INEQUALITIES FOR THE DUAL RELATIVE OPERATOR ENTROPY

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Abstract. In this paper, we introduce the concept of dual relative entropy defined by
\[ D(A|B) := A^{1/2} \left( A^{-1/2} B A^{-1/2} \ln \left( A^{-1/2} B A^{-1/2} \right) \right) A^{1/2} \]
for positive invertible operators \( A \) and \( B \) and establish various upper and lower bounds for the error operator in approximating the \( D(A|B) \) by
\[ m \ln m \left( MA - B \right) + M \ln M \left( B - mA \right) \]
under the natural assumption \( mA \leq B \leq MA \) for some \( m, M \) with \( 0 < m < M \). Applications for the operator entropy are also given. Some trace inequalities are derived as well.

Kamei and Fujii [8], [9] defined the relative operator entropy \( S(A|B) \), for positive invertible operators \( A \) and \( B \), by
\[ S(A|B) := A^{\frac{1}{2}} \left( \ln A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}, \] (1)
which is a relative version of the operator entropy considered by Nakamura-Umegaki [16].

In general, we can define for positive operators \( A, B \)
\[ S(A|B) := s - \lim_{\varepsilon \to 0^+} S(A + \varepsilon I|B) \]
if it exists, here \( I \) is the identity operator.

For the entropy function \( \eta(t) = -t \ln t \), the operator entropy has the following expression:
\[ \eta(A) = -A \ln A = S(A|I) \geq 0 \]
for positive contraction \( A \). This shows that the relative operator entropy (1) is a relative version of the operator entropy.

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Following [10, p. 149-p. 155], we recall some important properties of relative operator entropy for $A$ and $B$ positive invertible operators:

(i) We have the equalities

$$S(A|B) = -A^{1/2} \left( \ln A^{1/2} B^{-1} A^{1/2} \right) A^{1/2} = B^{1/2} \eta \left( B^{-1/2} A B^{-1/2} \right) B^{1/2};$$

(ii) We have the inequalities

$$S(A|B) \leq A \left( \ln \|B\| - \ln A \right) \text{ and } S(A|B) \leq B - A;$$

(iii) For any $C$, $D$ positive invertible operators we have that

$$S(A + B|C + D) \geq S(A|C) + S(B|D);$$

(iv) If $B \leq C$ then

$$S(A|B) \leq S(A|C);$$

(v) If $B_n \downarrow B$ then

$$S(A|B_n) \downarrow S(A|B);$$

(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B);$$

(vii) For every operator $T$ we have

$$T^* S(A|B) T \leq S(T^* A T|T^* B T).$$

The relative operator entropy is jointly concave, namely, for any positive invertible operators $A$, $B$, $C$, $D$ we have

$$S(tA + (1-t)B|tC + (1-t)D) \geq t S(A|C) + (1-t) S(B|D)$$

for any $t \in [0,1]$.

For other results on the relative operator entropy see [6], [12], [13], [15] and [17].

In the recent paper [5] we have obtained amongst other the following result in approximating the relative operator entropy $S(A|B)$ by some simpler quantity:

**Theorem 1.** Let $A$, $B$ be two positive invertible operators such that the condition

$$mA \leq B \leq MA,$$

for some $m$, $M$ with $0 < m < M$, is valid, then we have

$$\frac{1}{2M^2} (B - mA) A^{-1} (MA - B) \leq S(A|B) - \frac{\ln m}{M - m} (MA - B) - \frac{\ln M}{M - m} (B - mA) \leq \frac{1}{2m^2} (B - mA) A^{-1} (MA - B).$$
In particular, we have the following result for the operator entropy:

**Corollary 1.** Assume that \( pI \leq C \leq PI \) for some constants \( p, P \) with \( 0 < p < P \). Then we have for operator entropy \( \eta(C) = -C \ln C \) that

\[
\frac{p}{2P} (IP - C) C^{-1} (C - Ip) \leq \eta(C) + \frac{P \ln P}{P - p} (C - pI) + \frac{p \ln p}{P - p} (PI - C) \leq \frac{P}{2p} (IP - C) C^{-1} (C - Ip).
\]

Taking into account the above, we can introduce the concept of dual relative entropy defined by

\[
D(A|B) := A^{1/2} \left( A^{-1/2} BA^{-1/2} \ln \left( A^{-1/2} BA^{-1/2} \right) \right) A^{1/2}
\]

for positive invertible operators \( A \) and \( B \).

Observe that, if we replace in (2) \( B \) with \( A \), then we get

\[
S(B|A) = A^{1/2} \eta \left( A^{-1/2} BA^{-1/2} \right) A^{1/2} = A^{1/2} \left( -A^{-1/2} BA^{-1/2} \ln \left( A^{-1/2} BA^{-1/2} \right) \right) A^{1/2},
\]

therefore we have

\[
A^{1/2} \left( A^{-1/2} BA^{-1/2} \ln \left( A^{-1/2} BA^{-1/2} \right) \right) A^{1/2} = -S(B|A)
\]

for positive invertible operators \( A \) and \( B \), which shows that the dual relative entropy has the following representation in terms of the relative entropy:

\[
D(A|B) = -S(B|A)
\]

for positive invertible operators \( A \) and \( B \). It is also well known that, in general \( S(A|B) \) is not equal to \( S(B|A) \).

Motivated by the above results, we establish in this paper some error bounds in approximation of the dual relative entropy \( D(A|B) \) with the simpler quantity

\[
\frac{m \ln m}{M - m} (MA - B) + \frac{M \ln M}{M - m} (B - mA)
\]

under the natural assumptions (3) for the operators \( A \) and \( B \), namely \( mA \leq B \leq MA \), for some \( m, M \) with \( 0 < m < M \). For this purpose, we use some scalar inequalities for convex functions from [2], [3] and [4]. Applications for the operator entropy \( \eta(C) = -C \ln C = S(C|I) \) under the natural assumption \( pI \leq C \leq PI \) for some constants \( p, P \) with \( 0 < p < P \), are also provided.
1. Absolute Value Upper and Lower Bounds

With the assumption that the operators $A$ and $B$ satisfy the condition $mA \leq B \leq MA$, for some $m, M$ with $0 < m < M$, define the error operator

$$E_{m,M}(A, B) := \frac{m \ln m}{M - m} (MA - B) + \frac{M \ln M}{M - m} (B - mA) - D(A|B), \quad (8)$$

which represent the error in approximating the dual relative operator entropy by the operator from (7).

The next result provided some upper and lower bounds for the error operator $E_{m,M}(A, B)$.

**Theorem 2.** Let $A, B$ be two positive invertible operators such that the condition (3) is valid, then we have

$$2 \left( \frac{1}{2} A - \frac{1}{M - m} A^{1/2} \right) A^{-1/2} \left( B - \frac{m + M}{2} A \right) A^{-1/2} \left( A^{1/2} \right) K(m, M)$$

$$\leq E_{m,M}(A, B)$$

$$\leq 2 \left( \frac{1}{2} A + \frac{1}{M - m} A^{1/2} \right) A^{-1/2} \left( B - \frac{m + M}{2} A \right) A^{-1/2} \left( A^{1/2} \right) K(m, M), \quad (9)$$

where

$$K(m, M) := \left[ \frac{m \ln m \ln M}{2} - \left( \frac{m + M}{2} \right) \ln \left( \frac{m + M}{2} \right) \right]$$

$$= \ln \left( \frac{G(m^m, M^M)}{[A(m, M)]^A(m, M)} \right)$$

and $G(a, b) := \sqrt{ab}$ is the geometric mean while $A(a, b) := \frac{a + b}{2}$ is the arithmetic mean of positive numbers $a, b$.

**Proof.** Recall the following result obtained by the author in 2006 [2] that provides a refinement and a reverse for the weighted Jensen’s discrete inequality:

$$n \min_{j \in \{1, 2, \ldots, n\}} \{p_j\} \left[ \frac{1}{n} \sum_{j=1}^{n} \Phi(x_j) - \Phi \left( \frac{1}{n} \sum_{j=1}^{n} x_j \right) \right]$$

$$\leq \frac{1}{P_n} \sum_{j=1}^{n} p_j \Phi(x_j) - \Phi \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right)$$

$$\leq n \max_{j \in \{1, 2, \ldots, n\}} \{p_j\} \left[ \frac{1}{n} \sum_{j=1}^{n} \Phi(x_j) - \Phi \left( \frac{1}{n} \sum_{j=1}^{n} x_j \right) \right], \quad (10)$$
where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset $C$ of the linear space $X$, $\{x_j\}_{j \in \{1, 2, \ldots, n\}}$ are vectors in $C$ and $\{p_j\}_{j \in \{1, 2, \ldots, n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$. For $n = 2$, we deduce from (10) that

\[
2r \left[ \Phi(x) + \Phi(y) - \Phi \left( \frac{x+y}{2} \right) \right] 
\leq \nu \Phi (x) + (1 - \nu) \Phi (y) - \Phi \big[ \nu x + (1 - \nu) y \big] 
\leq 2R \left[ \frac{\Phi(x) + \Phi(y)}{2} - \Phi \left( \frac{x+y}{2} \right) \right]
\]

for any $x, y \in \mathbb{R}$ and $\nu \in [0, 1]$, where $r = \min \{\nu, 1 - \nu\}$ and $R = \max \{\nu, 1 - \nu\}$.

Now, if we take in (11) the convex function $\Phi(t) = t \ln t$, $t > 0$, then we get

\[
2r \left[ \frac{x \ln x + y \ln y}{2} - \left( \frac{x+y}{2} \right) \ln \left( \frac{x+y}{2} \right) \right] 
\leq \nu x \ln x + (1 - \nu) y \ln y - \left[ \nu x + (1 - \nu) y \right] \ln \left[ \nu x + (1 - \nu) y \right] 
\leq 2R \left[ \frac{x \ln x + y \ln y}{2} - \left( \frac{x+y}{2} \right) \ln \left( \frac{x+y}{2} \right) \right]
\]

for any $x, y > 0$ and $\nu \in [0, 1]$.

This is an inequality of interest in itself as well.

Now, if we take in (12) $x = m$, $y = M$ and $\nu = \frac{M-u}{M-m} \in [0, 1]$ with $u \in [m, M]$ then we get

\[
2 \min \left\{ \frac{M-u}{M-m}, \frac{u-m}{M-m} \right\} \times \left[ \frac{m \ln m + M \ln M}{2} - \left( \frac{m+M}{2} \right) \ln \left( \frac{m+M}{2} \right) \right] 
\leq \frac{M-u}{M-m} m \ln m + \frac{u-m}{M-m} M \ln M - u \ln u 
\leq 2 \max \left\{ \frac{M-u}{M-m}, \frac{u-m}{M-m} \right\} \times \left[ \frac{m \ln m + M \ln M}{2} - \left( \frac{m+M}{2} \right) \ln \left( \frac{m+M}{2} \right) \right].
\]

Since

\[
\min \left\{ \frac{M-u}{M-m}, \frac{u-m}{M-m} \right\} = \frac{1}{2} - \left| \frac{u-m+M}{M-m} \right|
\]

and

\[
\max \left\{ \frac{M-u}{M-m}, \frac{u-m}{M-m} \right\} = \frac{1}{2} + \left| \frac{u-m+M}{M-m} \right|,
\]
then from (13) we have
\[
2 \left( \frac{1}{2} - \frac{1}{M - m} \right) \left| u - \frac{m + M}{2} \right| K(m, M) \leq \frac{M - u}{M - m} m \ln m + \frac{u - m}{M - m} M \ln M - u \ln u \\
\leq 2 \left( \frac{1}{2} + \frac{1}{M - m} \right) \left| u - \frac{m + M}{2} \right| K(m, M)
\]
for any \( u \in [m, M] \).

Using the continuous functional calculus we have from (14) that
\[
2 \left( \frac{1}{2} \right) \left| X - \frac{m + M}{2} \right| K(m, M) \leq m \ln m M - X + M ^{\frac{1}{2}} M \ln M - X \ln X \\
\leq 2 \left( \frac{1}{2} \right) \left| X - \frac{m + M}{2} \right| K(m, M)
\]
for any selfadjoint operator \( X \) with the property that \( mI \leq X \leq MI \).

Multiplying both sides of (3) by \( A^{-1/2} \) we get
\[
mI \leq A^{-1/2} BA^{-1/2} \leq MI
\]
and by replacing \( X \) by \( A^{-1/2} BA^{-1/2} \) in (15) we obtain
\[
2 \left( \frac{1}{2} \right) \left| A^{-1/2} BA^{-1/2} - \frac{m + M}{2} \right| K(m, M) \leq m \ln m \frac{MI - X}{M - m} + M \ln M \frac{X - mI}{M - m} - (X - mI) \ln (A^{-1/2} BA^{-1/2}) \\
\leq 2 \left( \frac{1}{2} \right) \left| A^{-1/2} BA^{-1/2} - \frac{m + M}{2} \right| K(m, M).
\]

Multiplying both sides of (16) by \( A^{1/2} \) we get the desired result (9). \( \square \)

**Remark 1.** One can observe that the inequalities (10) are a simple consequence of Theorem 1, p.717 from [14]. Similar scalar inequalities as those in the proof of the theorem were obtained in [1] and [11].

**Remark 2.** If \( A \) and \( B \) commute, then
\[
A^{1/2} \left| A^{-1/2} \left( B - \frac{m + M}{2} A \right) A^{-1/2} \right| A^{1/2} = \left| B - \frac{m + M}{2} A \right|,
\]
\[
S(B|A) = B \ln (A - B)
\]
and by (9) we have

\[
(0 \leq 2 \left( \frac{1}{2} A - \frac{1}{M-m} \left| B - \frac{m+M}{2} A \right| \right) K (m,M)
\]

\[
\leq m \ln \frac{m}{M-m} \left( MA - B \right) + \frac{M \ln M}{M-m} \left( B - mA \right) - B (\ln B - \ln A)
\]

\[
\leq 2 \left( \frac{1}{2} A + \frac{1}{M-m} \left| B - \frac{m+M}{2} A \right| \right) K (m,M).
\]

The above result can be applied for the operator entropy

\[
\eta (C) = -C \ln C = S (C|I)
\]

as follows:

**Corollary 2.** Assume that \( pI \leq C \leq PI \) for some \( p, P \) with \( 0 < p < P \). Then we have that

\[
(0 \leq 2 \left( \frac{1}{2} I - \frac{1}{P-p} \left| C - \frac{p+P}{2} I \right| \right) K (p,P)
\]

\[
\leq \frac{p \ln p}{P-p} (PI - C) + \frac{P \ln P}{P-p} (C - pI) + \eta (C)
\]

\[
\leq 2 \left( \frac{1}{2} I + \frac{1}{P-p} \left| C - \frac{p+P}{2} I \right| \right) K (p,P).
\]

2. **An Upper Bound in Terms of Logarithm**

We have the following inequality of interest for convex functions, see for instance [3]:

**Lemma 1.** Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a convex function on the interval \( I \), \( a, b \in I \), the interior of \( I \) and \( \nu \in [0,1] \). Then

\[
0 \leq (1 - \nu) f (a) + \nu f (b) - f ((1 - \nu) a + \nu b)
\]

\[
\leq \nu (1 - \nu) (b - a) \left[ f' (b) - f' (a) \right].
\]

In particular, we have

\[
0 \leq \frac{f (a) + f (b)}{2} - f \left( \frac{a + b}{2} \right) \leq \frac{1}{4} (b - a) \left[ f' (b) - f' (a) \right].
\]

The constant \( \frac{1}{4} \) is best possible in both inequalities from (20).

We can state the following result:
Theorem 3. Let $A, B$ be two positive invertible operators such that the condition (3) is valid, then we have

\[
(0 \leq) E_{m,M}(A,B) \leq \frac{\ln M - \ln m}{M - m}(B - mA)A^{-1}(MA - B) \tag{21}
\]
\[
\leq \frac{1}{4}(M - m)(\ln M - \ln m)A.
\]

Proof. If we consider the convex function $f(t) = t \ln t, t > 0$, then $f'(t) = \ln t + 1$ and by (19) we have

\[
0 \leq (1 - \nu) a \ln a + \nu b \ln b - ((1 - \nu) a + \nu b) \ln ((1 - \nu) a + \nu b) \tag{22}
\]

for any $a, b > 0$ and $\nu \in [0,1]$.

On applying the inequality (22) on the interval $[m,M]$ and for $\nu = \frac{x - m}{M - m} \in [0,1]$ with $x \in [m,M]$ then we get

\[
0 \leq m \ln m - x + M \ln M x - m - x \ln x \tag{23}
\]

Using the continuous functional calculus we have from (23) that

\[
0 \leq m \ln m \frac{MI - X}{M - m} + M \ln M \frac{X - mI}{M - m} - X \ln X \tag{24}
\]

\[
\leq (\ln M - \ln m) \frac{(X - mI)(M - XI)}{M - m} \leq \frac{1}{4}(M - m)(\ln M - \ln m)I
\]

for any selfadjoint operator $X$ with the property that $mI \leq X \leq MI$.

By replacing $X$ by $A^{-1/2}BA^{-1/2}$ in (15) we get

\[
0 \leq m \ln m \frac{MI - A^{-1/2}BA^{-1/2}}{M - m} + M \ln M \frac{A^{-1/2}BA^{-1/2} - mI}{M - m} \tag{25}
\]

\[
- A^{-1/2}BA^{-1/2} \ln(A^{-1/2}BA^{-1/2})
\]

\[
\leq (\ln M - \ln m) \frac{(A^{-1/2}BA^{-1/2} - mI)(MI - A^{-1/2}BA^{-1/2})}{M - m}
\]

\[
\leq \frac{1}{4}(M - m)(\ln M - \ln m)I.
\]

Multiplying both sides of (25) by $A^{1/2}$ we get the desired result (21).

Corollary 3. Assume that $pI \leq C \leq PI$ for some $p, P$ with $0 < p < P$.

Then we have that

\[
(0 \leq) \frac{p\ln p}{P - p}(PI - C) + \frac{P\ln P}{P - p}(C - pI) + \eta(C) \tag{26}
\]
\[ \leq \frac{\ln P - \ln p}{P - p} (C - pI) (PI - C) \leq \frac{1}{4} (P - p) (\ln P - \ln p). \]

3. Further Lower and Upper Bounds

We have the following result, see for instance [4]:

**Lemma 2.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a twice differentiable function on the interval \( \tilde{I} \), the interior of \( I \). If there exists the constants \( d, D \) such that
\[ d \leq f''(t) \leq D \text{ for any } t \in \tilde{I}, \tag{27} \]
then
\[ \frac{1}{2} \nu (1 - \nu) d (b - a)^2 \leq (1 - \nu) f(a) + \nu f(b) - f((1 - \nu) a + \nu b) \tag{28} \]
\[ \leq \frac{1}{2} \nu (1 - \nu) D (b - a)^2 \]
for any \( a, b \in \tilde{I} \) and \( \nu \in [0, 1] \).

In particular, we have
\[ \frac{1}{8} (b - a)^2 d \leq \frac{f(a) + f(b)}{2} - f\left(\frac{a + b}{2}\right) \leq \frac{1}{8} (b - a)^2 D, \tag{29} \]
for any \( a, b \in \tilde{I} \).

The constant \( \frac{1}{8} \) is best possible in both inequalities in (29).

If \( D > 0 \), the second inequality in (28) is better than the corresponding inequality obtained by Furuichi and Minculete in [7] by applying Lagrange’s theorem two times. They had instead of \( \frac{1}{2} \) the constant 1. Our method also allowed to obtain, for \( d > 0 \), a lower bound that can not be established by Lagrange’s theorem method employed in [7].

We can state the following result:

**Theorem 4.** Let \( A, B \) be two positive invertible operators such that the condition (3) is valid, then we have
\[ (0 \leq) \frac{1}{2M} (B - mA) A^{-1} (MA - B) \leq E_{m,M} (A, B) \tag{30} \]
\[ \leq \frac{1}{2m} (B - mA) A^{-1} (MA - B). \]

**Proof.** If we consider the convex function \( f(t) = t \ln t, t > 0 \), then \( f''(t) = \frac{1}{t} \) and by (19) we have
\[ \frac{1}{2} \nu (1 - \nu) \frac{1}{\max \{a, b\}} (b - a)^2 \tag{31} \]
\[ \leq (1 - \nu) a \ln a + \nu b \ln b - ((1 - \nu) a + \nu b) \ln ((1 - \nu) a + \nu b) \]
\[ \leq \frac{1}{2} \nu (1 - \nu) \frac{1}{\min \{a, b\}} (b - a)^2 \]
for any \( a, b > 0 \) and \( \nu \in [0, 1] \).

On applying the inequality (31) on the interval \([m, M]\) and for \( \nu = \frac{x-m}{M-m} \in [0, 1] \) with \( x \in [m, M] \) then we get
\[
\frac{1}{2M} (x - m)(M - x) \leq \frac{M - x}{M - m} m \ln m + \frac{x - m}{M - m} M \ln M - x \ln x
\]
\[
\leq \frac{1}{2m} (x - m)(M - x).
\]

Using the continuous functional calculus we have from (32) that
\[
\frac{1}{2M} (X - mI)(M - XI) \leq \frac{MI - X}{M - m} m \ln m + \frac{X - mI}{M - m} M \ln M - X \ln X
\]
\[
\leq \frac{1}{2m} (X - mI)(M - XI)
\]
for any selfadjoint operator \( X \) with the property that \( mI \leq X \leq MI. \)

Now, on using a similar argument to the one in the proof of Theorem 3 we deduce the desired result (30). \( \square \)

Finally, we have

**Corollary 4.** Assume that \( pI \leq C \leq PI \) for some \( p, P \) with \( 0 < p < P \). Then we have the inequalities
\[
0 \leq \frac{1}{2P} (C - pI)(PI - C) \leq \frac{p \ln p}{P - p} (PI - C) + \frac{P \ln P}{P - P} (C - pI) + \eta(C)
\]
\[
\leq \frac{1}{2p} (C - pI)(PI - C).
\]

4. **Applications for Trace Inequalities**

If \( \{e_i\}_{i \in I} \) is an orthonormal basis of \( H \), we say that \( A \in \mathcal{B}(H) \) is **trace class** provided
\[
\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty.
\]

The definition of \( \|A\|_1 \) does not depend on the choice of the orthonormal basis \( \{e_i\}_{i \in I} \). We denote by \( \mathcal{B}_1(H) \) the set of trace class operators in \( \mathcal{B}(H) \).

The following properties are also well known:

(i) We have
\[
\|A\|_1 = \|A^*\|_1
\]
for any \( A \in \mathcal{B}_1(H) \);

(ii) \( \mathcal{B}_1(H) \) is an **operator ideal** in \( \mathcal{B}(H) \), i.e.
\[
\mathcal{B}(H) \mathcal{B}_1(H) \mathcal{B}(H) \subseteq \mathcal{B}_1(H);
\]

(iii) \( (\mathcal{B}_1(H), \|\cdot\|_1) \) is a Banach space.
We define the *trace* of a trace class operator \( A \in \mathcal{B}_1 (H) \) to be
\[
\text{tr} (A) := \sum_{i \in I} \langle Ae_i, e_i \rangle ,
\]
where \( \{e_i\}_{i \in I} \) is an orthonormal basis of \( H \). Note that this coincides with the usual definition of the trace if \( H \) is finite-dimensional. We observe that the series (1) converges absolutely and it is independent from the choice of basis.

The following results collects some properties of the trace:
(i) If \( A \in \mathcal{B}_1 (H) \) then \( A^* \in \mathcal{B}_1 (H) \) and
\[
\text{tr} (A^*) = \overline{\text{tr} (A)} ;
\]
(ii) If \( A \in \mathcal{B}_1 (H) \) and \( T \in \mathcal{B} (H) \), then \( AT, TA \in \mathcal{B}_1 (H) \) and
\[
\text{tr} (AT) = \text{tr} (TA) \quad \text{and} \quad |\text{tr} (AT)| \leq \|A\|_1 \|T\| ;
\]
(iii) \( \text{tr} (\cdot) \) is a bounded linear functional on \( \mathcal{B}_1 (H) \) with \( \|\text{tr}\| = 1 \);
(iv) \( \mathcal{B}_{\text{fin}} (H) \), the space of operators of finite rank, is a dense subspace of \( \mathcal{B}_1 (H) \).

We recall that Specht’s ratio is defined by [18]
\[
S (h) := \begin{cases} \frac{\frac{1}{h}-1}{\ln \left( \frac{1}{h}-1 \right)} & \text{if } h \in (0, 1) \cup (1, \infty) \\ 1 & \text{if } h = 1. \end{cases}
\]

It is well known that \( \lim_{h \to 1} S (h) = 1 \), \( S (h) = S \left( \frac{1}{h} \right) > 1 \) for \( h > 0, h \neq 1 \).

The function is decreasing on \( (0, 1) \) and increasing on \( (1, \infty) \).

We consider the Kantorovich’s constant defined by
\[
K (h) := \frac{(h + 1)^2}{4h}, \quad h > 0.
\]

The function \( K \) is decreasing on \( (0, 1) \) and increasing on \( [1, \infty) \), \( K (h) \geq 1 \) for any \( h > 0 \) and \( K (h) = K \left( \frac{1}{h} \right) \) for any \( h > 0 \).

In the recent paper [5] we have showed amongst other that
\[
(0 \leq) S (A|B) - \frac{\ln m}{M - m} (MA - B) - \frac{\ln M}{M - m} (B - mA) \leq \ln S \left( \frac{M}{m} \right) A ,
\]
\[
(0 \leq) S (A|B) - \frac{\ln m}{M - m} (MA - B) - \frac{\ln M}{M - m} (B - mA) \leq \frac{4}{(M - m)^2} \left( K \left( \frac{M}{m} \right) - 1 \right) (B - mA) A^{-1} (MA - B)
\]
\[
\leq \frac{1}{2M^2} (B - mA) A^{-1} (MA - B)
\]
\[ \leq S(A|B) - \frac{\ln m}{M - m} (MA - B) - \frac{\ln M}{M - m} (B - mA) \]
\[ \leq \frac{1}{2m^2} (B - mA) A^{-1} (MA - B) \]

for positive invertible operators \(A\) and \(B\) that satisfy the condition (3).

Observe that, if \(A, B \in B_1(H)\) with \(\text{tr} (A) = \text{tr} (B) = 1\) and satisfy (3), then we must assume \(m \leq 1 \leq M\) and by trace properties we have
\[ \text{tr} [(B - mA) A^{-1} (MA - B)] = \text{tr} [(m + M) B - mMA - BA^{-1}B] \]
\[ = m + M - mM - \text{tr} (A^{-1}B^2) \]
\[ = (M - 1) (1 - m) - \chi^2 (B, A), \]

where \(\chi^2 (B, A) =: \text{tr} (A^{-1}B^2) - 1 \geq 0\).
We also have
\[ \ln m (M - 1) + \ln M (1 - m) = \ln \left( m \frac{M-1}{M-m} M \frac{1-m}{M-m} \right). \]

We can state the following result:

**Proposition 1.** Let \(A, B \in B_1(H)\) with \(\text{tr} (A) = \text{tr} (B) = 1\) that satisfy (3) for some \(m, M\) with \(0 < m < 1 < M\). Then we have the inequalities
\[ (0 \leq) \ln m (M - 1) + \ln M (1 - m) \leq \ln \left( m \frac{M-1}{M-m} M \frac{1-m}{M-m} \right) \]
\[ (0 \leq) \text{tr} S(A|B) - \ln \left( m \frac{M-1}{M-m} M \frac{1-m}{M-m} \right) \]
\[ \leq \frac{4}{(M - m)^2} \left( K \left( \frac{M}{m} \right) - 1 \right) [(M - 1) (1 - m) - \chi^2 (B, A)] \]
\[ \text{and} \]
\[ \frac{1}{2M^2} [(M - 1) (1 - m) - \chi^2 (B, A)] \leq \text{tr} S(A|B) - \ln \left( m \frac{M-1}{M-m} M \frac{1-m}{M-m} \right) \]
\[ \leq \frac{1}{2m^2} [(M - 1) (1 - m) - \chi^2 (B, A)]. \]

Observe that
\[ \frac{m \ln m}{M - m} (M - 1) + \frac{M \ln M}{M - m} (1 - m) = \ln \left( m \frac{M-1}{M-m} M \frac{1-m}{M-m} \right), \]
then by taking the trace in the inequalities (21) and (30) we can state the following result as well:

**Proposition 2.** Let \(A, B \in B_1(H)\) with \(\text{tr} (A) = \text{tr} (B) = 1\) that satisfy (3) for some \(m, M\) with \(0 < m < 1 < M\). Then we have the inequalities
\[ (0 \leq) \ln \left( m \frac{M-1}{M-m} M \frac{1-m}{M-m} \right) - \text{tr} D(A|B) \]
\[ \leq \frac{\ln M - \ln m}{M - m} \left[ (M - 1)(1 - m) - \chi^2(B, A) \right] \]

and
\[ \frac{1}{2M} \left[ (M - 1)(1 - m) - \chi^2(B, A) \right] \leq \ln \left( m \frac{m(M-1)}{M-m} \frac{M(1-m)}{M-m} \right) - \text{tr} D(A|B) \]
\[ \leq \frac{1}{2m} \left[ (M - 1)(1 - m) - \chi^2(B, A) \right]. \]

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References


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