SUBNORMALITY OF OPERATOR VALUED WEIGHTED SHIFTS

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In this paper we shall answer on the following question: Which operator valued weighted shifts are subnormal?

Throughout this paper H will be a separable Hilbert space. By an operator on Hilbert space H we mean a bounded linear transformation of H into itself. The Hilbert space will be always over the field of complex numbers.

An operatior is normal if $T^*T=TT^*$. By a subspace we mean a closed linear manifold of a given Hilbert space H. A subspace L of H is invariant under T if $Tx\in L$ whenever $x\in L$. If L is an invariant subspace for T, then $T|L$ denotes the operator T restricted to L.

An operator T acting on a Hilbert space H is subnormal if there exists a Hilbert space K H, and a normal operator N on K such that H is invariant under N and $N|H=T$. The space K is called the extension space of H and operator N is called a normal extension of T. The minimal extension is unique up to unitary equivalence.

An operator T is hyponormal if $TT^*\leq T^*T$. It is easy to show that every normal operator is subnormal and also every subnormal operator is hyponormal. The positive square root of T^*T isdenoted by $|T|$.

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1.Definition. Let H be a complex Hilbert space and let $\{A_o, A_1, A_2, \ldots\}$ be a uniformly bounded sequence of bounded operators on H. Let $H^{(1)} = \bigoplus_{n=0}^{\infty} H_n$, $H_n = H$, be a Hilbert space of all sequences of vectors $\{f_n\}_{n=o}^{\infty}$, $f_n \in H$ such that $\|f\|^2 = \sum_{n=o}^{\infty} \|f_n\|^2 < \infty$

The scalar product of two vectors $f=(f_n)$ and $g=(g_n)$ is defined by $(f,g) = \sum_{n=o}^{\infty} (g_n, g_n)$.

An operator valued weighed shift is an operator defined on $H^{(1)}$ by the formula $T(f_o, f_1, f_2, \ldots) =$. $(O, A_o f_o, A_1 f_1, A_2 f_2, \ldots)$. The operator T has a matrix form as follows

$$\begin{bmatrix} O & O & O & \\ A_o & O & O & \\ O & A_1 & O & O \\ O & O & A_2 & O \\ & & \cdot & \cdot \\ & & & \cdot \end{bmatrix}$$

In tehe sequel, we are going to build up a normal extension of an operator valued weighted shift assuming it to be subnormal. The normal extension B will act on space K, containing $H^{(1)}$ as a subspace and the restriction of B onto $H^{(1)}$ is equal to T.

Since the operator T is subnormal it follows that T is hyponormal and it is natural to ask for necessary and sufficient conditions for T to be hyponormal with respect to weights A_n.

<u>2. Lemma.</u> A weighted shift T is hyponormal if and only if

$$A_i^* A_i \geq A_{i-1} A_{i-1}^* \text{, for } i = 1,2,\ldots.$$

This lemma is well known. It has been proved by Halmos [3] and used there to find an example of a hyponormal operator T such that T^2 is not hyponormal.

Direct computation shows

$$T^*T = \begin{bmatrix} A_0^*A_0 & 0 & 0 & \\ 0 & A_1^*A_1 & 0 & \\ 0 & 0 & A_2^*A_2 & \cdot \\ & & & \cdot \end{bmatrix} \text{ and } TT^* = \begin{bmatrix} 0 & 0 & 0 & \\ 0 & A_0A_0^* & 0 & \\ 0 & 0 & A_1A_1^* & \cdot \\ & & & \cdot \end{bmatrix}$$

from which the lemma is immediate.

Let B be a normal extension with $B|H = T$. The operator T has a matrix

$$\begin{bmatrix} 0 & A_0^* & 0 & 0 & \\ 0 & 0 & A_1^* & 0 & \\ 0 & 0 & 0 & A_2^* & \\ & & & \cdot & \cdot \\ & & & & \cdot \end{bmatrix}$$

In the sequel the vectors $f_j^{(1)}$ will be identified with the vectors $(0,0, \ldots, f_j^{(1)}, 0, \ldots)$ and some times we will use an upper index 1, to indicate that a vector belongs to $H^{(1)}$.

The immage of the vector $(o, \ldots, f_j^{(1)}, 0, \ldots)$ under B is the vector $(o, o, \ldots . A_j^{(1)} f_j^{(1)}, 0, \ldots)$ whose components are zero except for the $(j+1)^{st}$; hence we can write

$$Bf_j^{(1)} = A_j^{(1)} f_j^{(1)}$$

On the other hand we have

$$P^{(1)}B (o, \ldots, f_j^{(1)}, 0, \ldots) = A_{j-1}^{(1)} f_j^{(1)}$$

The last equality enables a us to write

$$(1) \qquad B f_j^{(1)} = A_{j-1}^{(1)} f_j^{(1)} + f_j^{(2)}$$

where $f_j^{(2)}$ is a vector in K orthogonal to $H^{(1)}$. The mapping $f_j^{(1)} \longrightarrow f_j^{(2)}$ is a bounded linear operator and that mapping will be denoted by $S_j^{(1)}$.

We will use $S_j^{(1)}$ and S_j interchangeably as a coninience.

3. Lemma. $f_j^{(2)} \perp f_j^{(2)}$, for $j \neq i$.

Proof: $(f_j^{(2)}, f_i^{(2)}) = (B^* f_j - A_{j-1}^* f_j, B^* f_i - A_{i-1}^* f_i)$

$= (B^* f_j, B^* f_i - A_{i-1}^* f) - (A_{j-1}^* f_j, f_i^{(2)}) =$

$= (Bf_j, Bf_i) - (f_j, BA_{i-1}^* f_i) = (A_j f_j, A_i f_i) -$

$- (f_j, A_{i-1} A_{i-1}^* f_i) = 0.$

First, we have used the fact that $f_j^{(2)} \perp A_{j-1}^* f_j$, then the normality of the operator B, the fact that $A_j f_j \perp A_i f_i$ for different i, j and finally $A_{i-1} A_{i-1} f_i \in H_i$; hence $f_j \perp A_{i-1} A_{i-1} f_i$.
Note that $\|S_j f_j^{(1)}\|^2 = \|f_j^{(2)}\|^2 =$

$$= (B^* f_j - A_{j-1}^* f_j, B^* f_j - A_{j-1}^* f_j) =$$

$$=(B^* f_j^{(1)}, B^* f_j^{(1)}) - (A_{j-1}^* f_j^{(1)}, B^* f_j^{(1)}) - (B^* f_j^{(1)}, A_{j-1}^* f_j^{(1)}) +$$

$$+ (A_{j-1} A_{j-1}^* f_j^{(1)}, f_j^{(1)}) =$$

$$= (A_j^* A_j f_j^{(1)}, f_j^{(1)}) - (B(A_{j-1}^* f_j^{(1)}), f_j^{(1)}) - (f_j, B(A_{j-1}^* f_j^{(1)}) +$$

$$+ (A_{j-1} A_{j-1}^* f_j^{(1)}, f_j^{(1)})$$

Since $A_{j-1} f_{j-1} \in H_{j-1}$ and $B(A_{j-1}^* f_j) = A_{j-1} A_{j-1}^* f_j \in H_j$, after cancelation we have

$$(2) \quad \|S_j f_j^{(1)}\|^2 = ((A_j^* A_j - A_{j-1} A_{j-1}^*) f_j^{(1)}, f_j^{(1)})$$

Denote by $H_j^{(2)}$ the closure of $S_j^{(1)} H_j^{(1)} = \mathrm{cl}\, S_j H$.

Lemma 3 implies tha the family $H_j^{(2)}$ is an orthogonal family of Hilbert spaces.
Let $H^{(2)} = \bigoplus_{j=0}^{\infty} H_j^{(2)}$.

Using the sequence of operators S_j we can define a bounded linear operator from $H^{(1)}$ into $H^{(2)}$ coordinatwise, namely

$$(Sf)_j = S_j f_j^{(1)}, \quad \text{for } f = (f_0^{(1)}, f_1^{(1)}, \ldots)$$

It is obvious that S is a linear operator, since all the "coordinate" operators are linear. Using the uniform boundness of the sequence of operators A_i we will show that S is a bounded operator.

From formula (2) we have

$$\|Sf\|^2 = \sum_{j=0}^{\infty} \|S_j f_j^{(1)}\|^2 = \sum_{j=0}^{\infty} (\|A_j f_j^{(1)}\|^2 - \|A_{j-1}^{*} f_j^{(1)}\|^2)$$

$$\leq \sum_{j=0}^{\infty} \|A_j f_j^{(1)}\|^2 \leq M^2 \sum_{j=0}^{\infty} \|f_j^{(1)}\|^2 = M^2 \|f\|^2$$

where $M = \sup \| A_j \|$, $j=0,1,2,\ldots$

<u>4. Definition.</u>

$$A_j^{(2)} = P_{j+1}^{(2)} B | H_j^{(2)} \text{ for } j=0,1,1 \ldots$$

Thus $A_j^{(2)} : H_j^{(2)} \longrightarrow H_{j+1}^{(2)}$ is a sequence of bounded operators, and moreover $\|A_j^{(2)}\| \leq \|B\|$ where $P_{j+1}^{(2)}$ denotes the orthogonal projection of K onto $H_{j+1}^{(2)}$. The operators $A_j^{(2)} S_j^{(1)}$ and $S_{j+1}^{(1)} A_j^{(1)}$ both map $H_j^{(1)}$ into $H_{j+1}^{(2)}$ and moreover we have following

<u>5. Lemma.</u>

$$A_j^{(2)} S_j^{(1)} = S_{j+1}^{(2)} A_j^{(1)}, \text{ for } j=0,1,2\ldots$$

<u>Proof:</u> Let $L = (Bf_{j+1}^{(2)}, Bf_j^{(1)})$, for a fixed j.

Then
$$L = (f_{j+1}^{(2)}, B^{*} A_j^{(1)} f_j^{(1)})$$

$$= (f_{j+1}^{(2)}, A_j^{(1)*} A_j^{(1)} f_j^{(1)} + S_{j+1}^{(1)} A_j^{(1)} f_j^{(1)}) =$$

$$= (f_{j+1}^{(2)}, S_{j+1}^{(1)} A_j^{(1)} f_j^{(1)}).$$

We have used formula (1) and the fact that $H^{(1)} \perp H^{(2)}$.

On the other hand, using the normality of B, invariance of $H^{(1)}$ under B, and again the orthogonality of $H^{(1)}$ and $H^{(2)}$ we have

$$L = (B^{*} f^{(2)}_{j+1}, A^{(1)*}_{j-1} f_j + S^{(1)}_j f_j^{(1)}) =$$

$$=(f^{(2)}_{j+1}, B(A^{(1)*}_{j-1} f_j^{(1)}) + (f^{(2)}_{j+1}, BS_j^{(1)} f_j^{(1)}) =$$

$$=(f^{(2)}_{j+1}, A_j^{(2)} S_j^{(1)} f_j^{(1)}).$$

Since $f^{(2)}_{j+1}$ was an arbitrary vector in $H^{(2)}_{j+1}$, the lemma is proved.

Using definition 4 we can define a bounded linear operator $A^{(2)}: H^{(2)} \longrightarrow H^{(2)}$ as follows

$$A^{(2)} f_j^{(2)} = A_j^{(2)} f_j^{(2)}, \text{ for } j = o,1,\ldots$$

Applying lemma 5 we will prove that the following diagram commutes

$$\begin{array}{ccc} H^{(1)} & \xrightarrow{\quad S \quad} & H^{(2)} \\ T \big\downarrow & & \big\downarrow A^{(2)} \\ H^{(1)} & \xrightarrow[\quad S^{(2)} \quad]{} & H^{(2)} \end{array}$$

Indeed, $ST(f_o,f_1,f_2, \ldots,) = S(0,A_of_o,A_1f_1,A_2f_2, \ldots)$

$= (0,S_1A_of_o,S_2A_1f_1, \ldots)$ and

$A^{(2)}S(f_o,f_1,f_2, \ldots) = A^{(2)}(S_of_o,S_1f_1,S_2f_2, \ldots)$

$= (0,A_o^{(2)}S_of_o,A_1^{(2)}S_1f_1,A_2^{(2)} S_2f_2, \ldots)$.

For short , $S^{(1)} = S$ and $A^{(1)} = T$.

6. Lemma. If $A_i^{(1)}H_i^{(1)}$ is a dense subset in $H_{i+1}^{(1)}$ then:

(1) $S_{i+1}^{(1)}A_iH_i^{(1)}$ is a dense subset in $H_{i+1}^{(2)}$, and

(2) $A_i^{(2)}H_i^{(2)}$ is a dense subset in $H_{i+1}^{(2)}$.

Proof:

Let $\varepsilon > 0$; $y \in H_{i+1}^{(2)}$; then there exists $y' \in S_{i+1}^{(1)}H_{i+1}^{(1)}$ such that $\|y-y'\| < \frac{\varepsilon}{2}$. Since $y' \in S_{i+1}^{(1)}H_{i+1}^{(1)}$ we have $y' = S_{i+1}^{(1)}x'$, where $x' \in H_{i+1}^{(1)}$. Since $x' \in H_{i+1}^{(1)}$ and $A_i^{(1)}H_i^{(1)}$ is a dense set in $H_{i+1}^{(1)}$ there exists $x'' \in A_i^{(1)}H_i^{(1)}$ such that $\|x - x''\| < \frac{\varepsilon}{2M}$ where $M = \sup \|A_i\|$ and $x'' = A_i^{(1)}x$.

Then $\|y - S_{i+1}^{(1)}A_i^{(1)}x\| \leq \|y - S_{i+1}^{(1)}x'\| + \|S_{i+1}^{(1)}A_i^{(1)}x' - S_{i+1}^{(1)}A_i^{(1)}x\|$

$$< \frac{\varepsilon}{2} + M\|x' - A_i^{(1)}x\| = \varepsilon$$

The proof of (2) is trivial since if $y \in H_{i+1}^{(2)}$ then by (1), for every $\varepsilon > 0$ there exists $y' \in S_{i+1}^{(1)}A_i^{(1)}H_i^{(1)}$ such that $\|y - y'\| < \varepsilon$, but $y' \in S_{i+1}^{(1)}A_i^{(1)}H_i^{(1)} = A_i^{(2)}S_i^{(1)}H_i^{(1)} \subseteq$ $\subseteq A_i^{(2)}H_i^{(2)}$, and $\|y-y'\| < \varepsilon$.

In the sequel we need Douglas decomposition theorem.

Theorem A. [2].

If A and Q are bounded operators on Hilbert space H then the following conditions are equivalent:

(1) $AA^* \leq \lambda^2 QQ^*$ for some constant $\lambda > 0$.

(2) $A = QC$ for some bounded operator C.

In subsequent analysis we will use Douglas' theorem when A and Q are bounded operators from given Hilbert spaces H_1 and H_2, respectively, into given Hilbert space H. Then operator C in the decomposition (2) will be an operator from H_1 into H_2.

Now we will give an equivalent form of lemma 5 which we will need later in the proof of main theorem.

7. Lemma. The equation $A_j^{(2)} S_j^{(1)} = S_{j+1}^{(1)} A_j^{(1)}$ together with the boundness of $A^{(2)}$ implies the existence of bounded sequence of positive numbers $\{\lambda_j^{(1)}\}$ such that the following inequality holds

$$(5) \quad A_j^* (|A_{j+1}|^2 - |A_j^*|^2) A_j \leq \lambda_j^{(1)} (|A_j|^2 - |A_{j-1}^*|^2)$$

where for the brevity we use

$$|A_{j+1}|^2 = A_{j+1}^{(1)*} A_{j+1}^{(1)}, \quad \text{and} \quad |A_j^{(1)*}|^2 = A_j^{(1)} A_j^{(1)*}$$

for $j = 0, 1, 2, ..$

Proof: Taking the adjoint of the equation

$A_j^{(2)} S_j^{(1)} = S_{j+1}^{(1)} A_j^{(1)}$ we see $A_j^* S_{j+1}^* = S_j^* A_j^{(2)*}$.

Set $A = A_j^* A_{j+1}^*$; $Q = S_j^*$ and $A_j^{(2)*} = C$.

By theorem A , the operator A decomposes as product of Q and C if and only if $\exists$ $\lambda_j^{(1)} \geq 0$ such that

$$AA^* \leq \lambda_j^{(1)} QQ^*,$$

or

$$(A_j^* S_{j+1}^* S_{j+1} A_j f_j, f_j) \leq \lambda_j^{(1)} (S_j^* S_j f_j, f_j)$$

for $j=0,1,2, \ldots$

or equivalently

$$(S_{j+1} A_j f_j, S_{j+1} A_j f_j) \leq \lambda_j^{(1)} ((A_j^* A_j - A_{j-1} A_{j-1}^*) f_j, f_j) .$$

Now, we are going to calculate left side of the last inequality. Since $A_j f_j \in H_{j+1}^{(1)}$, $S_{j+1} S_j f_j \in H_{j+1}^{(2)}$ and $A_j^* A_j f_j \in H_j^{(1)}$ we get

$$(5) \qquad S_{j+1} A_j f_j = B^* A_j f_j - A_j^* A_j f_j$$

Using equation (5) and the orthogonality of $H^{(1)}$ and $H^{(2)}$ and our construction we have

$$L = (S_{j+1}A_jf_j, S_{j+1}A_jf_j) = (S_{j+1}A_jf_j, B^*A_jf_j) =$$

$$= (B^*A_jf_j, B^*A_jf_j) - (A_j^* A_jf_j, B^*A_jf_j)$$

$$= (B A_jf_j, B A_jf_j) - (A_j^* A_jf_j, A_j^* A_j f_j)$$

$$= (A_{j+1}A_jf_j, A_{j+1}A_jf_j) - (A_j^* A_jA_j^* A_jf_j, f_j)$$

$$= (A_j^* (|A_{j+1}|^2 - |A_j^*|^2) A_jf_j, f_j)$$

Thus inequality (5) x is proved.

We rwmark that the operator $A^{(2)}$ is bounded if and only if the sequence $\{\lambda_j^{(1)}\}$ is bounded.

If $\{\lambda_j^{(1)}\}$ is bounded, then the solutions $A_j^{(2)}$ of the equation $X S_j^{(1)} = S_{j+1}^{(1)} A_j^{(1)}$ have norms less than $(\lambda_j^{(1)})^{1/2}$, so $\|A_j^{(2)}\|^2 \leq \lambda_j^{(1)} \leq M$.

Thus the operator $A^{(2)}$ defined on $H^{(2)}$ by the formula

$$A^{(2)} f_j^{(2)} = A_j^{(2)} f_j^{(2)} ,$$

$j = 0, 1, \ldots$ is a bounded linear operator on $H^{(2)}$.

The next step is to find out which kind of matrix the operator B has with respect to $H^{(1)}$ and $H^{(2)}$. We will also prove that $B(H^{(1)} \oplus H^{(2)}) \subseteq H^{(1)} \oplus H^{(2)}$.

8. Lemma. $(Bf_k^{(2)}, f_j^{(2)}) = 0$, for $j \neq k+1$.

Proof: Using formula (1), the normality of B, and the fact that $H^{(1)} \perp H^{(2)}$ we get

$$L = (f_j^{(2)}, B^*Bf_k^{(1)}) = (f_j^{(2)}, B^*A_k^{(1)}f_k^{(1)}) =$$

$$=(f_j^{(2)}, A_k^{(1)*}A_k^{(1)}f_k^{(1)}) + (f_j^{(2)}, S_{k+1}^{(1)}A_k^{(1)}f_k^{(1)})=0$$

for $j \neq k+1$.

On the other side we have

$$0 = L = (B^*f_j^{(2)}, B^*f_k^{(2)}) = (B^*f_j^{(2)}, A_{k-1}^{(1)*}f_k^{(1)}) +$$

$$+(f_j^{(2)}, B S_k^{(1)}f_k^{(1)}) = (f_j^{(2)}, Bf_k^{(2)})$$

Let $P^{(2)}$ be the orthogonal projection from K onto $H^{(2)}$ and let $P_j^{(2)}$ be the orthogonal projection from K onto

K onto the " coordinatwise" space $H_j^{(2)}$. Lemma 8 implies that $P_k^{(2)}Bf_j^{(2)} = 0$, for $k \neq j+1$.

Next we determine the projection of the vector $Bf_j^{(2)}$ into the subspace $H^{(1)}$ and in particular into the " coordinate" subspace $H_j^{(1)}$.

9. Lemma.

$$(Bf_j^{(2)}, f_k^{(1)}) = 0, \text{ for } j \neq k.$$

Proof: $(Bf_j^{(2)}, f_k^{(1)}) = (f_j^{(2)}, B^* f_k^{(1)}) =$

$$=(f_j^{(2)}, A_{k-1}^{(1)*} f_k^{(1)}) + (f_j^{(2)}, S_k^{(1)} f_k^{(1)}) =$$

$$= (f_j^{(2)}, f_k^{(2)}) = 0 \text{ , for } j \neq k.$$

10. Lemma.

$$P_j^{(1)} Bf_j^{(2)} = S_j^* f_j^{(2)}$$

Proof: $(P_j^{(1)} Bf_j^{(2)}, f_j^{(1)}) = (f_j^{(2)}, B^* f_j^{(1)}) =$

$$=(f_j^{(2)}, A_{j-1}^{(1)*} f_j^{(1)}) + (f_j^{(2)}, S_j^{(1)} f_j^{(1)}) = (S_j^{(1)*} f_j^{(2)}, f_j^{(1)})$$

and since $f_j^{(1)}$ is an arbitrary vector in $H_j^{(1)}$ the proof is complete.

Using definition 4 and lemma 10 we can write the decomposition of the vector $Bf_j^{(2)}$ as follows:

$$(6) \qquad Bf_j^{(2)} = S_j^{(1)*} f_j^{(2)} + A_j^{(2)} f_j^{(2)} + g_j$$

where g_j is a vector orthogonal to $H^{(1)} \oplus H^{(2)}$.

11. Lemma. In equation (6) $g_j = 0$.

<u>Proof:</u> $(Bg_j, Bf_k^{(1)}) = (Bg_j, A_k f_k^{(1)}) = (g_j, B^*(A_k^{(1)} f_k^{(1)})) = 0,$

since $A_k^{(1)} f_k^{(1)} \in H^{(1)}$, $B^*(A_k^{(1)} f_k^{(1)}) \in H^{(1)} \oplus H^{(2)}$ and

g_j is orthogonal to $H^{(1)} \oplus H^{(2)}$. Using the normality

and the fact $S_j^{(1)} f_j^{(1)} = f_j^{(2)}$ we have

$$0 = (B^* g_j, B^* f_j^{(1)}) = (g_j, B(A_j^{(1)*} f_j^{(1)} + S_j^{(1)} f_j^{(1)}) =$$

$$= (g_j, B(A_j^{(1)*} f_j^{(1)}) + (g_j, Bf_j^{(2)}) = (g_j, Bf_j^{(2)})$$

$$= (g_j, S_j^{(1)*} S_j^{(1)} f_j^{(1)}) + (g_j, A_j^{(2)} f_j^{(2)}) + \|g_j\|^2$$

Thus $\|g_j\|^2 = 0$ which implies that $g_j = 0$

Set $C_j^{(2)} = S_j^{(1)*}$ Thus $C_j^{(2)} : H_j^{(2)} \longrightarrow H_j^{(1)}$ and

$A_j^{(2)} : H_j^{(2)} \longrightarrow H_{j+1}^{(2)}$.

Now, the operator B restricted to the subspace

$H^{(1)} \oplus H^{(2)}$ has the following matrix representation

$$\left[\begin{array}{cccc|ccccc} 0 & 0 & 0 & & C_0^{(2)} & 0 & 0 & & \\ A_0^{(1)} & 0 & 0 & & 0 & C_1^{(2)} & 0 & & \\ 0 & A_1^{(1)} & 0 & & 0 & 0 & C_2^{(2)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & & & \cdot \\ \hline & & & & 0 & 0 & 0 & & \\ & & & & A_0^{(2)} & 0 & 0 & & \\ & 0 & & & 0 & A_1^{(2)} & 0 & & \\ & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & & \cdot \end{array}\right]$$

Suppose that we have already defined spaces $H^{(1)}, H^{(2)}, \ldots, H^{(n)}$, sequences of operators $A_i^{(k)} : H_i^{(k)} \longrightarrow H_{i+1}^{(k)}$, $S_i^{(k-1)} : H_i^{(k-1)} \longrightarrow H_i^{(k)}$

where $H_i^{(k)} =$ closure of $S_i^{(k-1)} H_i^{(k-1)}$ and $C_i^{(k)} = S_i^{(k-1)*}$

and moreover the operators satsfy the equation

$A_i^{(k+1)} S_i^{(k)} = S_{i+1}^{(k)} A_i^{(k)}$, and operator B restricted to the subspace $H^{(1)} \oplus H^{(2)} \oplus \cdot \cdot \oplus H^{(n)}$ has matrix of the following form

$$\left[\begin{array}{cccccc} A^{(1)} & G^{(2)} & 0 & & & \\ 0 & A^{(2)} & G^{(3)} & & & \\ & 0 & A^{(3)} & \cdot & \cdot & \\ & & & \cdot & \cdot & G^{(n)} \\ & & & & \cdot & A^{(n)} \end{array}\right]$$

$$A^{(n)} = \begin{bmatrix} 0 & 0 & 0 & & \\ A_0^{(n)} & 0 & 0 & & \\ 0 & A_1^{(n)} & 0 & \cdot & \cdot \\ & & & & \cdot \\ & & & & \cdot \end{bmatrix}, \quad G^{(n)} = \begin{bmatrix} C_0^{(n)} & 0 & 0 & & \\ 0 & C_1^{(n)} & 0 & & \\ 0 & 0 & C_2^{(n)} & \cdot & \cdot \\ & & & & \cdot \end{bmatrix}$$

The above can be considered to be the induction hypothesis.

We can set

$$(7) \quad B^{*}f_j^{(n)} = A_{j-1}^{(n)*} f_j^{(n)} + f_j^{(n+1)}$$

where $f_j^{(n+1)}$ is a vector orthogonal to $H^{(1)} \oplus \cdot \cdot \oplus H^{(n)}$.

Here we use notations $|A_j^{(k)}|^2 = A_j^{(k)*} A_j^{(k)}$ and $C_j^{(k)} = S_j^{(k-1)*}$ for $j = 0, 1, \ldots$ and $k = 1, 2, \ldots, n$.

The normality of B implies that $\|Bf_j^{(n)}\|^2 = \|B^{*}f_j^{(n)}\|^2$ or

$$(A_j^{(n)} f_j^{(n)} + C_j^{(n)} f_j^{(n)}, A_j^{(n)} f_j^{(n)} + C_j^{(n)} f_j^{(n)}) =$$

$$= (A_{j-1}^{(n)*} f_j^{(n)} + f_j^{(n+1)}, A_{j-1}^{(n)*} f_j^{(n)} + f_j^{(n+1)})$$

A computation shows that

$$\|A_j^{(n)} f_j^{(n)}\|^2 + \|C_j^{(n)} f_j^{(n)}\|^2 = \|A_{j-1}^{(n)*} f_j^{(n)}\|^2 + \|f_j^{(n+1)}\|^2$$

or equivalently

$$\left(\left(|A_j^{(n)}|^2 + |C_j^{(n)}|^2 - |A_{j-1}^{(n)*}|^2 \right) f_j^{(n)},\ f_j^{(n)} \right) = \|f_j^{(n+1)}\|^2$$

If we set $S_j^{(n)} f_j^{(n)} = f_j^{(n+1)}$ we get

$$|S_j^{(n)}|^2 = |A_j^{(n)}|^2 + |C_j^{(n)}|^2 - |A_{j-1}^{(n)*}|^2$$

So of necessity we have condition

$$(8) \quad |A_j^{(n)}|^2 + |C_j^{(n)}|^2 - |A_{j-1}^{(n)*}|^2 \geq 0$$

From (7) we se that $B f_j^{(n)} = A_{j-1}^{(n)*} f_j^{(n)} + S_j^{(n)} f_j^{(n)}$ where $S_j^{(n)} f_j^{(n)} \perp H^{(1)} \oplus H^{(2)} \oplus \cdots \oplus H^{(n)}$.

Let $H_j^{(n+1)}$ be the closure of $S_j^{(n)} H_j^{(n)}$, and set

$$H^{(n+1)} = \bigoplus_{j=0}^{\infty} H_j^{(m+1)}$$

Now we have the same lemmas which had been proved for n=1.

<u>12. Lemma.</u> $(f_i^{(n+1)}, f_j^{(n+1)}) = 0$ for $i \neq j$.

<u>Proof:</u> $(f_i^{(n+1)}, f_j^{(n+1)}) = (S_i^{(n)} f_i^{(n)}, S_j^{(n)} f_j^{(n)}) =$

$$= (B^* f_i^{(n)} - A_{i-1}^{(n)*} f_i^{(n)},\ S_j^{(n)} f_j^{(n)}) =$$

$$= (B^* f_i^{(n)}, S_j^{(n)} f_j^{(n)}) = (B^* f_i^{(n)}, B^* f_j^{(n)}) - (B^* f_i^{(n)}, A_{i-1}^{(n)*} f_j^{(n)})$$

$$= (B f_i^{(n)}, Bf_j^{(n)}) - (f_i^{(n)}, A_{j-1}^{(n)} A_{j-1}^{(n)*} f_j^{(n)} + S_{j-1}^{(n)*} A_{j-1}^{(n)*} f_j^{(n)})$$

=0, by very definition of all the subspaces.

Now we define an operator $A_j^{(n+1)}: H_j^{(n+1)} \longrightarrow H_{j+1}^{(n+1)}$ by the formula $A_j^{(n+1)} f_j^{(n+1)} = P_{j+1}^{(n+1)} Bf_j^{(n+1)}$, where $P_{j+1}^{(n+1)}$ is the orthogonal projection of K onto $H_{j+1}^{(n+1)}$.

The operators $S_{j+1}^{(n)} A_j^{(n)}$ and $A_j^{(n+1)} S_j^{(n)}$ both map the space $H_j^{(n)}$ into $H_{j+1}^{(n+1)}$, and moreover the folowing lemma is true

<u>13. Lemma.</u> $A_j^{(n+1)} S_j^{(n)} = S_{j+1}^{(n)} A_j^{(n)}$, for $j = 0,1,\ldots$ or equivalently

$$(f_{j+1}^{(n+1)}, Bf_j^{(n+1)}) = (f_j^{(n+1)}, S_{j+1}^{(n)} A_j^{(n)} f_j^{(n)})$$

<u>Proof:</u> Set $L = (Bf_{j+1}^{(n+1)}, Bf_j^{(n)}) = (Bf_{j+1}^{(n+1)}, S_{j-1}^{(n)*} f_j^{(n)} + A_j^{(n)} f_j^{(n)})$

$$= (f_{j+1}^{(n+1)}, B^*(S_j^{(n-1)} f_j^{(n)})) + (f_j^{(n+1)}, B^*(A_j^{(n)} f_j^{(n)}))$$

Since $S_j^{(n-1)*} f_j^{(n)} \in H_j^{(n-1)}$ and $B^* H^{(n-1)} \subseteq H^{(n-1)} \oplus H^{(n)}$, the first number is zero, so we have

$$L = (f_j^{(n+1)}, A_j^{(n)*} A_j^{(n)} f_j^{(n)} + S_{j+1}^{(n)} A_j^{(n)} f_j^{(n)})$$

and thus

$$L = (f_{j+1}^{(n+1)}, S_{j+1}^{(n)} A_j^{(n)} f_j^{(n)})$$

Using the normality of B we have

$$L = (B^* f_j^{(n+1)}, B^* f_j^{(n)}) = (B^* f_{j+1}^{(n)}, (A_{j-1}^{(n)*} + S_j^{(n)}) f_j^{(n)})$$

$$= (f_{j+1}^{(n+1)}, B(A_{j-1}^{(n)*} f_j^{(n)})) + (f_{j+1}^{(n+1)}, BS_j^{(n)} f_j^{(n)}) =$$

$$= (f_{j+1}^{(n+1)}, P_{j+1}^{(n+1)} BS_j^{(n)} f_j^{(n)}) = (f_{j+1}^{(n+1)}, A_j^{(n+1)} S_j^{(n)})$$

The sequence of operators $A_j^{(n+1)}$ is uniformly bounded in norm (namely by $\|B\|$) and the operator $A^{(n+1)} : H^{(n+1)} \longrightarrow H^{(n+1)}$ defined coordinatewise by $A^{(n+1)} f_j^{(n+1)} = A_j^{(n+1)} f_j^{(n+1)}$ is bounded. Now, we againare going to give an equivalent of lemma 13 which we will need later in proving the main theorem.

<u>14. Lemma.</u> The equation

$$(9) \qquad A_j^{(n+1)} S_j^{(n)} = S_{j+1}^{(n)} A_j^{(n)}$$

together with boundness of $A^{(n+1)}$ implies the existence of a bounded sequence of positive numbers $\{\lambda_j^{(n)}\}$ such that the following inequality is true

$$(10)\quad A_j^{(n)*}\left(\left|A_{j+1}^{(n)}\right|^2+\left|C_{j+1}^{(n)}\right|^2-\left|A_j^{(n)*}\right|^2\right)A_j^{(n)} \leq \lambda_j^{(n)}\left(\left|A_j^{(n)}\right|^2+\left|C_j^{(n)}\right|^2-\left|A_{j-1}^{(n)*}\right|^2\right)$$

for $j=0,1,\ldots$.

__Proof:__ The proof of this lemma is similar to the proof of lemma 7. By the same arguments as in lemma 7 we have a bounded e sequence of positive real numbers such that

$$(A_j^{(n)*}S_{j+1}^{(n)*}S_{j+1}^{(n)}A_j^{(n)}f_j^{(n)},\ f_j^{(n)}) \leq \lambda_j^{(n)}(S_j^{(n)*}S_j^{(n)}f_j^{(n)},f_j^{(n)})$$

in order to compute the left side L of the above inequality we first use definition (7), and obtain

$$\|S_{j+1}^{(n)}A_j^{(n)}f_j^{(n)}\|^2=(S_{j+1}^{(n)}A_j^{(n)}f_j^{(n)},B^*A_j^{(n)}f_j^{(n)})-$$

$$-(S_{j+1}^{(n)}A_j^{(n)}f_j^{(n)},\ A_j^{(n)*}A_j^{(n)}f_j^{(n)})=(B^*A_j^{(n)}f_j^{(n)}-A_j^{(n)*}A_j^{(n)}f_j\ ,B^*A_j^{(n)}f_j^{(n)})=$$

$$=(A_{j+1}^{(n+1)}A_j^{(n)}f_j^{(n)},\ A_{j+1}^{(n+1)}A_j^{(n)}f_j^{(n)})+(C_{j+1}^{(n)}A_j^{(n)}f_j^{(n)},C_j^{(n)}A_{j+1}^{(n)}f_j^{(n)})$$

$$-(A_j^{(n)}A_j^{(n)*}A_j^{(n)}f_j^{(n)},A_j^{(n)}f_j^{(n)})+(C_j^{(n)}A_j^{(n)*}A_j^{(n)}f_j^{(n)},A_j^{(n)}f_j^{(n)})$$

The last member of the prevous relation is equal to 0.

So we have

$$L=(A_j^{(n)*}(|A_{j+1}^{(n)}|^2+|C_{j+1}^{(n)}|^2-|A_j^{(n)*}|^2)A_j^{(n)}f_j^{(n)},\ f_j^{(n)})$$

On the other side we have

$$\left\|S_j^{(n)}f_j^{(n)}\right\|^2 = (S_j^{(n)}f_j^{(n)}, S_j^{(n)}f_j^{(n)}) =$$

$$= ((\left|A_j^{(n)}\right|^2 + \left|C_j^{(n)}\right|^2 - \left|A_{j-1}^{(n)*}\right|^2)f_j^{(n)}, f_j^{(n)})$$

thus inequality (7) is proved.

__15. Lemma.__ (1) If $A_j^{(n)}H_j^{(n)}$ is dense in $H_{j+1}^{(n)}$ then $A_j^{(n+1)}H_j^{(n+1)}$ is dense in $H_{j+1}^{(n+1)}$

(2) If $S_{j_o}^{(n)}=0$, then $S_p^{(n)}=0$, for $p\geq j_o$.

__Proof:__ (1) If $n=1$, this lemma coincides with lemma 6 and the proof is exactly the same as that of lemma 6.

(2) If $S_{j_o}^{(n)} = 0$, then $A_{j_o}^{(n+1)}S_{j_o}^{(n)} = S_{j_o}^{(n)}A_{j_o}^{(n)}$ hence $S_{j_o}^{(n)}A_{j_o}^{(n)} = 0$, and since $A_{j_o}^{(n)}H_{j_o}^{(n)}$ is dense in $H_{j_o+1}^{(n+1)}$ we get $\bar{S}_{j_o+1}^{(n)} = 0$.

The next step is to find the matrix of operator B with respect to $H^{(1)} \oplus H^{(2)} \oplus \dots \oplus H^{(n+1)}$. We have the following lemmas.

__16. Lemma.__

$$(B f_k^{(n+1)}, f_j^{(n+1)}) = 0 \text{ , for } j \neq k+1.$$

Proof: $L = (f_j^{(n+1)}, B^*B f_k^{(n)}) = (f_j^{(n+1)}, B^*(S_k^{(n-1)*} f_k^{(n)} + A_k^{(n)} f_k^{(n)}))$

$$= (f_j^{(n+1)}, A_k^{(n)*} A_k^{(n)} f_k^{(n)}) + (f_j^{(n+1)}, S_{k+1}^{(n)} A_k^{(n)} f_k^{(n)}) = 0$$

Lemma 13 had been used here.

On the other hand,

$$0 = L = (B^* f_j^{(n+1)}, B^* f_k^{(n)}) = (B^* f_j^{(n+1)}, A_{k-1}^{(n)*} f_k^{(n)}) +$$

$$+ (f_j^{(n+1)}, BS_k^{(n)} f_k^{(n)}) = (f_j^{(n+1)}, Bf_k^{(n+1)})$$

17. Lemma. $(Bf_j^{(n+1)}, f_k^{(n)}) = 0$, for $j \neq k$.

Proof: $(Bf_j^{(n+1)}, f_k^{(n)}) = (f_j^{(n+1)}, B^* f_k^{(n)}) =$

$$= (f_j^{(n+1)}, A_{k-1}^{(n)*} f_k^{(n)} + S_k^{(n)} f_k^{(n)}) = (f_j^{(n+1)}, f_k^{(n+1)})$$

and the last number is zero, by lemma 12 if $j \neq k$

18. Lemma. $P_j^{(n)} Bf_j^{(n+1)} = S_j^{(n)*} f_j^{(n+1)}$

Proof: $(P_j^{(n)} Bf_j^{(n+1)}, f_j^{(n)}) = (f_j^{(n+1)}, B^* f_j^{(n)}) =$

$$= (f_j^{(n+1)}, A_{j-1}^{(n)*} f_j^{(n)}) + (f_j^{(n+1)}, S_j^{(n)} f_j^{(n)}) =$$

$$= (S_j^{(n)*} f_j^{(n+1)}, f_j^{(n)})$$

From lemmas 16-18 and the definition of $A_j^{(n+1)}$ we can write a decomposition of the vecctor $Bf_j^{(n+1)}$ as follows:

$$Bf_j^{(n+1)} = A_j^{(n+1)} f_j^{(n+1)} + S_j^{(n)*} f_j^{(n+1)} + g_j$$

where g_j is a vector orthogonal to $H^{(1)} \oplus \ldots \oplus H^{(n+1)}$

<u>19. Lemma.</u> $g_j = 0$, in the above decomposition.

<u>Proof:</u> $(Bg_j, Bf_j^{(n)}) = (Bg_j, A_j^{(n)} f_j^{(n)} + S_j^{(n-1)*} f_j^{(n)})$

$$= (g_j, B^*(A_j^{(n)} f_j^{(n)}) = 0$$

The last follows from the facts $A_j^{(n)} f_j^{(n)} \in H^{(n)}$, $B(A_j^{(n)} f_j^{(n)}) \in H^{(n)} \oplus H^{(n+1)}$ and $g_j \perp H^{(1)} \oplus \ldots \oplus H^{(n+1)}$.

Using the normality of B we have

$$0 = (B^* g_j, B^* f_j^{(n)}) = (g_j, B(A_{j-1}^{(n)*} f_j^{(n)} + S_j^{(n)} f_j^{(n)})) =$$

$$= (g_j, B(A_{j-1}^{(n)*} f_j^{(n)} + A_j^{(n+1)} f_j^{(n+1)}) = (g_j, Bf_j^{(n+1)})$$

$$= (g_j, S_j^{(n)*} f_j^{(n+1)} + A_j^{(n+1)} f_j^{(n+1)} + g_j) = \|g_j\|^2$$

which implies that $g_j = 0$.

From the above results we can write the matrix of B with respect to $H^{(1)} \oplus H^{(2)} \oplus \ldots \oplus H^{(n+1)}$ as follows:

$$\begin{bmatrix} A^{(1)} & G^{(2)} & 0 & & & \\ 0 & A^{(2)} & G^{(3)} & & & \\ 0 & 0 & A^{(3)} & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & G^{(n+1)} \\ & & & & & A^{(n+1)} \end{bmatrix}$$

where

$$A^{(n)} = \begin{bmatrix} 0 & 0 & & \\ A_0^{(n)} & 0 & & \\ 0 & A_1^{(n)} & \cdot & \\ & & \cdot & \cdot \\ & & & \cdot \end{bmatrix}$$

and

$$G^{(n)} = \begin{bmatrix} C_0^{(n)} & 0 & & & \\ 0 & C_1^{(n)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix}$$

Note that $A^{(n)} : H^{(n)} \longrightarrow H^{(n)}$ and $G^{(n)} : H^{(n)} \longrightarrow H^{(n-1)}$ and in particular $A^{(1)} = T$.

20. Theorem. Let T be an operator valued weighted shift with weights $\left\{A_i^{(1)}\right\}_{i=0}^{\infty}$, where $\|A_i\| \le M$. Then the operator T is subnormal if and only if

(I) $$\left|A_j^{(n)}\right|^2 + \left(C_j^{(n-1)}\right)^2 - \left|A_{j-1}^{(n)*}\right|^2 \ge 0$$

(II) there exists assequence of positive real numbers $\{\lambda_j^{(n)}\}$ such that the following inequality holds

$$A_j^{(n)*}\left(\left|A_{j+1}^{(n)}\right|^2 + \left(C_{j+1}^{(n)}\right)^2 - \left|A_j^{(n)*}\right|^2\right)A_j^{(n)}$$

$$\le \lambda_j^{(n)}\left(\left|A_j^{(n)}\right|^2 + \left(C_j^{(n)}\right)^2 - \left|A_{j-1}^{(n)*}\right|^2\right)$$

(III) there exists a constant M such that $\left\|A_j^{(n)}\right\| \le M$

where $C_j^{(n)} = \left(\left|A_j^{(n)}\right|^2 + \left(C_j^{(n-1)}\right)^2 - \left|A_{j-1}^{(n)*}\right|^2\right)^{1/2}$ and $A_j^{(n+1)}$ is a solution of the equation

(11) $$XC_j^{(n)} = C_j^{(n)} A_j^{(n)} .$$

Proof: The necessity of condition (I) had been exhibited in inequality (8) and condition (II) is lemma 14.

Condition (III) follows from the boundness of the operator B which is a normal extension of T. Lemma 13 shows that $A_j^{(n+1)}$ satisfies equation (11).

Now we will prove that conditions (I),(II), and (III) are also sufficient for subnormality of the operat or T.

Since (I) holds, the operator $|A_j^{(n)}|^2 + (C_j^{(n-1)})^2 - |A_{j-1}^{(n)*}|^2$ is positive, therefore has a positive square root, and we denote that root by $C_j^{(n)}$.

Suppose we have operators $A_j^{(k)}$, $k = 1,2,\ldots n$, satisfying (11), and denote by $A_j^{(n+1)}$ the solution of the equation $XC_j^{(n)} = C_{j+1}^{(n)} A_j^{(n)}$, which exists by condition (11) and Douglas' theorem, whose application was shown in lemma 14.

Set $H^{(1)} = \bigoplus_{j=0}^{\infty} H_j^{(1)}$, $H_j^{(1)} = H$; $H^{(n)} = \bigoplus_{j=0}^{\infty} H_j^{(n)}$ and

$$\hat{H} = \bigoplus_{n=1}^{\infty} H^{(n)}$$

Now we define an operator B on H, such that B will be normal and $B|H^{(1)} = T$. We will explicitely writ dowm the matrix form of on H as follows:

$$\begin{bmatrix} A^{(1)} & G^{(2)} & 0 & & \\ 0 & A^{(2)} & G^{(3)} & & \\ & & \cdot & \cdot & \\ & & & \cdot & \\ & & & & \end{bmatrix}$$

where

$$A^{(n)} = \begin{bmatrix} 0 & 0 & 0 & & \\ A_0^{(n)} & 0 & 0 & & \\ 0 & A_1^{(n)} & 0 & & \\ & & \cdot & \cdot & \cdot \\ & & & & \cdot \end{bmatrix}$$

and

$$G^{(n)} = \begin{bmatrix} C_0^{(n-1)} & 0 & 0 & & & \\ 0 & C_1^{(n-1)} & 0 & & & \\ & 0 & C_2^{(n-1)} & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & & \cdot \end{bmatrix}$$

Thus each $G^{(n)}$ is selfadjoint.

Note that $A^{(n)} : H^{(n)} \longrightarrow H^{(n)}$ and $G^{(n)} : H^{(n)} \longrightarrow H^{(n-1)}$

Taking adjoint of the matrix B we have

$$\begin{bmatrix} A^{(1)} & 0 & & \\ G^{(2)} & A^{(2)} & 0 & \\ 0 & G^{(3)} & A^{(3)} & \\ & & \cdot & \cdot \\ & & & \cdot \end{bmatrix}$$

Direct computations show

$$B\ B = \begin{bmatrix} A^{(1)*}A^{(1)} & A^{(1)*}G^{(2)} & 0 & \\ G^{(2)}A^{(1)} & (G^{(2)})^2 + A^{(2)*}A^{(2)} & A^{(3)*}G^{(3)} & \\ 0 & G^{(3)}A^{(2)} & (G^{(3)})^2 + A^{(3)*}A^{(3)} & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \end{bmatrix}$$

$$BB = \begin{bmatrix} A^{(1)}A^{(1)*} + (G^{(2)})^2 & G^{(2)}A^{(2)*} & 0 & \\ A^{(2)}G^{(2)} & A^{(2)}A^{(2)*} + (G^{(3)})^2 & G^{(3)}A^{(3)*} & \\ 0 & A^{(3)}G^{(3)} & \cdot & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \end{bmatrix}$$

We will show that $B^*B = BB^*$ by showing that all corresponding entries are equal. First, on the lower diagonal we will show that $A^{(n)}G^{(n)} = G^{(n)}A^{(n-1)}$ for $n = 1,2,$. Multiplying the terms of this equation we see that it is equivalent to the equations $A_j^{(n)}C_j^{(n-1)} = C_{j+1}^{(n-1)}A_j^{(n-1)}$, for $j= 0,1, \ldots$, but the last equality is true since $A_j^{(n)}$ was a solution of equation (11) by definition.

Now, for the diagonal entries we have to show

(12) $A^{(n)}A^{(n)*} + (G^{(n+1)})^2 = A^{(n)}A^{(n)*} + (G^{(n)})^2$, for $n=1,2,\ldots$

The above equations are equivalent to the equations

$$A_{j-1}^{(n)}A_{j-1}^{(n)*} + (C_j^{(n)})^2 = A_j^{(n)*}A_j^{(n)} + (C_j^{(n-1)})^2 \text{ for } j=0,1,\ldots$$

and the latter is exactly the definition of $C_j^{(n)}$.

Finally, we show that B is bounded. Let $\hat{f} \in \hat{H}$, that is $\hat{f} = (f^{(1)}, f^{(2)}, \ldots)$. Then we have

$$B\hat{f} = (A^{(1)}f^{(1)} + G^{(2)}f^{(2)}, A^{(2)}f^{(2)} + G^{(3)}f^{(3)}, \ldots)$$

Using (III) we get

$$\|Bf\|^2 = \sum_{k=1}^{\infty} \|A^{(k)}f^{(k)} + G^{(k+1)}f^{(k+1)}\|^2$$

$$= \sum_{k=1}^{\infty} (\|A^{(k)}\| \|f^{(k)}\| + \|G^{(k+1)}\| \|f^{(k+1)}\|)^2$$

$$= M^2 \sum_{k=1}^{\infty} (\|f^{(k)}\| + \|f^{(k+1)}\|)^2$$

$$\leq 4M^2 \sum_{k=1}^{\infty} \|f^{(k)}\|^2 = 4M^2 \|f\|^2 .$$

thus $\|B\| \leq 2M$ which implies the boundness of the operator B.

<u>21. Corollary.</u> Let T be a subnormal operator valued weighted shift with positive invertible weights $\{A_i\}_{i=0}^{\infty}$.

If $A_{j_o} = A_{j_o+1}$ then $A_k = A_{j_o}$, for all $k \geq j_o$.

<u>Proof</u>: Write $H^{(1)} = \bigoplus_{j=0} H_j^{(1)}$, where $H_j^{(1)} = H$ and $A_j^{(1)} = A_j$. Since the $A_j^{(1)}$ are invertible we see that the spaces $A_j^{(1)}H_j^{(1)}$ are dense in(in fact equal to) the spaces $H_{j+1}^{(1)}$; and applying lemma 6 we get $A_j^{(2)}H_j^{(2)}$ is dense in $H_{j+1}^{(2)}$ for each j.

Since the $A_j^{(1)}$ are selfadjoint ; $A_{j_o}^{(1)} = A_{j_o+1}^{(1)}$ implies that $S_{j_o+1} = ((A_{j_o+1})^2 - (A_{j_o})^2)^{1/2} = 0$.

By lemma 5 we have

$$S_{j_o+2}^{(1)} A_{j_o+1}^{(1)} = A_{j_o+1}^{(2)} S_{j_o}^{(1)} = 0$$

Using the invertibility of $A_{j_o+1}^{(1)}$ we conclude that $S_{j_o+2} = 0$; which implies $(A_{j_o+2})^2 = (A_{j_o+1})^2 = (A_{j_o})^2$ and a square root we have $A_{j_o+2} = A_{j_o}$. This proof could be continued. Therefore $A_k = A_{j_o}$; for all $k \geq j_o$.

<u>22. Example</u>. Using corollary 21 we will give an example of hyponormal operator T, which i not subnormal and moreover all powers T^n, for $n \geq 1$ are hyponormal (see Halmos[5] problem 160 and Stampfli [9].

Let H be two dimensional Hilbert space and let T be an operator valued weighted shift with weights

$$A_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A_2, \quad A_n = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ for } n \geq 3.$$

Then T is hyponormal by lemma 2 but it is not subnormal by corollary 21. An easy computation show that

$$T^{*2}T^2 = \begin{bmatrix} (1/4)I & 0 & 0 & 0 & & \\ 0 & I & 0 & 0 & & \\ & 0 & 4I & 0 & & \\ & & 0 & 4I & \cdot & \\ & & & & \cdot & \cdot \\ & & & & & \cdot \end{bmatrix}$$

and

$$T^2T^{*2} = \begin{bmatrix} 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ 0 & 0 & (1/4)I & & & & \\ & & & I & & & \\ & & & & 4I & \cdot & \\ & & & & & \cdot & \cdot \\ & & & & & & \cdot \end{bmatrix}$$

Therefore T^2 is hyponormal. In the same manner it can be shown that T^n is hyponormal operator for every natural n .

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СУБНОРМАЛНОСТ НА ОПЕРАТОРСКО ТЕЖИНСКИТЕ ШИФТОВИ

Новак Ивановски

Во оваа работа се најдени потребните и доволните услови за да еден операторско тежински шифт е субнормален. Овој труд претставува генерализација на резултатот од Stampfli каде што е разгледуван шифт со тежини-скалари.