $a$, $b \in \mathring{I}$ and $\nu \in [0, 1]$.*

*In particular, we have*

$$\frac{1}{8} (b - a)^2 d \leq \frac{f (a) + f (b)}{2} - f \left(\frac{a + b}{2}\right) \leq \frac{1}{8} (b - a)^2 D, \tag{29}$$

*for any $a$, $b \in \mathring{I}$.*

*The constant $\frac{1}{8}$ is best possible in both inequalities in (29).*

If $D > 0$, the second inequality in (28) is better than the corresponding inequality obtained by Furuichi and Minculete in [7] by applying Lagrange's theorem two times. They had instead of $\frac{1}{2}$ the constant 1. Our method also allowed to obtain, for $d > 0$, a lower bound that can not be established by Lagrange's theorem method employed in [7].

We can state the following result:

**Theorem 4.** *Let $A$, $B$ be two positive invertible operators such that the condition (3) is valid, then we have*

$$(0 \leq) \frac{1}{2M} (B - mA) A^{-1} (MA - B) \leq E_{m,M} (A, B) \tag{30}$$
$$\leq \frac{1}{2m} (B - mA) A^{-1} (MA - B).$$

*Proof.* If we consider the convex function $f (t) = t \ln t$, $t > 0$, then $f'' (t) = \frac{1}{t}$ and by (19) we have

$$\frac{1}{2}\nu (1 - \nu) \frac{1}{\max \{a, b\}} (b - a)^2 \tag{31}$$
$$\leq (1 - \nu) a \ln a + \nu b \ln b - ((1 - \nu) a + \nu b) \ln ((1 - \nu) a + \nu b)$$
$$\leq \frac{1}{2}\nu (1 - \nu) \frac{1}{\min \{a, b\}} (b - a)^2$$

for any $a$, $b>0$ and $\nu\in[0,1]$.

On applying the inequality (31) on the interval $[m,M]$ and for $\nu=\frac{x-m}{M-m}\in[0,1]$ with $x\in[m,M]$ then we get

$$\frac{1}{2M}(x-m)(M-x)\leq\frac{M-x}{M-m}m\ln m+\frac{x-m}{M-m}M\ln M-x\ln x \tag{32}$$
$$\leq\frac{1}{2m}(x-m)(M-x).$$

Using the continuous functional calculus we have from (32) that

$$\frac{1}{2M}(X-mI)(M-XI)\leq\frac{MI-X}{M-m}m\ln m+\frac{X-mI}{M-m}M\ln M-X\ln X \tag{33}$$
$$\leq\frac{1}{2m}(X-mI)(M-XI)$$

for any selfadjoint operator $X$ with the property that $mI\leq X\leq MI$.

Now, on using a similar argument to the one in the proof of Theorem 3 we deduce the desired result (30). □

Finally, we have

**Corollary 4.** *Assume that $pI\leq C\leq PI$ for some $p$, $P$ with $0<p<P$. Then we have the inequalities*

$$(0\leq)\frac{1}{2P}(C-pI)(PI-C)\leq\frac{p\ln p}{P-p}(PI-C)+\frac{P\ln P}{P-p}(C-pI)+\eta(C) \tag{34}$$
$$\leq\frac{1}{2p}(C-pI)(PI-C).$$

## 4. Applications for Trace Inequalities

If $\{e_i\}_{i\in I}$ is an orthonormal basis of $H$, we say that $A\in\mathcal{B}(H)$ is *trace class* provided

$$\|A\|_1:=\sum_{i\in I}\langle|A|e_i,e_i\rangle<\infty. \tag{35}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following properties are also well known:

(i) We have

$$\|A\|_1=\|A^*\|_1$$

for any $A\in\mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ is an *operator ideal* in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H)\subseteq\mathcal{B}_1(H);$$

(iii) $(\mathcal{B}_1(H),\|\cdot\|_1)$ is a *Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$tr(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{36}$$

where $\{e_i\}_{i \in I}$ is an orthonormal basis of $H$. Note that this coincides with the usual definition of the trace if $H$ is finite-dimensional. We observe that the series (1) converges absolutely and it is independent from the choice of basis.

The following results collects some properties of the trace:

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$tr(A^*) = \overline{tr(A)};$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT$, $TA \in \mathcal{B}_1(H)$ and

$$tr(AT) = tr(TA) \text{ and } |tr(AT)| \leq \|A\|_1 \|T\|; \tag{37}$$

(iii) $tr(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|tr\| = 1$;

(iv) $\mathcal{B}_{fin}(H)$, the space of operators of *finite rank*, is a dense subspace of $\mathcal{B}_1(H)$.

We recall that *Specht's ratio* is defined by [18]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{38}$$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0. \tag{39}$$

The function $K$ is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

In the recent paper [5] we have showed amongst other that

$$(0 \leq) S(A|B) - \frac{\ln m}{M-m}(MA - B) - \frac{\ln M}{M-m}(B - mA) \leq \ln S\left(\frac{M}{m}\right) A, \tag{40}$$

$$\begin{aligned} &(0 \leq) S(A|B) - \frac{\ln m}{M-m}(MA - B) - \frac{\ln M}{M-m}(B - mA) \\ &\leq \frac{4}{(M-m)^2}\left(K\left(\frac{M}{m}\right) - 1\right)(B - mA) A^{-1}(MA - B) \end{aligned} \tag{41}$$

and

$$\frac{1}{2M^2}(B - mA) A^{-1}(MA - B) \tag{42}$$

$$\leq S(A|B) - \frac{\ln m}{M-m}(MA - B) - \frac{\ln M}{M-m}(B - mA)$$
$$\leq \frac{1}{2m^2}(B - mA)A^{-1}(MA - B)$$

for positive invertible operators $A$ and $B$ that satisfy the condition (3).

Observe that, if $A$, $B \in \mathcal{B}_1(H)$ with $tr(A) = tr(B) = 1$ and satisfy (3), then we must assume $m \leq 1 \leq M$ and by trace properties we have

$$\begin{aligned} tr\left[(B - mA)A^{-1}(MA - B)\right] &= tr\left[(m + M)B - mMA - BA^{-1}B\right] \\ &= m + M - mM - tr\left(A^{-1}B^2\right) \\ &= (M-1)(1-m) - \chi^2(B, A), \end{aligned}$$

where $\chi^2(B, A) =: tr\left(A^{-1}B^2\right) - 1 \geq 0$.

We also have

$$\frac{\ln m}{M-m}(M-1) + \frac{\ln M}{M-m}(1-m) = \ln\left(m^{\frac{M-1}{M-m}} M^{\frac{1-m}{M-m}}\right).$$

We can state the following result:

**Proposition 1.** *Let $A$, $B \in \mathcal{B}_1(H)$ with $tr(A) = tr(B) = 1$ that satisfy (3) for some $m$, $M$ with $0 < m < 1 < M$. Then we have the inequalities*

$$(0 \leq)\, trS(A|B) - \ln\left(m^{\frac{M-1}{M-m}} M^{\frac{1-m}{M-m}}\right) \leq \ln S\left(\frac{M}{m}\right), \tag{43}$$

$$\begin{aligned} &(0 \leq)\, trS(A|B) - \ln\left(m^{\frac{M-1}{M-m}} M^{\frac{1-m}{M-m}}\right) \\ &\leq \frac{4}{(M-m)^2}\left(K\left(\frac{M}{m}\right) - 1\right)\left[(M-1)(1-m) - \chi^2(B, A)\right] \end{aligned} \tag{44}$$

*and*

$$\begin{aligned} \frac{1}{2M^2}\left[(M-1)(1-m) - \chi^2(B, A)\right] &\leq trS(A|B) - \ln\left(m^{\frac{M-1}{M-m}} M^{\frac{1-m}{M-m}}\right) \\ &\leq \frac{1}{2m^2}\left[(M-1)(1-m) - \chi^2(B, A)\right]. \end{aligned} \tag{45}$$

Observe that

$$\frac{m \ln m}{M-m}(M-1) + \frac{M \ln M}{M-m}(1-m) = \ln\left(m^{\frac{m(M-1)}{M-m}} M^{\frac{M(1-m)}{M-m}}\right),$$

then by taking the trace in the inequalities (21) and (30) we can state the following result as well:

**Proposition 2.** *Let $A$, $B \in \mathcal{B}_1(H)$ with $tr(A) = tr(B) = 1$ that satisfy (3) for some $m$, $M$ with $0 < m < 1 < M$. Then we have the inequalities*

$$(0 \leq) \ln\left(m^{\frac{m(M-1)}{M-m}} M^{\frac{M(1-m)}{M-m}}\right) - trD(A|B) \tag{46}$$

$$\leq \frac{\ln M - \ln m}{M - m} \left[ (M-1)(1-m) - \chi^2(B,A) \right]$$

*and*

$$\frac{1}{2M} \left[ (M-1)(1-m) - \chi^2(B,A) \right] \leq \ln \left( m^{\frac{m(M-1)}{M-m}} M^{\frac{M(1-m)}{M-m}} \right) - trD(A|B) \tag{47}$$
$$\leq \frac{1}{2m} \left[ (M-1)(1-m) - \chi^2(B,A) \right].$$

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## References


[1] M. Klaričić Bakula, J. Pečarić, J. Perić, *On the converse Jensen inequality*, Appl. Math. Comp. **218** (2012), 6566–6575.

[2] S. S. Dragomir, *Bounds for the normalized Jensen functional*, Bull. Austral. Math. Soc. **74** (3)(2006), 417-478.

[3] S. S. Dragomir, *A note on Young's inequality*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM 111 (2017), no. 2, 349–354. Preprint RGMIA Res. Rep. Coll. **18** (2015), Art. 126. [http://rgmia.org/papers/v18/v18a126.pdf].

[4] S. S. Dragomir, *A note on new refinements and reverses of Young's inequality*, Transylv. J. Math. Mech. **8** (2016), no. 1, 45–49. Preprint RGMIA Res. Rep. Coll. **18** (2015), Art. 131. [http://rgmia.org/papers/v18/v18a131.pdf].

[5] S. S. Dragomir, *Some inequalities for relative operator entropy*, Preprint RGMIA Res. Rep. Coll. **18** (2015), Art. 145. [http://rgmia.org/papers/v18/v18a145.pdf].

[6] S. Furuichi, *Precise estimates of bounds on relative operator entropies*, Math. Ineq. Appl. **18** (2015), 869–877.

[7] S. Furuichi and N. Minculete, *Alternative reverse inequalities for Young's inequality*, J. Math Inequal. **5** (2011), Number 4, 595–600.

[8] J. I. Fujii and E. Kamei, *Uhlmann's interpolational method for operator means.* Math. Japon. **34** (1989), no. 4, 541–547.

[9] J. I. Fujii and E. Kamei, *Relative operator entropy in noncommutative information theory.* Math. Japon. **34** (1989), no. 3, 341–348.

[10] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for bounded selfadjoint operators on a Hilbert space.* Monographs in Inequalities, 1. Element, Zagreb, 2005. xiv+262 pp.+loose errata. ISBN: 953-197-572-8.

[11] R. Jakšić, J. Pečarić, *On some new converses of convex inequalities in Hilbert space*, Banach J. Math. Anal. b (2), (2015), 63-82.

[12] I. H. Kim, *Operator extension of strong subadditivity of entropy*, J. Math. Phys. 53(2012), 122204

[13] P. Kluza and M. Niezgoda, *Inequalities for relative operator entropies,* Elec. J. Lin. Alg. **27** (2014), Art. 1066.

[14] D. S. Mitrinović, ,J. E. Pečarić, A. M. Fink, *Classical and new inequalities in analysis.* Mathematics and its Applications, East European Series, 61. Kluwer Academic Publishers Group, Dordrecht, 1993.

[15] M. S. Moslehian, F. Mirzapour, and A. Morassaei, *Operator entropy inequalities.* Colloq. Math., **130** (2013), 159–168.
[16] M. Nakamura and H. Umegaki, *A note on the entropy for operator algebras.* Proc. Japan Acad. **37** (1961) 149–154.
[17] I. Nikoufar, *On operator inequalities of some relative operator entropies*, Adv. Math. **259** (2014), 376-383.
[18] W. Specht, *Zer Theorie der elementaren Mittel*, Math. Z. **74** (1960), pp. 91-98.

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