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ON HOLOMORPHIC RELATIVE INVERSES OF OPERATOR-VALUED FUNCTIONS

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Let G be a complex domain, X and Y Banach spaces and $A: G \to L(X, Y)$ holomorphic with $\operatorname{Ker} A(\lambda)$, $\operatorname{Im} A(\lambda)$ complemented, $\lambda \in G$. It is shown that the following conditions are equivalent: (1) A has a holomorphic relative inverse on G; (2) the function $\lambda \to \operatorname{Ker} A(\lambda)$ is locally holomorphic on G; (3) the function $\lambda \to \operatorname{Im} A(\lambda)$ is locally holomorphic on G. Based on this, it is shown that a semi-Fredholm-valued holomorphic function A has a holomorphic relative inverse on G if and only if $\dim \operatorname{Ker} A(\lambda)$ [$\operatorname{codim} \operatorname{Im} A(\lambda)$, respectively] is constant on G.

The latter result is a generalization of the well-known result of Allan on one-side holomorphic inverses.

1. Relative inverses. The notion of a relative inverse is known from algebra: an element x in a ring is said to be relatively invertible if there is another element y such that $xyx = x$. A ring in which every element is relatively invertible has been called a "regular ring" by von Neumann [12]; an example of such a ring is the algebra of all linear operators acting on a given finite dimensional Banach space. Later, Kaplansky [10], showed that, for a Banach algebra, regularity in this sense is a rather severe restriction; indeed, a regular Banach algebra is necessarily finite dimensional.

Thus, in the algebra of all operators acting on a given infinite dimensional Banach space, not every element is relatively invertible. Consequently, a relatively invertible operator might be expected to have some important special properties and this expectation has been proven right.

The first major step toward the investigation of relatively invertible operators on a Banach space was made by Atkinson [2]. Subsequently this proved to be a very useful concept, especially in applied mathematics [11]. The definition of regular invertibility in this context is somewhat stricter than the algebraic one, and it is introduced below. We also employ the useful concepts of "inner" and "outer" relative inverses as [11].

Throughout this article X and Y will be complex Banach spaces, and all mappings will be bounded linear operators.

DEFINITION 1.1. Let $A: X \to Y$ be given. If the operators B_1: $Y \to X$, B_2: $Y \to X$ satisfy the conditions $AB_1A = A$, $B_2AB_2 = B_2$, respectively, then B_1 is called an *inner relative inverse* of A, and B_2 is called an *outer relative inverse* of A. If the operator $B: Y \to X$ is at the same time an inner and an outer relative inverse of A, then B is called a *relative inverse* of A.

Let B be a relative inverse of A. The equalities $(BA)^2 = BA$ and $(AB)^2 = AB$ follow directly from Definition 1.1, so that BA and AB are bounded linear projectors in X and Y, respectively. From the inclusions $\operatorname{Im} A = \operatorname{Im} ABA \subset \operatorname{Im} AB \subset \operatorname{Im} A$ we conclude that $\operatorname{Im} A = \operatorname{Im} AB$. Furthermore, if $ABy = 0$ if follows that $BABy = 0$ or $By = 0$; in other words, $\operatorname{Ker} AB \subset \operatorname{Ker} B$. The opposite inclusion being obvious, we conclude that $\operatorname{Ker} B = \operatorname{Ker} AB$. Therefore, $\operatorname{Im} A$ and $\operatorname{Ker} B$ are mutually complementary closed subspaces of Y. It is now clear that $B \mid \operatorname{Im} A$—the restriction of B to $\operatorname{Im} A$—is a bijection from $\operatorname{Im} A$ onto $\operatorname{Im} B$. Interchanging the roles of A and B we similarly have that $\operatorname{Ker} A$ and $\operatorname{Im} B$ are mutually complementary closed subspaces of X and that $A \mid \operatorname{Im} B$ is a bijection from $\operatorname{Im} B$ onto $\operatorname{Im} A$. Since AB and BA are identities on $\operatorname{Im} A$ and $\operatorname{Im} B$, respectively, it follows that $A \mid \operatorname{Im} B$ and $B \mid \operatorname{Im} A$ are inverses to each other. Thus, in particular, if A has a relative inverse, then $\operatorname{Ker} A$ and $\operatorname{Im} A$ are complemented subspaces in X and Y, respectively.

Conversely, let $\operatorname{Ker} A$ and $\operatorname{Im} A$ be complemented subspaces and $X = \operatorname{Ker} A \oplus X_1$, $Y = \operatorname{Im} A \oplus Y_1$, where X_1, Y_1 are closed subspaces. Since $A \mid X_1$ is a continuous bijection between the Banach spaces X_1 and $\operatorname{Im} A$, by the Closed Graph Theorem there exists a continuous inverse $B_1: \operatorname{Im} A \to X_1$ of $A \mid X_1$. By the remarks in the previous paragraph, a possible relative inverse of A is necessarily an extension of B_1 to Y. Hence, if Q is the bounded projector of Y onto $\operatorname{Im} A$ along Y_1, it is easy to see that $B = B_1 Q$ is a relative inverse of A. Note also, that the relative inverse B, constructed in this way, is the unique relative inverse of A with the properties: the kernel is equal to Y_1 and the range is equal to X_1. Since the kernel of a relative inverse must be a complement of $\operatorname{Im} A$ (by the discussion in the previous paragraph), any relative inverse of A is obtained by the above construction.

These remarks give the following basic structural theorem.

THEOREM 1.2. *An operator $A: X \to Y$ has a relative inverse if and only if* $\operatorname{Ker} A$ *and* $\operatorname{Im} A$ *are complemented subspace of X and Y, respectively. For each decomposition of the spaces X and Y into topological direct sums $X = \operatorname{Ker} A \oplus X_1$, $Y = \operatorname{Im} A \oplus Y_1$, there is precisely one relative inverse B with the properties that* $\operatorname{Ker} B = Y_1$, $\operatorname{Im} B = X_1$, *and conversely. In this case, $P = BA$, $Q = AB$ are* *continuous projectors with the properties* $\operatorname{Im} P = \operatorname{Im} B$, $\operatorname{Ker} P = \operatorname{Ker} A$, $\operatorname{Im} Q = \operatorname{Im} A$, $\operatorname{Ker} Q = \operatorname{Ker} B$.

From Theorem 1.2 it follows that finite rank operators, Fredholm operators and projections are all relatively invertible. Also surjective operators with complemented kernels, as well as injective operators with complemented ranges. In fact, if $A: X \to Y$ is left (right) invertible, the relative inverses of A are precisely the left (right) inverses of A. (This easily verifiable fact will be used several times in the sequel.) Compact operators of infinite rank are not relatively invertible: the range of such an operator is not even closed (see [8; III. 1. 12]).

2. Some perturbation results. We now turn to some perturbation results, which will be essential for our treatment of holomorphic operator-valued functions in the next section. They also imply some results of J. Dieudonné (for the case $Y = X$; see [6; Propositions 2 and 3 and their corollaries]).

First we need the following two technical lemmas, which are essentially due to Atkinson [2] in the case $Y = X$. Here and later, $L(X, Y)$ denotes the space of all bounded linear operators from X to Y, and $L(X) = L(X, X)$.

LEMMA 2.1. *Let* $B \in L(Y, X)$, $U \in L(X, Y)$ *and* $\|U\| < 1/\|B\|$. *Then*

(1) $(I_X - BU)^{-1}B = B(I_Y - UB)^{-1}$

(2) $(I_Y - UB)^{-1}U = U(I_X - BU)^{-1}$.

Proof. Note that all indicated inverses exist. Factor $B\text{-}BUB$ in two ways and transfer terms to obtain (1); (2) is equivalent to (1).

For further use, we define $R(U)$ by

$$R(U) = (I_X - BU)^{-1}B = B(I_Y - UB)^{-1} \,. \tag{$*$}$$

LEMMA 2.2. *Let* A, U *belong to* $L(X, Y)$, *and let* $B \in L(Y, X)$ *be an outer relative inverse of* A. *Then, if* $\|U\| < 1/\|B\|$, *the operator* $R(U)$ *is an outer relative inverse of* $A - U$.

Proof. Since $BAB = B$, then $B(A - U)B = B(I_Y - UB)$. Therefore,

$$\begin{aligned} R(U)(A - U)R(U) &= (I_X - BU)^{-1}B(A - U)B(I_Y - UB)^{-1} \\ &= (I_X - BU)^{-1}B = R(U) \,. \end{aligned}$$

Before we proceed further, we consider some examples, which help to motivate the conditions we impose in the theorems below.

If $F \in L(X, Y)$ is Fredholm (and so relatively invertible) and $K \in L(X, Y)$ is compact, then $A = F + K$ is also Fredholm (see [14; p. 114]) and thus relatively invertible. Now, unless K has finite rank, then $A - F = K$ is not relatively invertible. Since $A - (1/2)F$ is relatively invertible, we shall say that the perturbation F of A does not have small enough norm.

If $A \in L(X, Y)$ is of finite rank (and so relatively invertible) and $K \in L(X, Y)$ is compact of infintie rank, note that $\operatorname{Ker}(\alpha K) \not\supset \operatorname{Ker} A$ and $\operatorname{Im}(\alpha K) \not\subset \operatorname{Im} A$. Hence $A - \alpha K$ is compact of infinite rank and is not relatively invertible for any $\alpha \neq 0$, although the perturbation αK of A may have arbitrarily small norm.

In the next two theorems we will show that, under certain conditions—one of which is B to be a relative inverse of A—the operator $R(U)$, defined in (*), is also an inner inverse of $A - U$. In other words, we will show that the operator $G(U) = A - U - (A - U)R(U)(A - U)$ is equal to zero. The following computation is a modification of that in [13; p. 371]; we obtain two representations of the operator $G(U)$ which will be used in the proofs of the announced theorems.

$$\begin{aligned} G(U) &= (A - U)[I_X - R(U)(A - U)] \\ &= (A - U)(I_X - BU)^{-1}[(I_X - BU) - B(A - U)] \\ &= (A - ABU + ABU - U)(I_X - BU)^{-1}(I_X - BA) \\ &= [A(I_X - BU) + (AB - I_Y)U](I_X - BU)^{-1}(I_X - BA) . \end{aligned}$$

Since $A(I_X - BU)(I_X - BU)^{-1}(I_X - BA) = A - ABA = 0$, we have

$$G(U) = (AB - I_Y)U(I_X - BU)^{-1}(I_X - BA) , \tag{α}$$

and from this, using Lemma 2.1(2), also

$$G(U) = (AB - I_Y)(I_Y - UB)^{-1}U(I_X - BA) . \tag{β}$$

THEOREM 2.3. *Let A, U belong to $L(X, Y)$ and let $B \in L(Y, X)$ be a relative inverse of A. If $\|U\| < 1/\|B\|$ and* $\operatorname{Ker} A \subset \operatorname{Ker} U$, *then $A - U$ has a relative inverse in $L(Y, X)$. Moreover,* $\operatorname{Ker}(A - U) = \operatorname{Ker} A$ *and* $\operatorname{Im}(A - U) \simeq \operatorname{Im} A$.

Proof. Since $I_X - BA$ is a projector onto $\operatorname{Ker} A$ (Theorem 1.2) and $\operatorname{Ker} A \subset \operatorname{Ker} U$, by ($\beta$) $G(U) = 0$; this and Lemma 2.2 imply that $R(U)$ is a relative inverse of $A - U$.

Further, note that

$\operatorname{Ker} R(U) = \operatorname{Ker}[(I_X - BU)^{-1}B] = \operatorname{Ker} B,$
$\operatorname{Im} R(U) = \operatorname{Im}[B(I_Y - UB)^{-1}] = \operatorname{Im} B.$

Therefore, the decomposition $X = \operatorname{Ker}(A - U) \oplus \operatorname{Im} R(U)$ (Theorem 1.2) can be rewritten as $X = \operatorname{Ker}(A - U) \oplus \operatorname{Im} B$. Thus, all subspaces $\operatorname{Ker}(A - U)$ have the same topological complement. Note that $\operatorname{Ker} A$ is one of these subspaces: $U = 0$ clearly satisfies the conditions of the theorem. Moreover, $\operatorname{Ker} A \subset \operatorname{Ker} U$ implies that $\operatorname{Ker} A \subset \operatorname{Ker}(A - U)$. We conclude that $\operatorname{Ker}(A - U) = \operatorname{Ker} A$.

Similarly, $Y = \operatorname{Im}(A - U) \oplus \operatorname{Ker} B$ and all the subspaces $\operatorname{Im}(A - U)$—one of them being $\operatorname{Im} A$—have a common topological complement. In particular $\operatorname{Im}(A - U) \simeq \operatorname{Im} A$.

COROLLARY 2.4. *Let $A \in L(X, Y)$ have a left inverse $B \in L(Y, X)$. Then, for all $U \in L(X, Y)$ such that $\|U\| < 1/\|B\|$, the operator $A - U$ has a left inverse and* $\operatorname{Im}(A - U) \simeq \operatorname{Im} A$.

Proof. Since $\{0\} = \operatorname{Ker} A \subset \operatorname{Ker} U$ (and relative inverse $\equiv$ left inverse in this case), Theorem 2.3 applies.

THEOREM 2.5. *Let A, U belong to $L(X, Y)$ and let $B \in L(Y, X)$ be a relative inverse of A. If $\|U\| < 1/\|B\|$ and* $\operatorname{Im} A \supset \operatorname{Im} U$, *then $A - U$ has a relative inverse in $L(Y, X)$ and* $\operatorname{Ker}(A - U) \simeq \operatorname{Ker} A$, $\operatorname{Im}(A - U) = \operatorname{Im} A$.

Proof. Consider the relation (α). Since, clearly, $U(I_X - BU)^{-1}(I_X - BA)(X) \subset \operatorname{Im} U$ and $\operatorname{Ker}(AB - I_Y) = \operatorname{Im} A$, the hypothesis $\operatorname{Im} U \subset \operatorname{Im} A$ implies $G(U) = 0$. Hence, $R(U)$ is indeed a relative inverse of $A - U$.

The decompositions $X = \operatorname{Ker}(A - U) \oplus \operatorname{Im} B$ and $Y = \operatorname{Im}(A - U) \oplus \operatorname{Ker} B$ are obtained in the same manner as in the proof of Theorem 2.3. Thus, in particular, the kernels $\operatorname{Ker}(A - U)$ – $\operatorname{Ker} A$ is among them – are all isomorphic.

Furthermore, $\operatorname{Im} A$ and $\operatorname{Im}(A - U)$ have a common topological complement. Since $\operatorname{Im} A \supset \operatorname{Im} U$ clearly implies $\operatorname{Im} A \supset \operatorname{Im}(A - U)$, the conclusion $\operatorname{Im} A = \operatorname{Im}(A - U)$ follows.

COROLLARY 2.6. *Let $A \in L(X, Y)$ have a right inverse $B \in L(Y, X)$. Then, for all $U \in L(X, Y)$ such $\|U\| < 1/\|B\|$, the operator $A - U$ has right inverse and* $\operatorname{Ker}(A - U)$ *is isomorphic to* $\operatorname{Ker} A$.

Proof. Since $Y = \operatorname{Im} A \supset \operatorname{Im} U$ (and relative inverse $\equiv$ right inverse in this case), Theorem 3.5 applies.

COROLLARY 2.7. *Let A, U belong to $L(X, Y)$, let A be relatively* *invertible, and let either* $\operatorname{Ker} A \subset \operatorname{Ker} U$ *or* $\operatorname{Im} A \supset \operatorname{Im} U$. *Then, if* $\|U\|$ *is small enough,* $\dim \operatorname{Ker}(A - U)$, $\operatorname{Codim} \operatorname{Ker}(A - U)$, $\dim \operatorname{Im}(A - U)$, *and* $\operatorname{Codim} \operatorname{Im}(A - U)$ *are all constant.*

REMARKS. From the proofs of the above propositions it is clear that we only need the existence of $R(U)$ for the conclusion. Thus these results hold whenever $R(U)$ is well defined—in particular, when $\|U\| < 1/\|B\|$ (in addition to the inclusions stated).

While preparing the final draft of [9], the article [4] was brought to our attention by Professor R. G. Bartle. Part of our results in this section are contained there, for the case $Y = X$.

3. Holomorphic relative inverses.

In this section we shall consider holomorphic operator-valued functions defined on a domain G in the complex plane. Our main concern will be existence of holomorphic relative inverses of such functions. From the previous sections we know the close relationship between relative inverses and kernels and ranges. This motivates the following definition.

Let $\Sigma(X)$ be the set of all linear (closed or not) subspaces of X and let $S: G \to \Sigma(X)$ be a subspace-valued function.

DEFINITION 3.1. A subspace-valued function $S: G \to \Sigma(X)$ is said to be *holomorphic at* λ_0, $\lambda_0 \in G$, if there exists a projection-valued function $P: G \to L(X)$ and a neighborhood V of λ_0 such that:

(1) The function P is holomorphic on V, and

(2) $\operatorname{Im} P(\lambda) = S(\lambda)$, $\lambda \in G$.

Let $\{S(\lambda): \lambda \in G\}$ be a family of subspaces of X holomorphic at a point $\lambda_0 \in G$ in the sense of Definition 3.1. Several remarks are in order. (The neighborhood V of λ_0 will be assumed connected.)

If $\lambda \in V$ then $S(\lambda) \simeq S(\lambda_0)$. In fact, for $|\lambda - \lambda_0|$ small enough $\|P(\lambda) - P(\lambda_0)\| < 1$, so that $P(\lambda)$ maps $S(\lambda_0)$ isomorphically onto $S(\lambda)$([16, p. 132]). In general, connect λ and λ_0 by a curve in V and to each λ_1 on that curve associate $V(\lambda_1) = \{\mu \in V: \|P(\mu) - P(\lambda_1)\| < 1\}$. The usual compactness argument gives the result. In particular, $\dim S(\lambda)$, $\lambda \in V$, is constant. Apply the same argument to $I - P(\lambda)$ to conclude that $\operatorname{codim} S(\lambda)$ is also constant on V.

Note that together with $\{S(\lambda): \lambda \in G\}$, the family of subspaces $\{\operatorname{Ker} P(\lambda): \lambda \in G\}$ is holomorphic at λ_0. Indeed, $I - P(\lambda)$, $\lambda \in G$, are the corresponding projectors. Moreover, $X = S(\lambda) \oplus \operatorname{Ker} P(\lambda)$, $\lambda \in G$.

THEOREM 3.2. *Let* $\{S(\lambda): \lambda \in G\}$ *and* $\{T(\lambda): \lambda \in G\}$ *be two families of subspaces of* X, *with* $S(\lambda_0) \oplus T(\lambda_0) = X$. *Then the following statements are equivalent.*

(1) *The families* $\{S(\lambda): \lambda \in G\}$ *and* $\{T(\lambda): \lambda \in G\}$ *are holomorphic at* λ_0.

(2) *There exists a neighborhood* V_1 *of* λ_0 *and a holomorphic operator-valued function* $A_1: V_1 \to L(X)$, *such that for each* $\lambda \in V_1$, *the operator* $A_1(\lambda)$ *is invertible,* $A_1(\lambda)[S(\lambda_0)] = S(\lambda)$ *and* $A_1(\lambda)[T(\lambda_0)] = T(\lambda)$.

(3) *There is a neighborhood* V_1 *of* λ_0 *such that* $X = S(\lambda) \oplus T(\lambda)$ *and the projector* $\Pi(\lambda)$ *with the properties* $\operatorname{Im} \Pi(\lambda) = S(\lambda)$, $\operatorname{Ker} \Pi(\lambda) = T(\lambda)$ *is holomorphic on* V_1.

Proof. (i): (1) $\Rightarrow$ (2). Let P, Q be the projectors corresponding to $\{S(\lambda): \lambda \in G\}$, $\{T(\lambda): \lambda \in G\}$, respectively, and holomorphic on a neighborhood V of λ_0. Let, further, Π_0 be the projector with the properties $\operatorname{Im} \Pi_0 = S(\lambda_0)$, $\operatorname{Ker} \Pi_0 = T(\lambda_0)$. Define A_1 on V to $L(X)$ by

$$A_1(\lambda) = P(\lambda)\Pi_0 + Q(\lambda)(I - \Pi_0), \ \lambda \in V .$$

Note that $P(\lambda_0)\Pi_0 = \Pi_0$ and $Q(\lambda_0)(I - \Pi_0) = I - \Pi_0$, so that $A_1(\lambda_0) = I$. Therefore $A_1(\lambda)$ is invertible in a neighborhood $V_0 \subset V$ of λ_0.

Further, there is a neighborhood $V_1 \subset V_0$ of λ_0 such that both $\|P(\lambda) - P(\lambda_0)\| < 1$ and $\|Q(\lambda) - Q(\lambda_0)\| < 1$ for $\lambda \in V_1$.

Thus, on V_1 $A_1(\lambda)$ is invertible, holomorphic and

$$(\alpha) \qquad A_1(\lambda)[S(\lambda_0)] = P(\lambda)\Pi_0[S(\lambda_0)] = P(\lambda)[S(\lambda_0)] = S(\lambda)$$

$$(\beta) \qquad A_1(\lambda)[T(\lambda_0)] = Q(\lambda)(I - \Pi_0)[T(\lambda_0)] = Q(\lambda)[T(\lambda_0)] = T(\lambda) .$$

(ii): (2) $\Rightarrow$ (3). Define $\Pi: V_1 \to L(X)$ by $\Pi(\lambda) = A_1(\lambda)\Pi_0 A_1^{-1}(\lambda)$, where Π_0 is as in (i). Clearly, Π is a projector, holomorphic on V_1. Moreover, from (α) and (β) we have

$$\operatorname{Im} \Pi(\lambda) = A_1(\lambda)\Pi_0 A_1^{-1}(\lambda)(X) = A_1(\lambda)[S(\lambda_0)] = S(\lambda) ,$$

$$\operatorname{Ker} \Pi(\lambda) = \operatorname{Ker} (A_1(\lambda)\Pi_0 A_1^{-1}(\lambda)) = \operatorname{Ker} [\Pi_0 A_1^{-1}(\lambda)] = T(\lambda) .$$

Thus, for $\lambda \in V_1$, $X = S(\lambda) \oplus T(\lambda)$.

(iii): (3) $\Rightarrow$ (1). This statement is obvious.

COROLLARY 3.3. *The family of subspaces* $\{S(\lambda): \lambda \in G\}$ *is holomorphic at* λ_0 *if and only if there exists a neighborhood* V_1 *of* λ_0 *and a holomorphic function* $A_1: V_1 \to L(X)$ *such that, for* $\lambda \in V_1$, $A_1(\lambda)$ *is invertible and* $A_1(\lambda)[S(\lambda_0)] = S(\lambda)$. *A possible representation of* A_1: $A_1(\lambda) = P(\lambda)P(\lambda_0) + (I - P(\lambda))(I - P(\lambda_0))$, *where* $P(\lambda)$ *is a holomorphic projector onto* $S(\lambda)$.

Proof. The statements, the equivalence of which is to be proven, become equivalent to (3) and (2) of Theorem 3.2, respectively, if we choose $T(\lambda) = \operatorname{Ker} P(\lambda)$. Note also that the projector Π_0 from part (i) of the proof of Theorem 3.2 coincides with $P(\lambda_0)$.

DEFINITION 3.4. Let $A: G \to L(X, Y)$ be holomorphic and let $\lambda_0 \in G$. We say that A has a holomorphic relative inverse at λ_0 if there is a neighborhood V of λ_0 and a holomorphic operator-valued function $B: V \to L(Y, X)$, such that $B(\lambda)$ is a relative inverse of $A(\lambda)$, for $\lambda \in V$.

THEOREM 3.5. *Let $A: G \to L(X, Y)$ be holomorphic at λ_0 and let $A(\lambda_0)$ be relatively invertible. Then the following statements are equivalent:*

(1) *$A(\lambda)$ has a holomorphic relative inverse at λ_0.*

(2) *The subspace-valued function $\lambda \to \operatorname{Ker} A(\lambda)$ is holomorphic at λ_0.*

(3) *The subspace-valued function $\lambda \to \operatorname{Im} A(\lambda)$ is holomorphic at λ_0.*

Proof. (i): (2) $\Rightarrow$ (1). In the notation of Corollary 3.3, with $S(\lambda) = \operatorname{Ker} A(\lambda)$, the holomorphic operator $A_1(\lambda)$ has the property $A_1(\lambda)[\operatorname{Ker} A(\lambda_0)] = \operatorname{Ker} A(\lambda)$, for $\lambda \in V_1$. This and the invertibility of A_1 imply that

$$(*) \qquad \operatorname{Ker}[A(\lambda)A_1(\lambda)] = \operatorname{Ker} A(\lambda_0) \ \text{ for } \lambda \in V_1 \, .$$

Note also that $A_1(\lambda_0) = I_X$. Consider the operator $U_1(\lambda) = A(\lambda_0) - A(\lambda)A_1(\lambda)$. From (*) it follows immediately that $\operatorname{Ker} U_1(\lambda) \supset \operatorname{Ker} A(\lambda_0)$. Furthermore, $U_1(\lambda_0) = 0$, so that if $B_0 \in L(Y, X)$ is a relative inverse of $A(\lambda_0)$, then $\| U_1(\lambda) \|$ will be less than $1/\| B_0 \|$ in a neighborhood $\tilde{V}_1 \subset V_1$ of λ_0.

Thus $U_1(\lambda)$, for $\lambda \in \tilde{V}_1$, satisfies the conditions of Theorem 2.3 and so $R(U_1(\lambda)) = B_0(I_Y - U_1(\lambda)B_0)^{-1} = (I_X - B_0U_1(\lambda))^{-1}B_0$ is a relative inverse of $A(\lambda_0) - U_1(\lambda) = A(\lambda)A_1(\lambda)$. In other words,

$$[A(\lambda)A_1(\lambda)]R(U_1(\lambda))[A(\lambda)A_1(\lambda)] = A(\lambda)A_1(\lambda) \, ,$$

$$R(U_1(\lambda))[A(\lambda)A_1(\lambda)]R(U_1(\lambda)) = R(U_1(\lambda)) \, .$$

Cancel $A_1(\lambda)$ in the first of these relations and multiply from the left by $A_1(\lambda)$ in the second to conclude that $A_1(\lambda)R(U_1(\lambda))$ is a relative inverse of $A(\lambda)$ for $\lambda \in \tilde{V}_1$. This relative inverse is obviously holomorphic.

(ii): (3) $\Rightarrow$ (1). Let $Q(\lambda)$ be a holomorphic projector onto $\operatorname{Im} A(\lambda)$ in a neighborhood of λ_0. As in the previous case, by

Corollary 3.3 the operator $A_2(\lambda) = Q(\lambda)Q(\lambda_0) + (I_Y - Q(\lambda))(I_Y - Q(\lambda_0))$ is holomorphic, invertible and $A_2(\lambda)[\operatorname{Im} A(\lambda_0)] = \operatorname{Im} A(\lambda)$, for λ in a neighborhood V_2 of λ_0. The last two properties imply that

$$(**) \qquad \operatorname{Im}[A_2^{-1}(\lambda)A(\lambda)] = \operatorname{Im} A(\lambda_0) \quad \text{for} \quad \lambda \in V_2 .$$

Consider the operator $U_2(\lambda) = A(\lambda_0) - A_2^{-1}(\lambda)A(\lambda)$. From (**) it follows immediately that $\operatorname{Im} U_2(\lambda) \subset \operatorname{Im} A(\lambda_0)$. Furthermore, since $A_2(\lambda_0) = I_Y$, $U_2(\lambda_0) = 0$. Similarly as in the previous case, if B_0 is a relative inverse of $A(\lambda_0)$, we conclude that there is a neighborhood $\tilde{V}_2$ of λ_0 such that $\|U_2(\lambda)\| < 1/\|B_0\|$ and $\operatorname{Im} U_2(\lambda) \subset \operatorname{Im} A(\lambda_0)$ for $\lambda \in \tilde{V}_2$.

By Theorem 2.5, $R(U_2(\lambda)) = B_0(I_Y - U_2(\lambda)B_0)^{-1} = (I_X - B_0 U_2(\lambda))^{-1}B_0$ is a relative inverse of $A(\lambda_0) - U_2(\lambda) = A_2^{-1}(\lambda)A(\lambda)$. In a similar way as in part (i), we derive that then $R(U_2(\lambda))A_2^{-1}(\lambda)$ is a holomorphic relative inverse of $A(\lambda)$, for $\lambda \in \tilde{V}_2$.

(iii): (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). If $B(\lambda)$ is a holomorphic relative inverse of $A(\lambda)$ at λ_0, then $P(\lambda) = I_X - B(\lambda)A(\lambda)$ and $Q(\lambda) = A(\lambda)B(\lambda)$ are holomorphic projectors onto $\operatorname{Ker} A(\lambda)$ and $\operatorname{Im} A(\lambda)$, respectively.

In order to facilitate further expression we introduce the following two definitions.

DEFINITION 3.6. Let $S: G \to \Sigma(X)$ be given. We say that S is *locally holomorphic on* G if it is holomorphic at each point $\lambda_0 \in G$ in the sense of Definition 3.1. We say that S is *globally holomorphic on* G, or simply *holomorphic on* G, if there is a projector-valued holomorphic function $P: G \to L(X)$, such that $\operatorname{Im} P(\lambda) = S(\lambda)$ for all $\lambda \in G$.

DEFINITION 3.7. Let $A: G \to L(X, Y)$ be holomorphic. We say that A has a *local holomorphic relative inverse on* G if A has a holomorphic relative inverse at each point $\lambda_0 \in G$ in the sense of Definition 3.4. We say that A has a global *holomorphic relative inverse on* G, or simply a *holomorphic relative inverse on* G, if there is a holomorphic function $B: G \to L(Y, X)$, such that $B(\lambda)$ is a relative inverse of $A(\lambda)$ for all $\lambda \in G$.

To prove a global version of Theorem 3.5, we will use in an essential way the following result from [15; p. 161].

THEOREM 3.8. (Šubin) *If the subspace-valued function* $S: G \to \Sigma(X)$ *is locally holomorphic on* G *it is holomorphic on* G.

THEOREM 3.9. *Let* $A: G \to L(X, Y)$ *be holomorphic. Then the following statements are equivalent*:

(1) *The function* A *has a holomorphic relative inverse on* G.

(2) *The function* A *has a local holomorphic relative inverse on* G.

(3) *The function* $\lambda \to \operatorname{Ker} A(\lambda)$ *is locally holomorphic on* G *and* $A(\lambda)$ *has a relative inverse for each* $\lambda \in G$.

(4) *The function* $\lambda \to \operatorname{Im} A(\lambda)$ *is locally holomorphic on* G *and* $A(\lambda)$ *has a relative inverse for each* $\lambda \in G$.

(5) *The function* $\lambda \to \operatorname{Ker} A(\lambda)$ *and* $\lambda \to \operatorname{Im} A(\lambda)$ *are holomorphic on* G.

Proof. By Theorem 3.5, (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). That (1) $\Leftrightarrow$ (5) is shown as in part (iii) of the proof of Theorem 3.5. The implication (5) $\Rightarrow$ (3) is clear. It remains to show that (3) $\Rightarrow$ (1).

If (3) holds, then (4) does too. By Šubin's result the functions $\lambda \to \operatorname{Ker} A(\lambda)$ and $\lambda \to \operatorname{Im} A(\lambda)$ are globally holomorphic on G.

Let $P: G \to L(X)$ be a holomorphic projector with $\operatorname{Im} P(\lambda) = \operatorname{Ker} A(\lambda)$, and let $Q: G \to L(Y)$ be a holomorphic projector with $\operatorname{Im} Q(\lambda) = \operatorname{Im} A(\lambda)$. Recall that (Theorem 1.2) there is precisely one relative inverse of $A(\lambda)$, for a given $\lambda \in G$, corresponding to the direct decompositions $X = \operatorname{Ker} A(\lambda) \oplus \operatorname{Ker} P(\lambda)$ and $Y = \operatorname{Ker} Q(\lambda) \oplus \operatorname{Im} A(\lambda)$. Hence, let B_λ be the relative inverse of $A(\lambda)$ satisfying the conditions $\operatorname{Ker} B_\lambda = \operatorname{Ker} Q(\lambda)$ and $\operatorname{Im} B_\lambda = \operatorname{Ker} P(\lambda)$ for each $\lambda \in G$.

We now show that the function $\lambda \to B_\lambda$ is holomorphic on G.

Let $\lambda_0 \in G$. In the proof of Theorem 3.5 (we use below the same notation as there) we showed that, in a neighborhood of λ_0, the operator $\tilde{R}_2(\lambda) = R(U_2(\lambda))A_2^{-1}(\lambda)$ is a holomorphic relative inverse of $A(\lambda)$. To finish the proof, it is enough to prove the following

Claim. $B_\lambda = A_1(\lambda)\tilde{R}_2(\lambda) = A_1(\lambda)R(U_2(\lambda))A_2^{-1}(\lambda)$, in a neighborhood of λ_0.

Note that

$$A_2(\lambda)[\operatorname{Ker} B_{\lambda_0}] = A_2(\lambda)[\operatorname{Ker} Q(\lambda_0)] = \operatorname{Ker} Q(\lambda) = \operatorname{Ker} B_\lambda ,$$

so that $A_2^{-1}(\lambda)[\operatorname{Ker} B_\lambda] = \operatorname{Ker} B_{\lambda_0}$. This and the obvious $\operatorname{Ker} R(U_2(\lambda)) = \operatorname{Ker} B_{\lambda_0}$ and $\operatorname{Im} R(U_2(\lambda)) = \operatorname{Im} B_{\lambda_0}$ imply the following equalities:

(a) $$\operatorname{Ker} \tilde{R}_2(\lambda) = \operatorname{Ker} [R(U_2(\lambda))A_2^{-1}(\lambda)] = \operatorname{Ker} B_\lambda ,$$

(b) $$\operatorname{Im} \tilde{R}_2(\lambda) = \operatorname{Im} [R(U_2(\lambda))A_2^{-1}(\lambda)] = \operatorname{Im} B_{\lambda_0} .$$

From (a) and (b) it follows

(c) $$\operatorname{Ker} [A_1(\lambda)\tilde{R}_2(\lambda)] = \operatorname{Ker} \tilde{R}_2(\lambda) = \operatorname{Ker} B_\lambda ,$$

(d)
$$\begin{aligned}\operatorname{Im}[A_1(\lambda)\tilde{R}_2(\lambda)] &= A_1(\lambda)[\operatorname{Im} B_{\lambda_0}] = A_1(\lambda)[\operatorname{Ker} P(\lambda_0)] \\ &= \operatorname{Ker} P(\lambda) = \operatorname{Im} B_\lambda .\end{aligned}$$

Hence B_λ and $A_1(\lambda)\tilde{R}_2(\lambda)$ have the same kernel and range. If we show that the latter is a relative inverse of $A(\lambda)$, the claim will follow from the uniqueness part of Theorem 1.2. The following simple identities are based on the above observations: $P(\lambda_0)\tilde{R}_2(\lambda) = 0$ (by (b)); $A(\lambda)P(\lambda) = 0$ (by the definition of P). Therefore,

(f)
$$\begin{aligned}A(\lambda)A_1(\lambda)\tilde{R}_2(\lambda) &= A(\lambda)[P(\lambda)P(\lambda_0) + (I - P(\lambda))(I - P(\lambda_0))]\tilde{R}_2(\lambda) \\ &= A(\lambda)\tilde{R}_2(\lambda).\end{aligned}$$

Multiply (f) by $A(\lambda)$ on the right to get

(g)
$$A(\lambda)[A_1(\lambda)\tilde{R}_2(\lambda)]A(\lambda) = A(\lambda)\tilde{R}_2(\lambda)A(\lambda) = A(\lambda) ,$$

since $\tilde{R}_2(\lambda)$ is relative inverse of $A(\lambda)$.

Similarly, again using (f),

(h)
$$\begin{aligned}[A_1(\lambda)\tilde{R}_2(\lambda)]A(\lambda)[A_1(\lambda)\tilde{R}_2(\lambda)] &= [A_1(\lambda)\tilde{R}_2(\lambda)]A(\lambda)\tilde{R}_2(\lambda) \\ &= A_1(\lambda)[\tilde{R}_2(\lambda)A(\lambda)\tilde{R}_2(\lambda)] = A_1(\lambda)\tilde{R}_2(\lambda) ,\end{aligned}$$

and the claim follows.

This concludes the proof of Theorem 3.9.

COROLLARY 3.10. *Let $A: G \to L(X, Y)$ be holomorphic, and let $A(\lambda)$ be right invertible for each $\lambda \in G$. Then there is a holomorphic function $B: G \to L(Y, X)$ such that $A(\lambda)B(\lambda) = I_Y$ for all $\lambda \in G$.*

Proof. Since $\operatorname{Im} A(\lambda) = Y$, the function $\lambda \to \operatorname{Im} A(\lambda)$—being constant—is holomorphic.

COROLLARY 3.11. *Let $A: G \to L(X, Y)$ be holomorphic and let $A(\lambda)$ be left invertible for each $\lambda \in G$. Then there is a holomorphic function $B: G \to L(Y, X)$ such that $B(\lambda)A(\lambda) = I_X$.*

Proof. Here $\operatorname{Ker} A(\lambda) = \{0\}$, so that the function $\lambda \to \operatorname{Ker} A(\lambda)$ is holomorphic.

REMARKS. In [3] the equivalence of the statements (1) and (5) of Theorem 3.9 is proven. In [15] the equivalence of (1) and (2) is proven.

The Corollaries 3.10 and 3.11 are also seen in [3] and [15] with different proofs. For the case $Y = X$ they were first proved in [1]. In the next section we will present far reaching generalization of these corollaries.

In view of Theorem 3.5 it seems natural that the operator B_λ from the proof of Theorem 3.9 is also locally equal to $A_1(\lambda)R(U_1(\lambda))A_2^{-1}(\lambda)$. It can be shown that this is not so.

4. **Semi-Fredholm operators.** We first recall some basic definitions. All operators which we consider in this section will be assumed to have closed ranges.

If $A \in L(X, Y)$, one defines

$$\alpha(A) = \dim \operatorname{Ker} A$$
$$\beta(A) = \operatorname{codim} \operatorname{Im} A = \dim (Y/\operatorname{Im} A) .$$

Here (A) and $\beta(A)$ may be finite natural numbers or $+\infty$. In terms of $\alpha(A)$ and $\beta(A)$ one defines three classes of operators, namely, *semi-Fredholm of the first kind, semi-Fredholm of the second kind*, and *Fredholm*, as *follows*:

$$\Phi_+(X, Y) = \{A \in L(X, Y): \alpha(A) < \infty\} ,$$
$$\Phi_-(X, Y) = \{A \in L(X, Y): \beta(A) < \infty\} ,$$
$$\Phi(X, Y) = \Phi_+(X, Y) \cap \Phi_-(X, Y) .$$

If $A \in \Phi(X, Y)$, then $\operatorname{Ker} A$ and $\operatorname{Im} A$ are complemented, so that (Theorem 1.2) every Fredholm operation is relatively invertible. However, this is not so for semi-Fredholm operators, in general. Since we are concerned with the existence of relative inverses, when considering semi-Fredholm operators, we shall restrict our attention to the following two subclasses of operators:

$$\Phi_+^r(X, Y) = \{A \in \Phi_+(X, Y): \operatorname{Im} A \text{ is complemented}\} ,$$
$$\Phi_-^r(X, Y) = \{A \in \Phi_-(X, Y): \operatorname{Ker} A \text{ is complemented}\} ,$$

the members of which are relatively invertible. Some authors call such operators *projective semi-Fredholm operators* of the first or second kind, respectively. Note that Fredholm operators are projective semi-Fredholm of both kinds.

Let $A: G \to \Phi_+^r(X, Y)$ be holomorphic. Assume further, that $A(\lambda)$ has a holomorphic relative inverse $B(\lambda)$. Then we know that $\dim \operatorname{Ker} A(\lambda)$ is constant on G. Similarly, if $A: G \to \Phi_-^r(X, Y)$ is holomorphic and possesses a holomorphic relative inverse, then codim $\operatorname{Im} A(\lambda)$ is constant on G. (Compare the comments after Definition 3.1.) These remarks give the "only if" part of the following two theorems.

THEOREM 4.1. *Let $A: G \to \Phi_+^r(X, Y)$ be holomorphic. Then A*

has a holomorphic relative inverse on G if and only if $\alpha(A(\lambda))$ *is constant of G.*

Proof. Let $\alpha(A(\lambda))$ be constant on G, and let $\lambda_0 \in G$. Further, let $X = \operatorname{Ker} A(\lambda_0) \oplus X_1$, $Y = Y_1 \oplus \operatorname{Im} A(\lambda_0)$, and let B_0 be the relative inverse of $A(\lambda_0)$ corresponding to these decompositions; i.e., $\operatorname{Ker} B_0 = Y_1$, $\operatorname{Im} B_0 = X_1$.

Consider the operator

$$J(\lambda) = I_X - B_0(A(\lambda_0) - A(\lambda)) .$$

Note that $J(\lambda_0) = I_X$, so that $J(\lambda)$, which is obviously holomorphic on G, is also invertible in a neighborhood of λ_0. Since $I_X - B_0A(\lambda_0)$ is a projector onto $\operatorname{Ker} A(\lambda_0)$ along X_1, the following inclusions are clear:

$$J(\lambda)(X_1) = B_0A(\lambda)(X_1) \subset X_1 \tag{1}$$

$$J(\lambda)(\operatorname{Ker} A(\lambda)) = (I_X - B_0A(\lambda_0))(\operatorname{Ker} A(\lambda)) \subset \operatorname{Ker} A(\lambda_0) . \tag{2}$$

When $J(\lambda)$ is invertible, the equality $\alpha(A(\lambda)) = \alpha(A(\lambda_0)) < \infty$ implies equality in (2). Defining $A_1(\lambda) = [J(\lambda)]^{-1}$, we have

$$A_1(\lambda)[\operatorname{Ker} A(\lambda_0)] = \operatorname{Ker} A(\lambda) ,$$

for λ in a neighborhood of λ_0. By Corollary 3.3 the function $\lambda \to \operatorname{Ker} A(\lambda)$ is holomorphic at λ_0.

Since λ_0 is an arbitrary point of G, the function $\lambda \to \operatorname{Ker} A(\lambda)$ is locally holomorphic on G. Thus the statement (3) in Theorem 3.9 holds, and hence A has a holomorphic relative inverse on G.

THEOREM 4.2. *Let* $A: G \to \Phi_-^r(X, Y)$ *be holomorphic. Then A has a holomorphic relative inverse on G if and only if* $\beta(A(\lambda))$ *is constant on G.*

Proof. Since $\beta(A) = \alpha(A^*)$, where A^* is the conjugate of A (see, for example, [5; p. 7]), $A^*(\lambda)$ is holomorphic and belongs to $\Phi_+^r(Y^*, X^*)$. By Theorem 4.1 A^* has a holomorphic relative inverse B^* on G. This in turn implies that A^{**} has a holomorphic relative inverse B^{**} on G. In particular (or better: equivalently), the function $\lambda \to \operatorname{Ker} A^{**}(\lambda)$ is holomorphic. Applying the canonical imbedding of X into X^{**}, we can view $A(\lambda)$ as the restriction of $A^{**}(\lambda)$ on X.

If $P: G \to L(X^{**}, Y^{**})$ is a holomorphic projector with $\operatorname{Im} P(\lambda) = \operatorname{Ker} A^{**}(\lambda)$, then $P(\lambda)|X$ is also a holomorphic projector and $\operatorname{Im}(P(\lambda)|X) = \operatorname{Ker}(A^{**}(\lambda)|X) = \operatorname{Ker} A(\lambda)$.

Thus, the function $\lambda \to \operatorname{Ker} A(\lambda)$ is holomorphic on G and the stated conclusion follows from Theorem 3.9.

Partial results for the semi-Fredholm case were obtained in [2; pp. 52-54]. A special case of our Theorem 4.1, with $A(\lambda) \in \Phi(X, Y)$, was obtained in [3; p. 192], [15; p. 164], and [7, p. 54].

Added in proof. Theorems 4.1 and 4.2 are contained (as the author subsequently learned) in the meromorphic result of H. Bart, M. A. Kaashoek, D. C. Lay ["Relative inverses of meromorphic operator functions and associated holomorphic projection function", Math. Ann. 218 (1975), 199-210], proved by different and more involved methods.

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