THE OPERATOR EQUATIONS $A^*A^2 = A$ AND $A^{*2}A^2 = A^*A$

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The object of this paper is to prove the following two theorems.

Theorem 1. Let A be a bounded linear operator on a Hilbert space H, with the property that

$$A^*A^2 = A \tag{1}$$

then A is a direct sum of a zero operator, a unitary operator and an operator which is unitarily equivalent to the operator valued weighted shift with weights $\{P, I, I, \ldots,\}$ where $P = (A_0^* A_0)^{1/2}$, $M = (\overline{AH})^{\perp}$, and A_0 is the operator A from M into AM.

Theorem 2. Let A be a bounded linear operator on a Hilbert space H, with the property that $A^{*2}A^2 = A^*A$, then A has the representation $\begin{bmatrix} O & O \\ C & U \end{bmatrix}$ where U is an isometry.

Throughout this paper H will be a separable Hilbert space over the field of complex numbers. If S is a subset of H, then the orthogonal complement of S within H will be denoted by $S^{\perp}$.

For a given set M in H, the closure of M is denoted by $\dot{M}$. For a definition of operator valued weighted shifts see [3] and [4].

In order to prove theorem 1 we need the following lemmas.

Lemma 1. $Ker A$ reduces A^*.

Proof: Let $x \in Ker A$, $x \neq 0$, then $x \notin AH$, because A is an isometry on AH. Therefore, $x \in \overline{AH}^{\perp} = Ker A^*$, which implies that $Ker A$ is invariant under A^*.

In the sequel, we make the assumption that $Ker A = \{O\}$. Set $M = H \ominus \overline{AH}$.

Lemma 2. $\{A^n M\}$ is a pairwise orthogonal family of linear manifolds.

Proof. By linear manifold we mean a subspace but not necessarily a closed subspace. The fact $M \perp A^n M$ for $n \geq 1$ is true from the very definition of M. However,

$$(Am, A^2m') = (m, A^*A^2m') = (m, Am') = 0,$$

for all m, m' in M. In order to show that $A^pM \perp A^qM$, for $p \neq q$, we need the following relation $A^{*n}A^n = A^*A$ which is true for $n = 1, 2$ and using the simple mathematical induction we can prove that it is true for every natural n.

Using this fact, we see that for $p < q$

$$(A^pm, A^qm') = (A^{*p}A^pmA^{q-p}m') = (A^*Am, A^{q-p}m') = (m, A^{q-p}m') = 0.$$

For notational convenience we set $L_n = A^nM$, $n = 0, 1, 2, \ldots$

Then lemma 2 implies that $L_n \perp L_m$ for $n \neq m$, so $\{L_n\}$ is a family of pairwise orthogonal subspaces. Denote

$$M^{(1)} = \bigoplus_{n=o}^{\infty} L_n = \bigvee_{n=o}^{\infty} L_n.$$

Now we envoke techniques developed by Halmos [2] for an isometry.

Using continuity of A, we get

$$A(L_n) = A(\overline{A^nM}) \subseteq \overline{A(A^nM)} = L_{n+1},$$

which implies that $M^{(1)}$ is an invariant subspace for A and the restriction of A to $M^{(1)}$ is denoted by U. Write $H = X \oplus M^{(1)}$, where

$$X = (\bigoplus_{n=o}^{\infty} \overline{A^nM})^{\perp} = (\bigvee_{n=o}^{\infty} \overline{A^nM})^{\perp}.$$

Set $X = H \ominus M^{(1)}$.

Lemma 3.

$$X = H \ominus M^{(1)} = (\bigvee_{n=o}^{\infty} \overline{A^nM}) = \bigcap_{n=o}^{\infty} \overline{A^nH}$$

Proof: From the definition of M we have $H = \overline{AH} \oplus M$. Suppose that $h \in (\bigvee_{n=o}^{\infty} \overline{A^nM})^{\perp}$. In particular $h \in M^{\perp}$ which implies that $h \in \overline{AH}$.

Applying operator A we obtain

$$AH = A(\overline{AH}) + AM \tag{2}$$

Let $x \in A(\overline{AH})$, $y \in AM$, then $x = \lim_n A(Ax_n)$, $y = Am$, $m \in M$, $(x, y) = \lim_n (A^2x_n, Am) = \lim_n (A^*A^2x_n, m) = \lim_n (Ax_n, m) = 0$, so the sum (2) is orthogonal.

Since $\overline{AH}$ is a closed subspace and $A|\overline{AH}$ is an isometry, the immage $A(\overline{AH})$ is closed. Since the space $A(\overline{AH})$ is closed and the sum in (2) is orthogonal, from (2) we get

$$\overline{AH} = A(\overline{AH}) \oplus \overline{AM} \tag{3}$$

From (3) we find that $h \in \overline{AM}^{\perp}$ implies that $h \in A(\overline{AH})$.

The following formula is true

$$A(\overline{AH}) = \overline{A^2(H)} \tag{4}$$

Notice $A(\overline{AH})$ is closed as has been pointed out above. Since $A^2(H) \subseteq (\overline{AH})$ it is clear that $\overline{A^2H} \subseteq A(\overline{AH})$

Using the continuity of operator A we have $A(\overline{AH}) \subseteq \overline{A^2H}$. So formula (4) is proved.

By induction one can show the following generalizations of (4).

$$A(\overline{A^nH}) = \overline{A^{n+1}(H)}, \text{ and } A(\overline{A^nM}) = (\overline{A^{n+1}M}) \tag{5}$$

Applying operator A to the formula (3) we get

$A(\overline{AH}) = A(\overline{A^2H}) \oplus A(\overline{AM})$, and using (5) we get $\overline{A^2H} = \overline{A^3H} \oplus (\overline{A^2M})$. Since $h \in \overline{A^2H}$ and $h \in (\overline{A^2M})^{\perp}$ implies $h \in \overline{A^3H}$. By the induction we have that $h \in X$ implies $h \in \bigcap_{n=0}^{\infty} \overline{A^nH}$, so $X \subseteq \bigcap_{n=0}^{\infty} \overline{A^nH}$.

Now assume $g \in \bigcap_{n=0}^{\infty} \overline{A^nH}$. If $g \in \overline{A^{n+1}H}$, then $g = \lim_k A^{n+1}x_k$. By use of $A^{*n}A^n = A^{*2}A^2 = A^*A$ we find $(g, A^nm) = \lim_k (Ax_k, m) = 0$ for every $m \in M$, due to $M \perp AH$. Therefore $g \perp A^nM$ which implies $g \in \overline{A^nM}^{\perp}$. Since n was arbitrary we obtain $g \in X$.

Lemma 4. $X = \bigcap_{n=0}^{\infty} \overline{A^nH}$ is an invariant subspace for A; moreover $A|X$ is a unitary operator.

Proof: The invariance of X follows from the fact that $A(\overline{A^nH}) = \overline{A^{n+1}H}$. The last equality implies $AX = X$, therefore $A|X$ is unitary.

Lemma 5. For $n \geqslant 1$ operator A is mapping isometrically the space $\overline{A^nM}$ onto the space $\overline{A^{n+1}M}$.

Proof: For $m \in M$, we have $\|A(Am)\|^2 = (A^2m, A^2m) = (A^*A^2m, Am) = (Am, Am) = \|Am\|^2$, and in general $\|A(A^nm)\|^2 = (A(A^nm), A(A^nm)) = (A^*A^2A^{n-1}m, A^nm) = (A^nm, A^nm) = \|A^nm\|^2$.

Therefore we can extend these isometries to $\overline{A^n M}$, and $A(\overline{A^n M}) \subseteq \overline{A^{n+1} M}$. But does equality sign hold in the last inclusion for $n \geqslant 1$? $A^{n+1}(M) = A(A^n M) \subseteq A(\overline{A^n M})$, and the set $A(\overline{A^n M})$ is closed because $A|\overline{A^n M}$ is an isometry, and we have $\overline{A^{n+1}(M)} \subseteq A(\overline{A^n M})$.

Lemma 6. The operator $U = A|M^{(1)}$ has the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \\ A_0 & 0 & 0 & 0 & \\ 0 & A_1 & 0 & 0 & \\ & & A_2 & \cdot & \\ & & & \cdot & \cdot \end{pmatrix}$$

where $A_i : \overline{A^i M} \to \overline{A^{i+1} M}$ are isometries for $i \geqslant 1$, and moreover the above matrix is unitarily equivalent to the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ P_0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ & & I & \cdot \\ & & & \cdot \end{pmatrix}$$

with respect to $M \oplus M \oplus \ldots$, where $P_0 = (A_0^* A_0)^{1/2}$.

Proof: Write $A = U_0 P_0$ where $P_0 = (A_0^* A_0)^{1/2}$ and U_0 is a partial isometry from $\overline{P_0 M}$ onto $\overline{AM}$. But, $\overline{P_0 M} = M$. For, if not then there exists $m \neq 0$, $m \in M$. and $m \in \overline{P_0 M}^{\perp} = Ker P_0^* = Ker P_0$, which implies $m \in Ker A_0 \subseteq Ker A$, but this contradicts the assumption that $Ker A = \{0\}$ stated after lemma 1. Thus U_0 is an isometry from M onto $\overline{AM}$.

Set

$$V = \begin{bmatrix} 0 & 0 & 0 & \\ U_0 & 0 & 0 & \\ 0 & A_1 & 0 & \\ & & A_2 & \cdot \\ & & & \cdot \end{bmatrix} \qquad P = \begin{bmatrix} P_0 & 0 & 0 & \\ 0 & I_1 & 0 & \\ & & I_2 & \\ & & & \cdot \end{bmatrix}$$

on $M^{(1)}$, where I_i's are the identities on L_i for $i \geqslant 1$.

Then we have $U=VP$.

Using operators U_0, A_i, $i \geqslant 1$ we can define an isomorphism from $M \oplus M \oplus \ldots$ onto $M^{(1)}$ as follows: Set $W_0=I$, $W_1=U_0$, $W_2=A_1U_0$ and in general, $W_n=A_{n-1}W_{n-1}$ for $n \geqslant 2$. Then $W_n: M \to L_n$ is an isomorphism.

Set

$$W=\sum_{n=0}^{\infty} \oplus W_n : M \oplus M \oplus M \oplus \ldots \to M^{(1)}$$

A direct computation shows that $W^{-1}VW$ has the matrix

$$\begin{bmatrix} 0 & 0 & 0 & \\ I & 0 & 0 & \\ 0 & I & 0 & \\ & & \cdot & \cdot \\ & & & \cdot \end{bmatrix}$$

on $M \oplus M \oplus \ldots$

However, $W^{-1}UW=(W^{-1}VW)(W^{-1}PW)$ has the matrix

$$\begin{bmatrix} 0 & 0 & 0 & \\ P_0 & 0 & 0 & \\ 0 & I & 0 & \\ & & I & \cdot \\ & & & \cdot \end{bmatrix}$$

on $M \oplus M \oplus \ldots$: This completes the proof.

Proof of theorem 1. Lemma 4 implies that the matrix representation of operator A on $H=X \oplus M^{(1)}$ is of the following form

$$\begin{bmatrix} C & O \\ O & U \end{bmatrix}$$

where C is a unitary operator on X. Using lemma 6 we can write down the matrix of operator A with the respect to $X \oplus M \oplus \overline{AM} \oplus \ldots$ as following

$$\begin{bmatrix} C & 0 & 0 & & \\ 0 & 0 & 0 & 0 & \\ 0 & A_0 & 0 & 0 & \\ 0 & 0 & A_1 & 0 & \cdot \\ & & & & \cdot \end{bmatrix}$$

or equivalently to the matrix

$$\begin{bmatrix} C & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & P_0 & 0 & & \\ 0 & 0 & I & \cdot & \\ & & & \cdot & \cdot \end{bmatrix}$$

on $X \oplus M \oplus M \oplus \dots$

The proof of theorem 1 is completed.

Remark: Now, we will show that the converse of theorem 1 is true i. e. if A is a direct sum of a zero operator, unitary, and an operator valued weighted shift with weights $\{P, I, I, \dots\}$ where P is Hermitean, then operator A satisfies equation (1).

The only thing to be cheked is to show that the operator with the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & & \\ P & 0 & 0 & 0 & & \\ 0 & I & 0 & 0 & & \\ & & I & 0 & \cdot & \\ & & & & \cdot & \cdot \\ & & & & & \cdot \end{bmatrix}$$

satisfies equation (1), which is obvious.

Corollary: Every operator satisfying equation (1) has a non-trivial invariant closed subspace.

Proof: If $\overline{AH} \neq H$, we are done, because $A(\overline{AH}) \subseteq \overline{AH} \neq H$.

If $\overline{AH} = H$, we have $(A^*A - I)\overline{AH} = \{0\}$; $A^*A - I = 0$, which implies A is an isometry.

Space AH is closed, and we have $AH = \overline{AH} = H$ so A is unitary and has a nontrivial invariant closed subspace by the spectral theorem.

Proof of theorem 2. Let A be an operator satisfying the equation

$$A^{*2}A^2 = A^*A. \tag{6}$$

Operator A will be decomposed as a two-by-two matrix as follows:

$$A = \begin{bmatrix} B & D \\ C & U \end{bmatrix}$$

where $U: \overline{Range\,A} \to \overline{Range\,A}$; $D: \overline{Range\,A} \to \overline{Range\,A}^{\perp}$; $B: \overline{Range\,A} \to$ $\to \overline{Range\,A}^{\perp}$ and $C: \overline{Range\,A}^{\perp} \to \overline{Range\,A}^{\perp}$. Since $\overline{Range\,A}$ is invariant under A we have $D = 0$. If $y = Ax$, using (6), we have $\|Ay\|^2 = (AAx,$ $AAx) = (A^{*2}A^2x, x) = \|y\|^2$, $U = A|\overline{Range\,A}$ is an isometry. Hence A maps $\overline{Range\,A}^{\perp}$ into the $\overline{Range A}$, therefore $B = 0$.

Conversely, if A is a two-by-two matrix $\begin{bmatrix} 0 & 0 \\ C & U \end{bmatrix}$ on the space $H_1 \oplus H_2$, H_1 and H_2 are closed nontrivial subspaces, with $C \in L(H_1)$ and U unitary on H_2. Then

$$A^*A = \begin{bmatrix} C^*C & C^*U \\ U^*C & U^*U \end{bmatrix}$$

and

$$A^{*2}A^2 = \begin{bmatrix} C^*U^*UC & C^*U^*U^2 \\ U^{*2}UC & U^{*2}U^2 \end{bmatrix},$$

implies $A^{*2}A^2 = A^*A$, $H_2 = \overline{Range\,A}$.

Theorem 2 enables us to construct an example of an operator satisfying (6) but not (1). Take $A = \begin{bmatrix} 0 & 0 \\ C & U \end{bmatrix}$ on $H \oplus H$, where U is unitary. Then we have

$$A^*A^2 = \begin{bmatrix} C^*U & C^*U^2 \\ C & U \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} = A$$

in general. It is enough to take for example $C \neq 0$ and U unitary.

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РЕЗИМЕ

ЗА ОПЕРАТОРСКИТЕ РАВЕНКИ $A^*A^2 = A$ и $A^{*2}A^2 = A^*A$

Новак ИВАНОВСКИ

Во овој труд се покажува дека ограничениот линеарен оператор A во Хилбертовиот простор H кој ја задоволува релацијата $A^*A^2 = A$ е сума од нула оператор, унитарен оператор и оператор кој е унитарно еквивалентен со операторско тежински шифт со специјални тежини.

Исто така е дадена декомпозицијата на ограничениот линеарен оператор кој ја задоволува равенката $A^{*2}A^2 = A^*A$.

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