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QUAZIPERIODICITY OF THE SOLUTIONS TO LINEAR NONHOMOGENEOUS DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. In this paper we give some conditions for existence of quasiperiodic solutions to the linear nonhomogenious ordinary differential equation of second order (2).The main results refers to quasiperiodic solutions with constant quasiperiod. We note that the considered problem in this paper is examined with a method different than the methods in [1],[3].

1. INTRODUCTION

Definition 1.1. *We say that $y = \varphi(x)$, $x \in I = (a, b) \subseteq D_\varphi$ is a quasi-periodic function (QPF) if there are: a function $\omega(x)$ and a coefficient $\lambda = \lambda(\omega(x))$ such that the relation*

$$\varphi(x + \omega(x)) = \lambda\varphi(x), \quad x, x + \omega(x) \in I \tag{1}$$

is satisfied. The function $\omega(x)$ is called a quasi-period (QP) and λ is said to be a quasi-periodic coefficient (QPC) of the function $\varphi(x)$.

Example 1.1. The function $\varphi(x) = e^{2x}\cos x$ is a quasi-periodic, with QP $\omega = 2\pi$ and QPC $\lambda = e^{4\pi}$, since:

$$\forall x \in \mathbb{R}, \ \ \varphi(x + 2\pi) = e^{2(x+2\pi)}\cos(x + 2\pi) = e^{4\pi}e^{2x}\cos x = e^{4\pi}\varphi(x).$$

Example 1.2. The function $\psi(x) = e^{x^2}\cos x^2$ is a quasi-periodic, with QP $\omega = -x + \sqrt{x^2 + 2\pi}$ and QPC $\lambda = e^{2\pi}$, since:

$$\forall x \in \mathbb{R}, \ \ \psi(x + \omega) = e^{(x+\omega)^2}\cos(x + \omega)^2 = e^{2\pi}e^{x^2}\cos x^2 = e^{2\pi}\psi(x).$$

Remarks

1. In the general case, when $\lambda = \lambda(x, \omega(x))$, the existence of the relation (1) is very complicated problem.

2. If $\omega(x) = \omega^* = konst.$ and $\lambda = 1$ for $x \in I$, then (1) is a definition for a periodic function in a classical sense.

3. If $\omega = \omega(x) \neq konst$ and $\lambda = 1$ for $x \in I$, then (1) is a generalization of the definition for a periodic function and in this case $\omega = \omega(x)$ is a function of "repeating values" of $y = \varphi(x)$.

Key words and phrases. differential equation, quasiperiod, quasiperiodic solution.

2. PROBLEM FORMULATION

In the paper [2], some conditions for existence of quasiperiodic solutions (QPS) to the linear nonhomogenous ordinary differential equation of the first order, are given.

Here in a similar way we find some conditions for existence of QPS to the linear nonhomogenious ordinary differential equation of the second order

$$y'' + f(x)y' + g(x)y = h(x), \tag{2}$$

where $f(x), g(x), h(x)$ are continuous and two times differentiable functions on $I \subseteq D_f \cap D_g \cap D_h \cap D_y$. We try to find a solution of (2) which satisfies the relation

$$y(x + \omega(x)) = \lambda y(x) \tag{3}$$

where $\omega = \omega(x) \in C_I^2$, $\lambda > 0$, $\lambda \neq 1$, x, $x + \omega(x) \in I$.

We describe the problem by the system

$$\begin{cases} y''(x) + f(x)y'(x) + g(x)y(x) = h(x) \\ y''(x+\omega) + f(x+\omega)y'(x+\omega) + g(x+\omega)y(x+\omega) = h(x+\omega) \\ y(x+\omega) = \lambda y(x) \\ y'(x+\omega)(1+\omega') = \lambda y'(x) \\ y''(x+\omega)(1+\omega')^2 + y'(x+\omega)\omega'' = \lambda y''(x) \end{cases} \tag{4}$$

If $1 + \omega' \neq 0$ we can eliminate $y(x+\omega)$ and its derivatives $y'(x+\omega)$, $y''(x+\omega)$ and $y''(x)$ and so we reduce the above system to the equation

$$\frac{\lambda(h(x) - f(x)y' - g(x)y) - \dfrac{\lambda y'}{1+\omega'} \cdot \omega''}{(1+\omega')^2} = -f(x+\omega) \cdot \frac{\lambda y'}{1+\omega'} - \lambda g(x+\omega) \cdot y + h(x+\omega)$$

If we rearrange the last equation to y and y' it will take the form

$$\begin{aligned} &\lambda y' \Big[f(x+\omega)(1+\omega')^2 - f(x)(1+\omega') - \omega'' \Big] + \\ &+ \lambda y \Big[g(x+\omega)(1+\omega')^3 - g(x)(1+\omega') \Big] - \\ &- \Big[h(x+\omega)(1+\omega')^3 - \lambda h(x)(1+\omega') \Big] = 0 \end{aligned} \tag{5}$$

The above argument is a proof of the following theorem.

Theorem 2.1. *Let the nonhomogeneous linear differential equation of second order (2) has QPS defined by the relation (3). Using the system (4), the equation (2) is reduced to a nonhomogeneous linear differential equation of first order given by (5).*

Example 2.1. If the equation

$$y'' - 2y' + e^x y = e^{-x}$$

has QPS with QP ω then, according to the Theorem 2.1., it is reduced to the equation

$$\left[\omega''+2\omega'^2+2\omega'\right]y'-\left[e^{\omega}(1+\omega')^2-1\right](1+\omega')e^{x}y+\left[\frac{1}{\lambda}e^{-\omega}(1+\omega')^2-1\right](1+\omega')e^{-x}=0.$$

In this paper we give some existence conditions for QPS with a given constant period and at the same time we find the form of the solution.

3. SOME EXISTENCE CONDITIONS FOR QPS WITH A CONSTANT QP

Lemma 3.1. *Let $y(x)$ be QPS to (2) with QP $\omega(x) = konst. = \varpi$ and QPC λ $(\lambda > 0, \lambda \neq 1)$. Then:*

$$\lambda y'[f(x+\varpi)-f(x)]+\lambda y[g(x+\varpi)-g(x)]-[h(x+\varpi)-\lambda h(x)]=0. \tag{6}$$

Proof. If $\omega(x) = konst = \varpi$ then $\varpi' = \varpi'' = 0$. Substituting $\varpi, \varpi', \varpi''$ in (5) we obtain (6). □

Using Lemma 3.1 we obtain the following theorem.

Theorem 3.1. *Let the coefficients $f(x), g(x), h(x)$ in (2) be QPF with a constant QP ϖ and QPC μ, ν, η respectively, such that $\mu \neq \nu$, $\nu \neq \eta$, $\mu \neq \eta$, and $\mu, \nu, \eta \neq \lambda$, $\nu \neq 1$. Then the equation (2) has QPS $y(x)$ with the same QP ϖ and QPC λ $(\lambda > 0, \lambda \neq 1)$, if the relation*

$$\left(\frac{h}{g}\right)''+\left(\frac{h}{g}\right)'f=0, \quad (f=f(x),\ g=g(x),\ h=h(x)) \tag{7}$$

is satisfied. Then the QPS is given in the form $y = \dfrac{h}{g}$.

Proof. Using the conditions

$$\begin{aligned} &f(x=\varpi)=\mu f(x),\ g(x+\varpi)=vg(x),\ h(x+\varpi)=\eta h(x), \\ &\mu\neq\nu,\quad \nu\neq\eta,\quad \mu\neq\eta,\quad \mu,\nu,\eta\neq\lambda,\quad \nu\neq 1. \end{aligned} \tag{8}$$

and Lemma 3.1. we reduce the equation (2) to the equation

$$\lambda(\mu-1)f(x)y'+\lambda(\nu-1)g(x)y-(\eta-\lambda)h(x)=0. \tag{9}$$

Depending on μ the following two cases are possible:

a) If $\mu = 1$, i.e. $f(x)$ is a periodic function, then (9) is equivalent to the equation

$$\lambda(\nu-1)g(x)y-(\eta-\lambda)h(x)=0 \tag{10}$$

whose solution is

$$y=\frac{(\eta-\lambda)}{\lambda(\nu-1)}\cdot\frac{h(x)}{g(x)},\quad \nu\neq 1,\ g(x)\neq 0. \tag{11}$$

The solution (11) is a QPF with QP ϖ and QPC $\lambda = \dfrac{\eta}{\nu}$, and:

$$\frac{\eta-\lambda}{\lambda(\nu-1)}=\frac{\eta-\dfrac{\eta}{\nu}}{\dfrac{\eta}{\nu}(\nu-1)}=1.$$

b) If $\mu \neq 1$, then we can write the equation (9) in the form

$$y' + G(x)y + H(x) = 0 \tag{12}$$

where $G(x) = \dfrac{\nu - 1}{\mu - 1} \cdot \dfrac{g(x)}{f(x)}$, $H(x) = \dfrac{\lambda - \eta}{\lambda(\mu - 1)} \cdot \dfrac{h(x)}{f(x)}$.

The coefficients $G(x)$ and $H(x)$ in (12) are QPF with QP ϖ and QPC $\dfrac{\nu}{\mu}$, $\dfrac{\eta}{\mu}$, respectively. Thus, (12) is reduced to the equation ([2]):

$$y[G(x+\varpi) - G(x)] - \Big[\frac{1}{\lambda}H(x+\varpi) - H(x)\Big] = 0$$

which has a solution

$$y = -\frac{(\lambda - \eta)(\eta - \lambda\mu)}{\lambda^2(\nu - 1)(\nu - \mu)} \cdot \frac{h}{g}, \quad \nu \neq 1, \; g(x) \neq 0 \tag{13}$$

that is QPF with QP ϖ and QPC $\lambda = \dfrac{\eta}{\nu}$ and:

$$\frac{(\lambda - \eta)(\eta - \lambda\mu)}{\lambda^2(\nu - 1)(\nu - \mu)} = -1.$$

Using the fact that the solution y determined by (11) or (13) is also a solution of (2), we obtain the relation (7) . □

Remark. It should be noteed that under the conditions of the theorem, the nature of the solution (11) or (13) of (2) does not depend on the coefficient $f(x)$ and it is QPF in the both cases, if $f(x)$ is a periodic or quasiperiodic function.

Example 3.1. The coefficients $f(x) = -2$, $g(x) = e^{-x}\sin x$, $h(x) = e^{x}\sin x$ in the equation

$$y'' - 2y' + e^{-x}\sin x \cdot y = e^{x}\sin x$$

are all QPF with the same QP $\varpi = 2\pi$ and they satisfy the relation (7). So the equation has a solution

$$y = \frac{h(x)}{g(x)} = e^{2x}$$

that is QPF with QP $\varpi = 2\pi$ and QPC $\lambda = e^{4\pi}$.

Theorem 3.2. *Let the coefficients $f = f(x)$, $g = g(x)$, $h = h(x)$ in (2) be QPF with constant QP ϖ and QPC μ, ν, η respectively, such that $\mu = \nu = \eta = \lambda$, $\lambda \neq 1$. The equation (2) has QPS with QP ϖ and QPC λ if:*

3.2.1. $\Phi(x+\varpi) - \Phi(x) = \ln\dfrac{1}{\lambda}$, *where* $\Phi(x) = \displaystyle\int \frac{g(x)}{f(x)}dx$; *and*

3.2.2. f, g, h *satisfy the relation*

$$\left(\frac{hf^2}{g^2 - g'f + gf'}\right)'' = h. \tag{14}$$

Then the QPS of (2) has the form

$$y = e^{-\int \frac{g(x)}{f(x)}dx} \tag{15}$$

Proof. Using the conditions

$$f(x+\varpi)=\lambda f(x),\quad g(x+\varpi)=\lambda g(x),\quad h(x+\varpi)=\lambda h(x) \tag{16}$$

and the relation (6) from Lemma 3.1 we obtain the equation

$$\lambda(\lambda-1)[f(x)y'+g(x)y]=0 \tag{17}$$

Since $\lambda(\lambda-1)\neq 0$ the last equation is equivalent to the equation

$$f(x)y'+g(x)y=0 \tag{18}$$

that for $f(x)\neq 0$ has a solution in the form (15).

The solution (15) is QPF with QP ϖ and QPC λ if the condition 3.2.1. holds. Indeed, from (15), under the condition 3.2.1., it follows that:

$$y(x+\varpi)=Ce^{-\int\limits_{x_0}^{x+\varpi}\frac{g(t)}{f(t)}dt}=Ce^{-\int\limits_{x_0}^{x}\frac{g(t)}{f(t)}dt-\int\limits_{x}^{x+\varpi}\frac{g(t)}{f(t)}dt}=y(x)\cdot e^{-(\Phi(x+\varpi)-\Phi(x))}=\lambda y(x).$$

Since the solution (15) is also a solution of (2), it has to satisfy the equations (17) and (2). Thus we have subsequently

$$y'=-\frac{g(x)}{f(x)}y$$

and

$$y''=h(x)$$

and after a short transformation we obtain that f,g,h have to satisfy the relation (14) i.e. the condition 3.2.2. □

Theorem 3.3. *Let $f=f(x)$, $g=g(x)$, $h=h(x)$ in (2) be QPF with constant QP ϖ and QPC μ,ν,η respectively, such that $\mu=\nu=\lambda\neq 1$. The equation (2) has QPS with QP ϖ and QPC λ if:*

3.3.1. $h(x)\equiv 0$;

3.3.2. $\Phi(x+\varpi)-\Phi(x)=\ln\frac{1}{\lambda}$, *where* $\Phi(x)=\int\frac{g(x)}{f(x)}$; *and*

3.3.3. f,g,h *satisfy the relation*

$$\left(\frac{g}{f}\right)^2-\left(\frac{g}{f}\right)'=0 \tag{19}$$

Then QPS of (2) has a form

$$y=Ce^{-\int\frac{g(x)}{f(x)}dx}. \tag{20}$$

Proof. Using the relations

$$f(x+\varpi)=\lambda f(x),\quad g(x+\varpi)=\lambda g(x)\quad h(x+\varpi)=h(x) \tag{21}$$

and the relation (6) in Lemma 2.1. we reduce the equation (2) to the equation

$$f(x)y'+g(x)y+\frac{1}{\lambda}h(x)=0, \tag{22}$$

whose QPS with QP ϖ and QPC λ is a solution to the equation ([2]):

$$y\Big[G^*(x+\varpi) - G^*(x)\Big] - \left[\frac{1}{\lambda}H^*(x+\varpi) - H^*(x)\right] = 0 \tag{23}$$

where $G^* = \dfrac{g(x)}{f(x)}$, $H^* = \dfrac{1}{\lambda}\dfrac{h(x)}{g(x)}$. Using again the relations (21) we obtain

$$h(x) \equiv 0 \tag{24}$$

Thus (22) gets a form

$$f(x)y' + g(x)y = 0 \tag{25}$$

and we have the solution (20), that under the conditions 3.3.2. is QPS.

Using (25), (24), (21) and (2), after some transformation, we obtain that f and g have to satisfy the relation (19). □

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