THE OPERATOR EQUATION $A^* A^2 = A$

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In [1] the following theorem is proved:

Theorem 1. Let A be an operator with the property $A^* A^2 = A$. Then A is subnormal if and only if A is a contraction i.e. $\|A\| \leqslant 1$.

In this paper we wili give another proof of this theorem.

Throughout this papar H will be a separable Hilbert space over the field of complex numbers.

We recall that a bounded linear operator T in the Hilbert space H is normal if $T^* T = TT^*$. An operator T on a Hilbert space H is called subnormal if there exists a Hilbert space $K \supseteq H$ and a normal operator B on K such that H is an invariant subspace of B and $B|H = T$.

Definition: Let H be a complex Hilbert space and let $\{A_0' A_1' A_2', \ldots,\}$ be a uniformly bounded sequence of bounded operators on H. Let $H^{(1)} = \bigoplus_{n=0}^{\infty} H_n$, $H_n = H$ be a Hilbert space of all sequences of vectors $\{f_n\}_{n=0}^{\infty}$, $f_n \in H$ such that

$$\|f\|^2 = \sum_{n=0}^{\infty} \|f_n\|^2 < \infty.$$

The scalar product of two vectors $f = \{f_n\}$ and $g = \{g_n\}$ is defined by

$$(f, g) = \sum_{n=0}^{\infty} (f_n, g_n)$$

An operator valued weightedt shift is an operator defined on $H^{(1)}$ by the formula

$$T(f_0, f_1, f_2, \ldots) = 0, A_0 f, A_1 f_1, A_2 f_2 \ldots)$$

In ordet to prove theorem 1 we will need the following two theorems.

Theorem A. [3]. Let T be an operator valued weighted shift with weights $\{A_i^{(1)}\}_{i=0}^{\infty}$, where $\| A_i^{(1)}\| \leqslant M$. Then the operator T is subnormal if and only if

(i) $| A_j^{(n)} |^2 + (C_j^{(n-1)})^2 - | A_{j-1}^{(n)*} |^2 \geqslant 0$

(ii) there exists a sequence of positive real numbers $\{\lambda_j^{(n)}\}$ such that the following inequality holds

$$A_j^{(n)*} (| A_{j+1}^{(n)} |^2 + (C_{j+1}^{(n)})^2 - | A_j^{(n)*} |^2) A_j \leqslant \tag{1}$$

$$\leqslant \lambda_j^{(n)} (| A_j^{(n)} |^2 + (C_j^{(n)})^2 - | A_{j-1}^{(n)*} |^2)$$

(iii) there exisis a constant M such that $\| A_j^{(n)} \| \leqslant M$ where

$$C_j^{(n)} = (| A_j^{(n)} |^2 + (C_j^{(n-1)})^2 - | A_{j-1}^{(n)*} |^2)^{1/2}$$

and $A_j^{(n+1)}$ is a solution of the equation

$$X C_j^{(n)} = C_{j+1}^{(n)} A_j^{(n)} \tag{2}$$

Theorem B. [4] Let A be a bounded linear operator on a Hilbert space H with the property that $A^* A^2 = A$, then A is a direct sum of a zero operator, an unitary operator and an operator which is unitarily equivalent to the operator valued weighted shift with weights $\{P, I, I, \dots\}$ where $P = (A_0^* A_0)^{1/2}$, $M = \overline{AH}^{\perp}$ and A_0 is operator A from M into AM.

Proof of theorem 1. Suppose that operator A has the preoperty $A^* A^2 = A$. Then, the generality will be not lost if we suppose that operator A is an operator valued weighted shift with weights $\{P, I, \dots\}$, where P is positive with dense range in M, and acts on $M^{(1)} = M \oplus M \oplus M \oplus \cdots$

Assume that A is subnormal, then A is hyponormal and applying lemma 2 of [3] we see that $P^2 \leqslant I$, therefore $\| A \| \leqslant 1$.

Conversely, suppose that A is operator valued weighted shift with weights $\{P, I, I, \dots\}$, then $\| A \| \leqslant 1$ implies $0 \leqslant P^2 \leqslant I$. Now we will apply theorem A. We set $A_0^{(1)} = P$, and $A_j^{(1)} = I$ for $j \geqslant 1$.

Using the lemma 14 [3], we have to find a bounded sequence of positive real numbers $\lambda_j^{(n)}$ such that (1) holds.

If $n = 1$ we see

$$C_0^{(1)} = (P^2)^{1/2} = P; \quad C_1^{(1)} = (I - P^2)^{1/2} \text{ and } C_j^{(1)} = 0$$

for $j \geqslant 2$.

Then (1) takes the following form for $j = 0$

$$P (I + I - P^2 - P^2) P \leqslant \lambda_0^{(1)} (P^2 + P^2)$$

or equivalently

$$P((\lambda_0^{(1)} - 1) I + P^2) P \geqslant 0$$

which is obviosely true for $\lambda_0^{(1)} \geqslant 1$
For $j = 1$, the inequality (1) is

$$I(I - P^2) \leqslant \lambda_1^{(2)} (I + I - P^2 - P^2)$$

or equivalently

$$(2\lambda_1^{(1)} - I)(I - P^2) \geqslant 0 \text{ for all } \lambda_1^{(1)} \geqslant 1/2.$$

Set $(A_0^{(2)} Px = (I - P^2)^{1/2} Px$. Denote $(I - P^2)^{1/2}$ by P'.
Thus $(A_0^{(2)} - P') Px = 0$, for all x in M and since P has a dense range, we obtain $A_0^{(2)} = P'$.

Then, $A^{(1)}$, $A^{(2)}$, and $G^{(2)}$, from the proof of theorem A have the following forms

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & & \\ P & 0 & 0 & & \\ 0 & I & 0 & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix} \qquad G^{(2)} = \begin{bmatrix} P & 0 & 0 & & \\ 0 & P' & 0 & & \\ 0 & 0 & 0 & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix}$$

and

$$A^{(2)} = \begin{bmatrix} 0 & 0 & 0 & & \\ P' & 0 & 0 & & \\ 0 & 0 & 0 & \cdot & \\ & & & & \cdot \end{bmatrix}$$

For $n = 2$ and $j = 0$, we see

$$C_0^{(2)} = (|A_0^{(2)}|^2 + (C_0^{(1)})^2)^{1/2} = (P'^2 + P^2)^{1/2} = I.$$

and

$$C_1^{(2)} = (|A_1^{(2)}|^2 + (C_1^{(1)})^2 - |A_0^{(2)}|^2)^{1/2} =$$
$$= (I - P^2 - (I - P^2))^{1/2} = 0$$

and all $C_j^{(2)}$ (2 = 0 for $j \geqslant 2$.

The inequality (1) in this case has following from

$$\lambda_0^{(2)} (2 - P^2)(I - P^2) \geqslant 0$$

for every positive $\lambda_0^{(2)}$, and now $A^{(3)}$ and $G^{(3)}$ in the proof of theorem A have following forms: $A^{(3)} = 0$ and

$$G^{(3)} = \begin{bmatrix} I & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & \cdot \quad \cdot \\ & & & \quad \cdot \end{bmatrix}$$

Then $C_0^{(n)} = I$, for $n \geqslant 3$ and $C_j^{(n)} = 0$ for all $j \geqslant 2$ and $n \geqslant 2$.

Write $\hat{M} = M^{(1)} \oplus M^{(2)} \oplus \cdots$

where $M^{(2)} = M \oplus \overline{P' M} \oplus 0 \oplus 0 \oplus \cdots$

$$M^{(n)} = M \oplus 0 \oplus \ldots, \text{ for } n \geqslant 3,$$

and define the operator N which will act on $\hat{M}$ by the following matrix

$$N = \begin{bmatrix} A^{(1)} & G^{(2)} & 0 & 0 & \\ 0 & A^{(2)} & G^{(3)} & 0 & \\ & 0 & 0 & G^{(3)} & \cdot \\ & & & & \cdot \end{bmatrix}$$

Taking the adjoint of N we see

$$N^* = \begin{bmatrix} A^{(1)*} & 0 & 0 & & \\ G^{(2)} & A^{(2)*} & 0 & & \\ 0 & G^{(3)} & 0 & & \\ & \cdot & G^{(3)} & \cdot & \\ & & 0 & \cdot & \cdot \end{bmatrix}$$

Since $G^{(2)}$ and $G^{(3)}$ are selfadjoint, theorem A forces the subnormality of N. Also, by direct computations of $N^* N$ and NN^* and comparing the coresponding entries one can show that $N^* N = NN^*$. Since $N | M^{(1)} = A^{(1)} = A$, we see that A is subnormal. The proof of theorem 1 is complete.

REFERENCES

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ЗА ОПЕРАТОРСКАТА РАВЕНКА $A^*A^2 = A$

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(Р е з и м е)

Во овој труд се дава нов доказ на теоремата на Ембри [1]. Користејќи ги резулатите од [3] и [4] се докажува следната теорема.

Теорема. Нека A е ограничен линеарен оператор во Хилбертов простор H кој ја задоволува равенката $A^* A^2 = A$. Тогаш операторот A е субнормален ако и само ако е контракција т.е. $\| A \| \leqslant 1$.