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# ON A CONVEX QUADRATIC PROGRAMMING PROBLEM

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We consider the quadratic programming problem minimize $f(x)=\frac{1}{2}x^TD^TDx$ constrained by $Cx \geqslant c$ for a nonsingular $n\times n$ matrix D. It is a consequence of the Kuhn-Tucker theorem that $\bar{x}$ is an optimal solution if and only if there exists a complementary feasible solution $(\bar{v};\bar{u})$ to $v=G^TGu-c$, where $G^T=CD^{-1}$. Then, $\bar{x}=D^{-1}G\bar{u}$ and $\bar{u}$ is an optimal solution for the quadratic programming problem maximize $g(u)=u^Tc-\frac{1}{2}u^TG^TGu, u\geqslant o$. We study the optimal solutions in the case when the elements of D and C are defined as follows:

$$d_{ij}=\begin{cases}k, & i=\pi(j)\\ -1, & i\neq\pi(j)\end{cases}, \quad c_{ij}=\begin{cases}k_1, & i=\pi_1(j)\\ -1, & i\neq\pi_1(j)\end{cases}$$

, where $\pi$ and $\pi_1$ are given permutations on $\{1,\dots,n\}$, $k$ and $k_1$ are nonzero reals.

Consider the quadratic programming problem (q.p.p.):

(1) minimize $f(x)=\frac{1}{2}x^TD^TDx$ constrained by $Cx\geqslant c$,

where D is a given nonsingular $n\times n$ matrix, C is a given $m\times n$ matrix and c is a given m-vector. This problem is equivalent to the q.p.p.

(2) minimize $f(y)=\frac{1}{2}y^Ty$ constrained by $G^Ty\geqslant c$,

considered by Tucker in [5]. Indeed, denoting $y=Dx$, $G^T=CD^{-1}$, (1) becomes the problem (2) and conversely, for any nonsingular matrix D, denoting $x=D^{-1}y$, $C=G^TD$, (2) becomes the problem (1). The matrix D defines a linear transformation in Euclidean n-space $E^{(n)}$, and the problem (1) seeks a point $\bar{x}$ of the polyhedral set $T=\{x;Cx\geqslant c\}$ in $E^{(n)}$, whose image $D\bar{x}$ is at least distance from the origin.

Let the matrix D in (1) be nonsingular and denote $G^T=CD^{-1}$. Then we can associate to (1) the linear complementarity problem (l.c.p.)

(3) $v=G^TGu-c, v\geqslant o, u\geqslant o, v^Tu=0$,

and the q.p.p. (dual to (1))

(4) maximize $g(u)=u^Tc-\frac{1}{2}u^TG^TGu, \ u\geqslant o$.

Conditions (3) are known as the Kuhn-Tucker conditions for (4). As (4) is a convex program, (3) are necessary and sufficient conditions for optimality.

<u>Theorem</u>. The point $\bar{x}$ is an optimal solution for (1) if and only if, there exists a complementary feasible solution (c.f.s.) $(\bar{v};\bar{u})$ for (3). Then $\bar{x}=D^{-1}G\bar{u}$, $\bar{u}$ is an optimal solution for (4) and $f(\bar{x})=g(\bar{u})=\bar{u}^Tc/2$.

**Proof.** Let the convex polyhedral set $T=\{x;Cx\geqslant c\}$ be nonempty, and suppose $\bar{x}$ is an optimal solution for (1). Denote $B=\{x;x^TD^TDx\leqslant 2f(\bar{x})\}$. It is easy to see that $B\cap T=\{\bar{x}\}$. Note that $\bar{x}=o$ if and only if $c\leqslant o$. If $\bar{x}\neq o$, the hyperplane $\{x;\bar{x}^TD^TD(x-\bar{x})=0\}$, tangent at $\bar{x}$ to B, bounds a half-space $H=\{x;\bar{x}^TD^TD(x-\bar{x})\geqslant 0\}$ that contains the set T. Partition $C\bar{x}\geqslant c$ into

(5) $C_I\bar{x}=c_I, \ C_J\bar{x}>c_J$.

The set I is nonempty, since $\bar{x}$ is optimal. H must contain the polyhedral set $\{x;C_I(x-\bar{x})\geqslant o\}$, which coincides with T near $\bar{x}$. This means that $C_I(x-\bar{x})\geqslant o$ implies $\bar{x}^TD^TD(x-\bar{x})\geqslant 0$. So, by Farkas theorem, there exists a vector $\bar{u}_I\geqslant o$ such that $\bar{u}_I^TC_I=\bar{x}^TD^TD$. Set $\bar{u}_J=o$ and $\bar{u}=\begin{bmatrix}\bar{u}_I\\ \bar{u}_J\end{bmatrix}$. Obviously, $\bar{u}\geqslant o$, and

(6) $\bar{u}^TC=\bar{x}^TD^TD$.

Then,

(7) $\bar{u}^T(C\bar{x}-c)=0$,

since (5). By (6) and the nonsingularity of D,

(8) $\bar{x}=(D^TD)^{-1}C^T\bar{u}$,

and, by (7), $f(\bar{x})=\bar{u}^Tc/2$.

The condition (8) and $\bar{x}\in T$ imply

(9) $CD^{-1}(CD^{-1})^T\bar{u}-c\geqslant o$,

and (7) can be written as

(10) $\bar{u}^T(CD^{-1}(CD^{-1})^T\bar{u}-c)=0$.

Set $G^T=CD^{-1}$ and $\bar{v}=G^TG\bar{u}-c$. Then (9), (10) and $\bar{u}\geqslant o$ mean that $(\bar{v};\bar{u})$ is a c.f.s. for (3).

Now, let $(\bar{v};\bar{u})$ be a c.f.s. for (3), where $G^T=CD^{-1}$. Denote $\bar{x}=(D^TD)^{-1}C^T\bar{u}$, $g(\bar{u})=\bar{u}^Tc-\frac{1}{2}(G\bar{u})^TG\bar{u}$. Then, $C\bar{x}=G^TG\bar{u}\geqslant c$, which means that $\bar{x}\in T$. For any $x\in T$, $f(x)-g(\bar{u})=\frac{1}{2}(Dx-G\bar{u})^T(Dx-G\bar{u})+\bar{u}^T(G^TDx-c)\geqslant 0$, or, $f(x)\geqslant g(\bar{u})=\bar{u}^Tc/2=f(\bar{x})\geqslant g(u)$, where $g(u)=u^Tc-\frac{1}{2}(Gu)^TGu$ for any $x\in T$ and $u\geqslant o$. So, $\bar{x}$ is an optimal solution for (1), $\bar{u}$ is an optimal solution for (4), and $f(\bar{x})=g(\bar{u})$.

<u>Corollary 1</u>. If the matrix C has rank m, then the problems (1) and (4) have optimal solution for any vector c.

<u>Corollary 2</u>. If the problem (3) has no feasible solution, then the problem (1) is infeasible, and the problem (4) has no finite optimal solution.

The l.c.p., as well as the q.p.p. have been studied by many authors. P.Wolfe, E.Beale and others have devised methods of solving the quadratic programs (see [4]). C.Lemke, R.Cottle and G.Dantzig, and others have devised methods of solving the l.c.p. for various matrices, including positive semidefinite (see [1], [2]).

We shall describe a modification of the algorithm, proposed by the author of this paper, for solving the l.c.p.

(11) $w=q+Mu, \ w\geqslant o, u\geqslant o, \ w^Tu=0$,

for an nxn matrix M whose elements are

$$(12)\quad m_{ij}=\begin{cases} k, & i=\pi(j)\\ -1, & i\neq\pi(j)\end{cases},$$

where k is any real greater then n-2, $\pi$ is a given permutation on $\{1,\dots,n\}$ (see [3]), which processes (1) and (4) when the elements of D and C are of type (12).

Let D and C in (1) be nxn matrices, whose elements are, respectively,

$$d_{ij}=\begin{cases} k, & i=\pi(j)\\ -1, & i\neq\pi(j)\end{cases},\quad c_{ij}=\begin{cases} k_1, & i=\pi_1(j)\\ -1, & i\neq\pi_1(j)\end{cases},$$

where k and $k_1$ are any nonzero reals, $\pi$ and $\pi_1$ are given permutations on $\{1,\dots,n\}$. If $k\neq-1$ and $k\neq n-1$, then D is nonsingular and the inverse of D is the following matrix:

$$D^{-1}=\frac{1}{(k+1)(k-n+1)}\left[\mu_{ij}\right],\text{ where }\mu_{ij}=\begin{cases} k-n+2, & j=\pi(i)\\ 1, & j\neq\pi(i)\end{cases}$$

Now, going back to problem (3) we find

$$D^{-1}(D^{-1})^T=\frac{1}{(k+1)^2(k-n+1)^2}\left[\xi_{ij}\right],$$

where

$$\xi_{ij}=\begin{cases} (k-n+2)^2+n-1, & i=j\\ 2k-n+2, & i\neq j\end{cases},$$

and

$$G^TG=CD^{-1}(CD^{-1})^T=\frac{1}{(k+1)^2(k-n+1)^2}\left[\eta_{ij}\right],$$

where

$$\eta_{ij}=\begin{cases} (k_1k-n+1-(n-2)k_1)^2+(n-1)(k_1-k)^2, & i=j\\ (k_1-k)(2(k_1k-n+1)-(n-2)(k_1+k)), & i\neq j\end{cases}.$$

If $k_1=k$, or $2(k_1k-n+1)-(n-2)(k_1+k)=0$, then $G^TG=\alpha^2I$, where I is the unit matrix and

$$\alpha^2=\frac{(k-n+1-(n-2)k)^2}{(k+1)^2(k-n+1)^2}\quad\text{in the case } k_1=k, \text{ or}$$

$$\alpha^2=\frac{(k_1k-n+1-(n-2)k_1)^2+(n-1)(k_1-k)^2}{(k+1)^2(k-n+1)^2}\quad\text{in the case}$$

$2(k_1k-n+1)-(n-2)(k_1+k)=0$, and obviously, $\bar v_i=-c_i$, $\bar u_i=0$ for $c_i\leq0$, $\bar v_i=0$, $\bar u_i=c_i/\alpha^2$ for $c_i\geq0$ give a c.f.s. for (3).

Now, suppose $k_1\neq k$ and $2(k_1k-n+1)-(n-2)(k_1+k)\neq0$, and denote

$$\beta=(k_1k-n+1-(n-2)k_1)^2+(n-1)(k_1-k)^2,$$
$$\gamma=(k_1-k)(2(k_1k-n+1)-(n-2)(k_1+k)).$$

Then

$$(13)\quad G^TG=\frac{|\gamma|}{(k+1)^2(k-n+1)^2}\left[\gamma_{ij}\right],$$

where

$$\gamma_{ij}=\begin{cases} \beta/|\gamma|, & i=j\\ \gamma/|\gamma|, & i\neq j\end{cases}.$$

Note that $G^TG$ is a singular matrix if and only if $k_1=-1$, or n-1. For $k_1=-1$,

$$G^TG=\frac{n}{(k-n+1)^2}E,$$

where E is an nxn matrix all of whose elements are the unity, and we can find a c.f.s. for (3) easily. Let $k_1\neq-1$. There are two possibilities:

Case I. $\gamma<0$. Then, in (13), $\gamma_{ij}=\begin{cases} -\beta/\gamma, & i=j\\ -1, & i\neq j\end{cases}$.

Denoting $w=\dfrac{(k+1)^2(k-n+1)^2}{-\gamma}v$, $q=\dfrac{(k+1)^2(k-n+1)^2}{\gamma}c$,

$$k^*=-\beta/\gamma,\quad M=\begin{bmatrix} k^* & -1 & \dots & -1\\ -1 & k^* & \dots & -1\\ \dots & \dots & \dots & \dots\\ -1 & -1 & \dots & k^*\end{bmatrix}$$

the problem (3) becomes

$$(14)\quad w=q+Mu,\ w\geq0,\ u\geq0,\ w^Tu=0.$$

The element $k^*$ of M must be greater, or equal to n-1, because $G^TG$ is positive semidefinite. So, (14) is a special case of the problem (11). In [3] we have shown that for

$$q_r=\max_{1\leq j\leq n}\{q_j\}\quad\text{and}\quad s=\sum_{j=1}^{n}q_j+(k^*-n+1)q_r$$

if $s<0$, then (14) is infeasible in the case $k^*=n-1$; otherwise ($k^*>n-1$) the pair $(\bar w;\bar u)=(0;-M^{-1}q)$ is a c.f.s.

If $s\geq0$, we get a c.f.s. for (14) applying the following algorithm.

Step0. Initialize $\nu=0$, $I^{(\nu)}=\{1,\dots,n\}$, $s^{(\nu)}=s$, $r^{(\nu)}=r$.

Step 1. Increase $\nu=\nu+1$, set $I^{(\nu)}=I^{(\nu-1)}-\{r^{(\nu-1)}\}$ and find

$$q_{r^{(\nu)}}=\max_{j\in I^{(\nu)}}\{q_j\},\quad s^{(\nu)}=\sum_{j\in I^{(\nu)}}q_j+(k^*-n+1+\nu)q_{r^{(\nu)}}.$$

Step 2. Test $\nu<n-1$

2.1 If yes, go to step 3.

2.2 If not, the scalar pivot on $m_{r^{(\nu)}r^{(\nu)}}=k^*$ in (14) gives a c.f.s. Stop!

Step 3. Test $s^{(\nu)}>0$.

3.1 If yes, go to step 1.

3.2 If not, the pivot on $M_{I^{(\nu)}I^{(\nu)}}$ [1] in (14) gives a c.f.s. Stop!

Case II. $\gamma>0$. Then, in (13) $\gamma_{ij}=\begin{cases} \beta/\gamma, & i=j\\ 1, & i\neq j\end{cases}$.

Now, denoting

$$w=\frac{(k+1)^2(k-n+1)^2}{\gamma}v,\quad q=\frac{(k+1)^2(k-n+1)^2}{\gamma}(-c)$$

$$k^*=\beta/\gamma,\quad M=\begin{bmatrix} k^* & 1 & \dots & 1\\ 1 & k^* & \dots & 1\\ \dots & \dots & \dots & \dots\\ 1 & 1 & \dots & k^*\end{bmatrix},$$

(3) becomes the problem (14) with $k^*\geq1$ in M, since $G^TG$ is positive semidefinite. The case $k^*=1$ is trivial. Therefore, suppose $k^*>1$. Then M is nonsingular and the inverse of M is the matrix

$$M^{-1}=\frac{1}{(k^*-1)(k^*+n-1)}\begin{bmatrix} k^*+n-2 & -1 & \dots & -1\\ -1 & k^*+n-2 & \dots & -1\\ \dots & \dots & \dots & \dots\\ -1 & -1 & \dots & k^*+n-2\end{bmatrix}$$

Also, M is positive definite matrix. So, (14) has a c.f.s. for any q (see [1]). We get it applying the following algorithm:

Step 0. Initialize $\nu=0$, $I^{(\nu)}=\{1,\dots,n\}$.

Step 1. Find

$$r^{(\nu)}=\max_{j\in I^{(\nu)}}\{q_j\},\quad s^{(\nu)}=\sum_{j\in I^{(\nu)}}q_j-(k^*+n-1-\nu)q_{r^{(\nu)}}.$$

Step 2. Test $s^{(\nu)}<0$.

2.1 If yes, increase $\nu=\nu+1$ and go to step 3.

2.2 If not, the pivot on $M_{I^{(\nu)}I^{(\nu)}}$ in (14) gives a c.f.s. Stop!

Step 3. Test $\nu<n-1$

3.1 If yes, set $I^{(\nu)}=I^{(\nu-1)}-\{r^{(\nu-1)}\}$ and go to step 1.

3.2 If not, the scalar pivot on $m_{r^{(\nu)}r^{(\nu)}}=k^*$ gives a c.f.s. Stop!

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1) $M_{I^{(\nu)}I^{(\nu)}}$ is a principal submatrix obtained from M by deleting all rows and columns corresponding to indices in $\{1,\dots,n\}-I^{(\nu)}$