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OBTAINING THE DISTRIBUTION ASSOCIATED TO NONLINEAR CONTROL SYSTEM

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Abstract. Consider the nonlinear control system

$$\dot{x} = f(x,u) \ , \ x \in M \ , \ u \in \Omega \subseteq \mathcal{R}^m$$

where M is an n-dimensional manifold, $n \geq m$, and f is a vector field on M, for each $u \in \Omega$. The problem solved in this paper is obtaining the representation

$$f(x,u) = \sum_{i=n-r+1}^{n} h_i(x) \cdot \gamma_i(x,u) \quad \text{almost everywhere on } M \times \Omega$$

where $h_{n-r+1}(x), \dots, h_n(x)$ are vector fields on M and $\gamma_{n-r+1}(x,u), \dots, \gamma_n(x,u)$ are real functions, such that r is minimal.

1. Introduction.

Let be given a nonlinear control system

$$\dot{x} = f(x,u) \ , \ x \in M \ , \ u \in \Omega \tag{1.1}$$

where M is an n-dimensional manifold, Ω is an open subset of $\mathcal{R}^m$, $m \leq n$ and f is a continuous vector field having continuous partial derivatives by u of arbitrary order in a neighborhood of $x^0 \in M$, for every $u \in \Omega$.

In this paper we define the rank of a matrix function on $M \times \Omega$ as its maximal rank over $M \times \Omega$. The singular cases are accounted for. We also pose and solve the problem of obtaining the representation

$$f(x,u) = \sum_{i=n-r+1}^{n} h_i(x) \cdot \gamma_i(x,u) \quad \text{almost everywhere on } M \times \Omega \tag{1.2}$$

where $h_{n-r+1}(x), \dots, h_n(x)$ are vector fields on M and $\gamma_{n-r+1}(x,u), \dots, \gamma_n(x,u)$ are real functions, such that r is minimal. This problem is connected with obtaining the distribution on M

$$\mathcal{D}(x) = \text{span}\{f(x,u), \ u \in \Omega\} = \text{span}\{h_i(x), \ i = n-r+1, \dots, n\} \ , \quad x \in M$$

where r is the dimension of the distribution $\mathcal{D}$ and $h_i(x)$, $i = n-r+1, \dots, n$ are its generating vector fields. The motivation for the paper comes from the

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linear-in-control system

$$\dot{x} = a(x) + \sum_{j=1}^{m} b_j(x)\cdot u_j \tag{1.3}$$

where $a(x), b_1(x), \ldots, b_m(x)$ are vector fields. Systems (1.3) are the most frequently treated nonlinear systems in control theory and practice [1], and the best results of the nonlinear control theory are obtained for the linear-in-control systems. The system (1.3) has the form (1.2), where $m = r + 1$.

2. Main result

To obtain vector fields $h_{n-r+1}, \ldots, h_n$ satisfying

$\mathrm{span}\{f(x,u),\ u \in \Omega\} = \mathrm{span}\{h_i(x),\ i = n-r+1, \ldots, n\}$almost everywhere on M

we can use direct approach, i.e., application of Taylor expansion of $f(x,u)$ by u in a neighborhood of $0 \in \mathcal{R}^m$

$$f(x,u) = \sum_{w_1=0}^{\infty}\sum_{w_2=0}^{\infty}\cdots\sum_{w_m=0}^{\infty} \frac{u_1^{w_1}}{w_1!}\cdot\frac{u_2^{w_2}}{w_2!}\cdots\frac{u_m^{w_m}}{w_m!}\cdot F^{<w_1,w_2,\ldots,w_m>}(x)$$

where $u = [u_1, u_2, \ldots, u_m]^T$ and $F^{<w_1,w_2,\ldots,w_m>}$, $w_1, w_2, \ldots, w_m = 1, 2, \ldots$ are some vector fields on M, and then, among these vector fields to choose the vector fields $h_{n-r+1}, \ldots, h_n$. But such procedure is not easy to formalize in algorithmic terms.

Thus we use an indirect approach. Firstly we solve the equation

$$\omega^T(x)\cdot f(x,u) = 0\ , \quad u \in \Omega \tag{2.1}$$

for the unknown vector-function $\omega(x)$. Moreover, we need all linearly-independent solutions $\omega_1(x), \ldots, \omega_{n-r}(x)$. Then the vector-functions $h_{n-r+1}(x), \ldots, h_n(x)$ can be found from the linear-homogeneous system

$$\omega_i^T\cdot h_j = 0, \quad i = 1, 2, \ldots, n-r, \quad j = n-r+1, \ldots, n\,. \tag{2.2}$$

Let introduce the following notation, instead of (2.1)

$$R_0(x,u)\cdot\omega(x) = 0\ , \tag{2.3}$$

where $R_0(x,u) = f^T(x,u)$. The algorithm for solving the system (2.3), as well as a method for obtaining the vector fields $h_{n-q+1}, \ldots, h_n$ without solving the linear-homogeneous system (2.2), is presented in Appendix A (see [2],[3], also).

It remains to find the functions $\gamma_{n-r+1}(x,u), \ldots, \gamma_n(x,u)$. For the specific choice of vectors $h_{n-q+1}, \ldots, h_n$ found in Appendix A, the functions $\gamma_{n-r+1}(x,u), \ldots, \gamma_n(x,u)$ are simple. That claim is content of the following

Proposition 2.1. *Let* $\omega_i^T \cdot h_j = 0, \quad i = 1, 2, \ldots, n-r, \quad j = n-r+1, \ldots, n$ *for some vector fields* $h_{n-r+1}, \ldots, h_n$ *on* M *and let us introduce the following notations*

$$A = \begin{bmatrix} \Omega_1^T \\ \Omega_2^T \end{bmatrix}, \quad \Omega_1 = [\omega_1, \ldots, \omega_{n-r}], \quad \Omega_2^T = \begin{bmatrix} \omega_{n-r+1}^T \\ \vdots \\ \omega_N^T \end{bmatrix} \stackrel{def}{=} [I_r, 0] \ . \tag{2.4}$$

If the matrix A *with the choice (2.4) is nonsingular, then* $A^{-1} = [H_1, H_2]$, *where* $H_1 = \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}$ *and* $H_2 = [h_{n-r+1}, \ldots, h_n]$. *Moreover, in this case,*

$$\gamma_{n-r+1} = f_1 \ , \ldots, \gamma_n = f_r \ ,$$

where $f_1, \ldots, f_r$ *are the first* r *elements of the vector* f.

Remark. By pre-numeration of the indices of ω^T and indices of f, we can achieve non-singularity of A. The proof of Proposition 2.1 is given in Appendix B.

We have obtained a representation for f

$$f(x,u) = \sum_{i=n-r+1}^{n} h_i(x) \cdot f_i(x,u) \quad \text{almost everywhere on } M \times \Omega \tag{2.5}$$

in which appear, virtually, only r components of f. We can use this representation to simplify the function f. After we find the integer r, we choose r functions among $f_1, \ldots, f_n$, with lowest degrees of u, ideally, linear in u. Then we can permute the indices of f and apply (2.5).

In the case there are r linear-in-control functions among $f_1, \ldots, f_n$, then the system (1.1) is of the class (1.3).

Remark. The integer r satisfies $q \le r$, where $n - q$ is the number of linearly independent solutions for the unknown n-dimensional vector-column function $\omega(x)$ and real function $\psi(x)$ of the following system

$$\omega^T(x) \cdot f(x,u) = \psi(x), \quad u \in \Omega$$

(which could be solved by the algorithm in Appendix A, also). More specific, r is equal to q or $q+1$.

Remark. It can be proved that the integers r and q are invariants of the nonlinear control system (1.1) under coordinate transformation and full feedback control $u = \varphi(x, v)$.

Appendix A We formulate an algorithm for solving the matrix system

$$R_0(x,u) \cdot \omega(x) = 0 \tag{A.1}$$

(see [2],[3] also). If the rows of the matrix R_0 are linearly dependent, let separate the linear independent rows in the matrix $\bar{R}_0$. Then, instead of (A.1), take the reduced equation

$$\bar{R}_0(x,u) \cdot \omega(x) = 0 \ . \tag{A.2}$$

By differentiation of the equation (A.2) with respect to u_i, $i = 1, \dots, m$, one obtains

$$\begin{bmatrix} \bar{R}_0 \\ \frac{\partial \bar{R}_0}{\partial u_1} \\ \vdots \\ \frac{\partial \bar{R}_0}{\partial u_m} \end{bmatrix} \cdot \omega \stackrel{\text{def}}{=} R_1 \cdot \omega = 0 \,. \tag{A.3}$$

Let denote by $\bar{R}_1$ the matrix consisting of all linearly independent rows of the matrix R_1. The order of rejecting of the linearly dependent rows of R_1 is from the top of R_1 to the bottom. It means that the first rows of the matrix $\bar{R}_1$ are the rows of $\bar{R}_0$. Then, instead of (A.3), take the reduced equation

$$\bar{R}_1 \cdot \omega = 0 \,. \tag{A.4}$$

Applying the differential operators $\frac{\partial}{\partial u_i}$ on this system, one obtains, analogously $R_2 \cdot \omega = 0$ and $\bar{R}_2 \cdot \omega = 0$. The procedure continues.

Let denote $r_i =$ row dim $\bar{R}_i$. It is clear that r_i, $i = 1, 2, \dots$ is non-decreasing series which is limited from above by n (the number of columns of R_0). Hence, there exists least integer k such that $r_k = r_{k+1}$, and in this case holds $\bar{R}_k = \bar{R}_{k+1} = \bar{R}_{k+2} = \cdots$. Thus, if a solution of the system (A.1) exists, it has to satisfy

$$\bar{R}_k \cdot \omega = 0 \tag{A.5}$$

$$\frac{\partial \bar{R}_k}{\partial u_i} = K_i \cdot \bar{R}_k \,, \quad i = 1, \dots, m \tag{A.6}$$

for some matrix functions $K_i(x, u)$, $i = 1, \dots, m$. Further on, we show that if $r_k < n$ then there exist $n - r$, $r \stackrel{\text{def}}{=} r_k$ vector functions-solutions of the system (A.1), $\omega_1, \omega_2, \dots, \omega_{n-r}$, defined and linearly independent almost everywhere on M, which can be obtained by linear algebraic operations only.

Since the rows of the matrix $\bar{R}_k$ are linearly independent, it can be partitioned on sub-matrices

$$\bar{R}_k = [\bar{R}'_k, \ \bar{R}''_k]$$

so that the matrix $\bar{R}'_k$ is square and nonsingular. (If $\bar{R}'_k$ is singular, then by permutation of the columns of $\bar{R}_k$ and pre-numeration of the indices of x, we can achieve non-singularity of $\bar{R}'_k$.) Partitioning the conditions (A.6), we obtain

$$\frac{\partial \bar{R}'_k}{\partial u_i} = K_i \cdot \bar{R}'_k \,, \quad \frac{\partial \bar{R}''_k}{\partial u_i} = K_i \cdot \bar{R}''_k \,, \quad i = 1, \dots, m \,. \tag{A.7}$$

Proposition A.1. *The matrix function $P = \bar{R}'^{-1}_k \cdot \bar{R}''_k$ does not depend on u, i.e.*

$$\frac{\partial P}{\partial u_i} = 0 \,, \quad i = 1, \dots, m \,. \tag{A.8}$$

Proof. Applying the conditions (A.7), one obtains

$$\frac{\partial P}{\partial u_i} = \frac{\partial}{\partial u_i} \left(\bar{R}'^{-1}_k \cdot \bar{R}''_k \right) = -\bar{R}'^{-1}_k \cdot \frac{\partial \bar{R}'_k}{\partial u_i} \cdot \bar{R}'^{-1}_k \cdot \bar{R}''_k + \bar{R}'^{-1}_k \cdot \frac{\partial \bar{R}''_k}{\partial u_i} =$$

$$= -\bar{R}_k'^{-1} \cdot K_i \cdot \bar{R}_k' \cdot \bar{R}_k'^{-1} \cdot \bar{R}_k'' + \bar{R}_k'^{-1} \cdot K_i \cdot \bar{R}_k'' = 0 \ . \qquad \blacksquare$$

The system (A.5), in partitioned form, is

$$\bar{R}_k' \cdot \omega' + \bar{R}_k'' \cdot \omega'' = 0 \qquad (A.9)$$

where the vector ω is partitioned on

$$\omega = \begin{bmatrix} \omega' \\ \omega'' \end{bmatrix} .$$

From the equation (A.9) it follows

$$\omega' = -P \cdot \omega'' \ . \qquad (A.10)$$

Therefore, the requiring solutions of (A.5) are the columns of the matrix

$$\begin{bmatrix} -P \\ I_{n-r} \end{bmatrix} = [\omega_1(x), \omega_2(x), \ldots, \omega_{n-r}(x)] \ . \qquad (A.11)$$

A consequence of the algorithm is the matrix equality

$$R_0(x,u) = K(x,u) \cdot \bar{R}_k(x,u)$$

for some matrix function $K(x,u)$. Thus the solutions $\omega_1, \omega_2, \ldots, \omega_{n-r}$ of (A.5) are also solutions of (A.1).

Let us discuss singular cases.

(i) Consider the set of points $(x,u) \in M \times \Omega$ where the matrix functions $K_i(x,u)$, $i = 1,2,\ldots,m$ and $K(x,u)$ are not defined. By (A.6) and (A.11) that point set is given by

$$\mathcal{G} = \left\{(x,u) \in M \times \Omega : \ \det\left(\bar{R}_k(x,u) \cdot \bar{R}_k^T(x,u)\right) = 0\right\} \ . \qquad (A.12)$$

All steps of the algorithm are valid for all $(x,u) \notin \mathcal{G}$ (which means that feedback control $u = \sigma(x)$ satisfying (A.12) is not allowed to be applied). If there exists a point $x^s \in M$ such that $\det\left(\bar{R}_k(x^s,u) \cdot \bar{R}_k^T(x^s,u)\right) = 0$, $\forall u \in \Omega$, then we name the point x^s as *singular point* for our problem.

(ii) The points of M on which the function $P(x)$ is not defined are also singular points for our algorithm.

In this paper we suppose that x^0 is not a singular point.

Remark. It can be proved that, under the conditions (A.6), that there exists a square nonsingular matrix function $Q(x,u)$ such that the matrix function $Q(x,u) \cdot \bar{R}_k(x,u)$ does not depend on u. Then the system (A.5) is

$$\bar{Q}(x) \cdot \omega(x) = 0,$$

where $\bar{Q}(x) = Q(x,u) \cdot \bar{R}_k(x,u)$. Then the points of M on which the function $P(x)$ is not defined are given by

$$\det\left(\bar{Q}(x) \cdot \bar{Q}^T(x)\right) = 0 \ .$$

But obtaining the function $Q(x,u)$ (and $\bar{Q}(x)$) is via solving a system of PDE, which is more complicated than the proposed matrix-function inversion contained in Proposition A.1.

Remark. The matrices K_i, $i = 1, \dots, m$ satisfy

$$K_i \cdot K_j + \frac{\partial K_i}{\partial u_j} = K_j \cdot K_i + \frac{\partial K_j}{\partial u_i}\,, \quad i, j = 1, \dots, m\,.$$

Since by (A.11)

$$\begin{bmatrix} \omega_1^T \\ \vdots \\ \omega_{n-r}^T \end{bmatrix} \cdot \begin{bmatrix} I_r \\ P^T \end{bmatrix} = [-P^T, I_{n-r}] \cdot \begin{bmatrix} I_r \\ P^T \end{bmatrix} = -P^T + P^T = 0\,,$$

for the unknown vector fields $h_{n-r+1}, \dots, h_n$ (which satisfy (2.2)), we take

$$[h_{n-r+1}, \dots, h_n] = \begin{bmatrix} I_r \\ P^T \end{bmatrix}.$$

Appendix B

Proof of Proposition 2.1. By the algorithm in Appendix A, we have

$$\Omega_1^T = \begin{bmatrix} \omega_1^T \\ \vdots \\ \omega_{n-r}^T \end{bmatrix} = [-P^T, I_{n-r}]\,.$$

Then from $H_2 = \begin{bmatrix} I_r \\ P^T \end{bmatrix}$ it follows that

$$A \cdot [H_1, H_2] = \begin{bmatrix} \Omega_1^T \\ \Omega_2^T \end{bmatrix} \cdot [H_1, H_2] = \begin{bmatrix} \Omega_1^T \cdot H_1 & \Omega_1^T \cdot H_2 \\ \Omega_2^T \cdot H_1 & \Omega_2^T \cdot H_2 \end{bmatrix} = \begin{bmatrix} I_{n-r} & 0 \\ 0 & I_r \end{bmatrix} = I_n$$

because $\Omega_1^T \cdot H_2 = 0$ by assumption, and

$$\Omega_1^T \cdot H_1 = [-P^T, I_{n-r}] \cdot \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = I_{n-r}\,,$$

$$\Omega_2^T \cdot H_1 = [I_r, 0] \cdot \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = 0\,,$$

$$\Omega_2^T \cdot H_2 = [I_r, 0] \cdot \begin{bmatrix} I_r \\ P^T \end{bmatrix} = I_r\,.$$

Now, consider the presentation

$$f(x, u) = \sum_{i=n-q+1}^{n} h_i \cdot \gamma_i = H_2 \cdot \gamma = [H_1, H_2] \cdot \begin{bmatrix} 0 \\ \gamma \end{bmatrix}, \qquad (B.1)$$

where the vector function $\gamma = [\gamma_{n-r+1}, \dots, \gamma_n]^T$ is introduced. Multiplying the identity (B.1) by the nonsingular matrix $[H_1, H_2]$, we obtain

$$\begin{bmatrix} 0 \\ \gamma \end{bmatrix} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix} \cdot f = \begin{bmatrix} 0 \\ \Omega_2 \cdot f \end{bmatrix}.$$

Consequently,

$$\gamma_{n-r+1} = f_1\,, \dots, \gamma_n = f_r\,. \quad \blacksquare$$

Conclusion. The vector variable x may be considered as "parameter", except in the case when by the function f is given a nonlinear dynamical system (1.1). In particular, the results of the paper are valid for representation of the vector function $f(u)$ as a linear combination of r vectors, i.e. $f(u) = \sum_{i=n-r+1}^{n} h_i \cdot \gamma_i(u)$, where $\gamma_i(u)$ are some real functions. Thus the result of the paper can be of interest for the pure mathematics, also.

References

[1] Isidori A. *"Nonlinear Control Systems: An introduction"*, New York, Springer-Verlag, 1985.

[2] Stefanovski J., Trenčevski K., Covachev V., "Cauchy problem and reduction of a system of partial differential equations", *Tamkang Journal of Mathematics*, **30**, No.4 (1999), 315-329.

[3] Stefanovski J., Trenčevski K., Covachev V., "Charpit system of partial equations with algebraic constraints", *Journal of Partial Differential Equations*, **13** No.2 (2000), 151-162.

[4] Stefanovski J., Trenčevski K., "Analytic solutions of systems", Journal of Dynamical and Control Systems", **8** No.4 (2002), 463-486.

[5] Stefanovski J., "Feedback affinization of nonlinear control systems", Systems & Control Letters **46** (2002), 207-217.

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