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ON BERBERIAN'S REPRESENTATION

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In this paper we shall show some new properties on the faithful star representation introduced by Berberian [2].

Throughout the paper H will be a separable Hilbert Space. We denote the algebra of bounded linear operators on H by $B(H)$.

Definition 1. *Let m be the Banach space of all bounded complex sequences (sup norm). If $s=\{\lambda_n\}\in m$, let $s^+=\{\lambda_{n+1}\}$. Also let $1\in m$ be the sequence consisting of ones. Then a generalized limit (for short) $g\lim$ is a linear functional on m such that*

(*a*) $L(1)=1$

(*b*) $L(s)=L(s^+)$ *for all* $s\in m$, *(translation invariance)*

(*c*) $L(\{a_n\})>0$ *if* $a_n>0$; *for* $n>0$.

Banach [1] *showed that generalized limits exist. Moreover if $\{a_n\}\in m$ is a real sequence, it is easy to show that*

$$\liminf_n a_n \leqslant g\lim(\{a_n\}) \leqslant \limsup_n a_n$$

and if $\lim \lambda_n=\alpha$, *then* $g\lim(\{\lambda_n\})=\alpha$.
The following result is taken from Berberian [2].

Teorem A. *Let H be a separable Hilbert space. Then there exists a Hilbert space K and a faithful star representation $\varphi: B(H)\to B(K)$ with the following properties:*

(1) $\varphi(S+T)=\varphi(S)+\varphi(T)$

(2) $\varphi(S\cdot T)=\varphi(S)\cdot\varphi(T)$

(3) $\varphi(T^*)=\varphi(T)^*$

(4) $\|\varphi(T)\|=\|T\|$

(5) $\varphi(I)=$ *identity on* K.

(6) $\sigma(\varphi(T))=\sigma(T)$

(7) $a(T)=a(\varphi(T))=\sigma_p(\varphi(T))$

(8) *for any bounded in norm sequence $\{x_n\}$ in H there exists a vector $u\in K$, such that*

$$\|\varphi(T)u\|=g\lim_n\|Tx_n\| \text{ for all } T\in B(H).$$

Using theorem A (7) Berberian gave a very elementary proof of the fact that every normal operator has an approximate point value λ, such that $|\lambda| = ||T||$. The representation $T \to \varphi(T)$ has very nice properties.

Theorem 1. *Let T be normal, quasi-normal, subnormal, hyponormal, paranormal, normaloid operator, then $\varphi(T)$ is normal, quasi-normal, subnormal, hyponormal, paranormal and normaloid operator respectively.*

Remark: Before proving this theorem we give some notations from Berberian [2]..

B denotes the set of all sequences $s = \{x_n\}$ with $x_n \in H$ and $\{||x_n||\}$ bounded. If $s = \{x_n\}$ and $t = \{y_n\}$ are elements of B then formula $\psi(s, t) = g \lim (x_n, y_n)$ defines a positive symmetric bilinear form on B. By N, we denote the set $\{s \in B;$ such that $\psi(s, s) = 0\}$. The quotient space B/N is denoted by P, in which the scalar product is defined by $(s', t') = \psi(s, t)$, where $s' = s + N$ and $t' = t + N$. The completion of P is a Hilbert space K.

Proof: The properties of being normal and quasi-normal are preserved because of properties (2) and (3) of theorem A. For subnormality we will use Halmos-Bram characterization, (see Halmos [4] and Bram [3]. For notational convenience we set $\varphi(T) = T^\circ$. For every finite family of vectors $u^{(0)}, u^{(1)}, \dots u^{(n)}$ in K we have to prove that

$$\lambda = \sum_{i,j=0}^{n} (T^{oj} u^{(i)}, T^{oi} u^{(j)}) \geqslant 0$$

Suppose not, then there exists a positive real number $\varepsilon > 0$ such that

$$S(\lambda, \varepsilon) \cap R^+ = \varnothing,$$

where $S(\lambda, \varepsilon) = \{\mu \in C)$, such that $|\mu - \lambda| < \varepsilon\}$ and R^+ is the set of nonnegative real numbers. Without loss of generality we may assume that $\varepsilon < m$, where $m = \max\limits_{0 \leqslant k \leqslant n} ||u^{(k)}||$.

Since the set P is dense in K; there exists a finite set of vectors

$$x^{(0)}, x^{(1)}, \dots, x^{(n)}$$

in P, such that

$$||u^{(k)} - x^{(k)}|| < \frac{\varepsilon}{3m ||T||^{2n} (n+1)^2}, \quad k = 0, 1, 2, \dots n.$$

Then we have

$$|\lambda - \sum_{i,j=0}^{n} (T^{0j} x^{(i)}, T^{0i} x^{(j)})| = |\sum_{i,j=0}^{n} (T^{0j} u^{(i)}, T^{0i} u^{(j)}) - (T^{0j} x^{(i)}, T^{0i} x^{(j)})|$$

$$\leqslant \sum_{i,j=0}^{n} (||T^{0j}|| \cdot ||u^{(i)} - x^{(i)}|| \cdot ||T^{0i} u^{(j)}|| + ||T^{0j} x^{(i)}|| \, ||T^{0i}|| \, ||u^{(j)} - x^{(j)}||)$$

$$\leqslant \sum_{i,j=0}^{n} (||T||^{2n} m ||u^{(i)} - x^{(i)}|| + ||T||^{2n} \cdot 2m . \; ||u^{(j)} - x^{(j)}||) \leqslant$$

$$< ||T||^{2n} 3m \cdot (n+1)^2 \cdot \frac{\varepsilon}{3m ||T||^{2n} (n+1)^2} = \varepsilon,$$

But, $x^{(i)} = '\{y_n^{(i)}\} + N$, where $'\{||y_n^{(k)}||\}$ is bounded and $T^{0j} x^{(i)} = \{T^j y_n^{(i)}\} + N$.

From the definition of scalar product we see

$$(T^{0j}, x^{(i)}, T^{0i} x^{(j)}) = g \lim_n (T^j y_n^{(i)}, T^i y_n^{(j)})$$

Applying the linearity of g lim, we obtain

$$\sum_{i,j=0}^{n} (T^{0j} x^{(i)}, T^{0i} x^{(j)}) = g \lim_n \sum_{i,j=0}^{n} (T^j y_n^{(i)}, T^i y_n^{(j)}))$$

and the last summation is positive by the Halmos-Bram theorem; which is a contradiction to the fact that $S(\lambda, \varepsilon) \cap R^+ = \varnothing$.

If T is a hyponormal, then $T^* T - TT^* = S^2$, where S is positive. Then for $u \in K$ we have

$$(\varphi(T)^* \varphi(T) - \varphi(T) \varphi(T)^*) u, u) = (\varphi(S^2) u, u)$$
$$= (\varphi(S) \cdot \varphi(S) u, u) = \|\varphi(S) u\|^2 \geqslant 0$$

So $\varphi(T)$ is hyponormal.

If T is paranormal, i.e. $\|T^2 x\| \geqslant \|Tx\|^2$, for $\|x\| = 1$ then using the following lemma of T. Andő. (see [6] Lectures Notes in Mathematics no. 247, pp. 547). T is paranormal if and only if

$$T^{*2} T^2 - 2\lambda T^* T + \lambda^2 I \geqslant 0 \text{ for positive } \lambda.$$

Paranormality of $\varphi(T)$ follows from 2) and 3) of theorem A, and the property c) of g lim and the above mentioned lemma of T. Andö.

If T is normaloid, i.e. $\|T\| = \text{sub}\{|\lambda|, \lambda \in \sigma(T)\}$ then by 4) of Theorem A we have

$$\|\varphi(T)\| = \|T\| = \text{sub}\{|\lambda|, \lambda \in \sigma(T)\} = \sup\{|\lambda|; \lambda \in \sigma(\varphi(T))\}.$$

All the properties of the operator T, are not preserved under the representation $T \to \varphi(T)$. We have the following:

Theorem 2. *If T is compact operator then $\varphi(T)$ is not necessarily compact.*

Proof: We will construct an operator with finite dimensional range (namely one dimensional range) such that $\varphi(T)$ is not compact. Let $\{e_n\}_{n=1}^{\infty}$ be the orthonormal basis of H. Set $Te_1 = e_1$; $Te_k = 0$, for $k > 1$. T is the projection from H onto one dimensional space spanned by e_1. We define a sequence of vectors in K, such that every vector of that sequence is an eigenvector for $\varphi(T)$ corresponding to 1, and moreover this sequence is an orthonormal and infinite.

Set $\hat{x}_{2(0)} = \{e_1, e_1, e_1, e_1, \ldots\}$

$\hat{x}_{2(1)} = \{e_1, e_1, -e_1, -e_1, e_1, e_1, \ldots\}$

$\hat{x}_{2(i)} = \{e_1, \ldots, e_1, -e_1, \ldots -e_1, e_1, \ldots e_1, \ldots\}$

Twe vectors $\hat{x}_{2(i)}$ have first $2^{(i)'}s$ coordinates e_1, second $2^{i'}s$ coordinates $-e_1$, and so on alternatively.

Two things are easy to check: First $\hat{x}_{2(i)}$ are unit vectors in K; since $\|\hat{x}_{2(i)}\|^2 = g \lim (1) = 1$. Second $\varphi(T) \hat{x}_{2(i)} = \hat{x}_{2(i)}$; which follows from the fact that

$$\|\varphi(T) \hat{x}_{2(i)} - \hat{x}_{2(i)}\| = g \lim \|(\pm 1)(Te_1 - e_1)\| = g \lim 0 = 0.$$

Using two times translation invariance of g lim we have

$$(\hat{x}_{2(0)}, \hat{x}_{2(1)}) = g\lim(1, 1, -1, -1, \dots) =$$
$$= g\lim(-1, -1, 1, 1, \dots) = (-1)g\lim(1, 1, -1, \dots).$$

So $(\hat{x}_{2(0)}, \hat{x}_{2(1)}) = 0$.

The proof of orthogonality of the family $\hat{x}_{2(i)}$ will be shown by induction. Suppose that $\hat{x}_{2(0)}, \dots, \hat{x}_{2(n)}$ is an orthogonal family. We want to show that the family of vectors $\{\hat{x}_{2(0)}, \dots, \hat{x}_{2(n)}, \hat{x}_{2(n+1)}\}$ is orthogonal, too. It is sufficient to show that $(\hat{x}_{2(i)}, \hat{x}_{2(n+1)}) = 0$

$$\forall i \in \{0, 1, 2 \dots n\}$$

$$(\hat{x}_{2(0)}, \dot{x}_{2(n+1)} = g\lim(1, \dots, 1, -1, \dots, -1, 1, \dots 1, \dots) =$$
$$= g\lim(-1, \dots, -1, 1, \dots 1, \dots) = (-1)g\lim, (1, \dots, 1-1, \dots 1, \dots)$$

$(\hat{x}_{2(0)}, \hat{x}_{2(n+1)}) = 0$. Translation invariance of g lim had been used 2^{n+1} times.

Let $1 < i < n$; $2^{n+1} = 2^i \cdot 2^{n+1-i}$.

For short set $a = (1, 1, \dots 1, -1, \dots -1, \dots, 1, \dots 1, -1, \dots, -1)$ a row of length 2^{n+1} where the first 2^i are ones, and then alternatively.

Then we have

$$(\hat{x}_{2(i)}, \hat{x}_{2(n+1)}) = g\lim(a, -a, a, \dots)$$
$$= g\lim(-a, a, \dots) = (-1)g\lim(a, -a, a, \dots)$$
$$(x_{2(i)}, x_{2(n+1)}) = 0.$$

Translation invariance of g lim had been applied 2^{n+1} times.

Therefore $\lambda = 1$ is an eigenvalue for $\varphi(T)$ with infinite multiplicity which shows that $\varphi(T)$ is not compact.

The construction in Theorem 2 enables us to prove following

Theorem 3. *If T and $\varphi(T)$ are compact operators then $T = 0$.*

Proof: From the polar decomposition we can assume that T° is a positive compact operator, $\neq 0$ and moreover $\|T^\circ\| = 1$.

Then $T \in B(H)$ and T is positive, compact operator. Then there exists a vector $e \neq 0$, such that $Te = e$; Set $e_0 = \frac{e}{\|e\|}$. Denote by P the projection onto the subspace spanned by e_0. Then $PT = TP$. It is easy to check that $T - P > 0$. However, P^0 is not compact by the same arguments used in Theorem 2. The last implies that T^0 is not compact which is a contradiction.

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