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ON THE DUAL PAIR OF LP-PROBLEMS IN CANONICAL FORM WITH NONNEGATIVE INVERSE MATRIX

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Abstract. The dual pair of LP-problems

$\min\{z = c^T x \mid Ax \geq b,\ x \geq o\}$, $\max\{w = b^T y \mid A^T y \leq c,\ y \geq o\}$

where A is an nxn matrix with inverse $A^{-1} \geq 0$, is solvable iff $c^T A^{-1} \geq o$. Otherwise, the primal objective function z is unbounded on the feasible region. Some special cases of the matrix A are discussed.

1. General conclusions

We consider the dual pair of LP-problems

$$\min\{z = c^T x \mid Ax \geq b,\ x \geq o\} \qquad (P)$$

$$\max\{w = b^T y \mid A^T y \leq c,\ y \geq o\} \qquad (D)$$

where A is a given real nxn matrix with inverse $A^{-1} \geq 0$, $b=[b_i]$ and $c=[c_j]$ are given n-vectors, x and y are the vectors of variables in Euclidean space R^n.

Let N denote the set of integers $\{1,2,\ldots,n\}$, and

$(A^{-1})_j$ denote the j-th column of A^{-1}.

Define

$$x_o = \sum_{j \in N} (A^{-1})_j$$

$$y_o = (A^{-1})^T c$$

$$b_o = A^{-1} b$$

The assumption $A^{-1} \geq 0$, which means $(A^{-1})_j \geq o$, $j \in N$ and implies $x_o \geq o$, plays an essential role. Almost immediately can be stated that:

(i) The feasible region T of (P) is unbounded;

(ii) (P) and (D) are solvable iff $y_o \geq o$.

Indeed, we can choose k in N satisfying the condition

$$b_k = \max_{i \in N}\{b_i\}. \quad (1)$$

In the case when $b_k \le 0$, at least the points of the ray

$$\{x = \lambda x_o \mid \lambda \ge 0\}$$

satisfy the constraints of (P), and clearly, T is undbounded. In the opposite case, $b_k > 0$, the points of the half-line

$$\{x = (b_k+\lambda)x_o \mid \lambda \ge 0\}$$

belong to T, and therefore T must be undbounded.

The proof of (ii) also is evident. If $y_o \ge o$, then y_o is a feasible solution to (D) and, by duality theorem, there exists a pair $\hat{x},\hat{y}$ of optimal solutions to (P) and (D), respectively. If $y_o \not\ge o$, and for example its ℓ-th component is negative, $(y_o)_\ell < 0$, then for the points of the half-line

$$\{x = b_k x_o + \lambda(A^{-1})_\ell \mid \lambda \ge 0\} \subseteq T$$

we have

$$z = c^T x = b_k c^T x_o + \lambda(y_o)_\ell \to -\infty \text{ for } \lambda \to +\infty.$$

So, in this case (P) has no finite optimal solution and (D) is infeasible.

Now, it is easy to make the following conclusion:

(iii) For (P) and (D) if $y_o \ge o$ and $b_o \ge o$, then the vectors b_o, y_o are optimal solusions.

Indeed, b_o is feasible solution to (P), y_o is feasible solution to (D) and moreover $z_o = c^T b_o = c^T A^{-1} b = ((A^{-1})^T c)^T b = y_o^T b = b^T y_o = w_o$.

In the case when $y_o \ge o$, but $b_o \not\ge o$, a pair of optimal solutions $\hat{x},\hat{y}$ to (P) and (D) can be found applying simplex algorithm for the equivalent standard form of (D):

$$\max\{w = b^T y \mid A^T y + v = c, \quad y \ge o, \ v \ge o\}. \quad (D_s)$$

Since $A^{-1} \geq 0$ is known, it is convenient to start with the basis $B=A^T$ and the corresponding reduced form of (D_s):

$$\max\{w-b^T y_o = -b_o^T v \mid y+(A^T)^{-1} v = y_o, \quad y \geq o, \ v \geq o\}.$$

2. Some special cases

a) The trivial case A=I. For the identity matrix I the condition $y_o \geq o$ means $c \geq 0$, and under this assumption the vectors

$$\hat{x} = [\hat{x}_j]_{n\times 1}, \text{ where } \hat{x}_j = \begin{cases} b_j & \text{if } b_j > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{y} = [\hat{y}_j]_{n\times 1}, \text{ where } \hat{y}_j = \begin{cases} c_j & \text{if } b_j > 0 \\ 0 & \text{otherwise} \end{cases}$$

are optimal solutions; $\hat{z} = \hat{w} = \sum_{j:b_j>o} c_j b_j$ is the optimal value of z and w.

b) $A = \frac{1}{n-1}E - I$. The nxn matrix with elements all equal to 1 is denoted by E. For the inverse of A we have $A^{-1} = E-I \geq 0$ and therefore, the necessary and sufficient conditions for solvability of (P) and (D) can be stated as

$$c^T e \geq c_k \tag{2}$$

where e denotes the vector, whose components are all equal to 1, and k is defined by $c_k = \max_{j\in N}\{c_j\}$.

In the case (2), if $e^T b \geq \max_{i\in N}\{b_i\} = b_\ell$, then

$\hat{x} = \beta_o e-b$, where $\beta_o = e^T b$, is an optimal solution to (P),

$\hat{y} = \gamma_o e-c$, where $\gamma_o = e^T c$, is an optimal solution to (D),

and $\hat{z} = \beta_o \gamma_o - c^T b = \hat{w}$ is the common optimal value of z and w. If $\beta_o < b_\ell$, then it is useful to continue solving the reduced form of (D):

$$\max\{w-\hat{w} = -(\beta_o e-b)^T v \mid y+(E-I)v = \hat{y}, \ y \geq o, \ v \geq o\}.$$

Analogous conclusions can be made for the more general case

$$A = \frac{1}{a}(I - \frac{1}{a+n}E), \quad A^{-1} = aI+E, \quad 0 < |a| \le 1.$$

Namely,

$$\min\{c_j\} \ge -\frac{1}{a}\gamma_o \text{ if } 0 < a \le 1$$

or,

$$\max\{c_j\} \le -\frac{1}{a}\gamma_o \text{ if } -1 \le a < 0$$

is a necessary and cufficient condition for solvatibily of (P) and (D) is this case. Moreover, if

$$\min\{b_j\} \ge -\frac{1}{a}\beta_o \text{ for } 0 < a \le 1,$$

or,

$$\max\{b_j\} \le -\frac{1}{a}\beta_o \text{ for } -1 \le a < 0$$

then $\hat{x} = \beta_o e+ab$, $\hat{y} = \gamma_o e+ac$ is a pair of optimal solutions.

c) $A = \begin{bmatrix} -(I+E)_{J_1J_1} & E_{J_1J_2} \\ E_{J_2J_1} & (I-E)_{J_2J_2} \end{bmatrix}$

This block form of A is corresponding to the subsets of N defined by:

$$J_1 = \{2j-1, j=1,\dots,m\}$$

$$J_2 = \{2j, j=1,\dots,m\}$$

where

$$n = 2m.$$

For the inverse of A we have

$$A^{-1} = \begin{bmatrix} (E-I)_{J_1J_1} & E_{J_1J_2} \\ E_{J_2J_1} & (E+I)_{J_2J_2} \end{bmatrix}$$

and so, the condition $c^TA^{-1} \ge o$ for solvability of (P) and (D) reduces to

$$\gamma_o \ge \max\{c_{j_1}, -c_{j_2}\}, \tag{3}$$

where

$$\gamma_o = c^T_{J_1} e_{J_1} + c^T_{J_2} e_{J_2}, \quad c_{j_1} = \max_{j\in J_1} \{c_j\}, \quad c_{j_2} = \min_{j\in J_2} \{c_j\}$$

In the case (3), if

$$\beta_o \geq \max\{b_{j_1}, -b_{j_2}\},$$

where

$$\beta_o = b^T_{J_1} e_{J_1} + b^T_{J_2} e_{J_2}, \quad b_{j_1} = \max_{j\in J_1} \{b_j\}, \quad b_{j_2} = \min_{j\in J_2} \{b_j\},$$

then the vectors

$$\begin{bmatrix} \hat{x}_{J_1} \\ \hat{x}_{J_2} \end{bmatrix} = \begin{bmatrix} \beta_o e_{J_1} - b_{J_1} \\ \beta_o e_{J_2} + b_{J_2} \end{bmatrix}, \qquad \begin{bmatrix} \hat{y}_{J_1} \\ \hat{y}_{J_2} \end{bmatrix} = \begin{bmatrix} \gamma_o e_{J_1} - c_{J_1} \\ \gamma_o e_{J_2} + c_{J_2} \end{bmatrix}$$

are optimal solutions, and

$$\hat{z} = \hat{w} = \beta_o \gamma_o - c^T_{J_1} b_{J_1} + c^T_{J_2} b_{J_2}$$

is the optimal value of z and w.

If $\beta_o < \max\{b_{j_1}, -b_{j_2}\}$, then it can be considered and solved the reduces form of (D_s):

$$\max\left\{ w - \hat{w} = -\hat{x}^T_{J_1} v_{J_1} - \hat{x}^T_{J_2} v_{J_2} \;\middle|\; \begin{array}{l} y_{J_1} + (E-I)_{J_1 J_1} v_{J_1} + E_{J_1 J_2} v_{J_2} = \hat{y}_{J_1},\ y_{J_1} \geq o,\ v_{J_1} \geq o \\ y_{J_2} + (E_{J_2 J_1} v_{J_1} + (E+I)_{J_2 J_2} v_{J_2} = \hat{y}_{J_2}, y_{J_2} \geq o,\ v_{J_2} \geq o \end{array} \right\}$$

d) $\underline{A = \frac{1}{s}[t_{ij}]}$, where $s = \sum_{j=o}^{n-2} a^j$, $t_{ij} = \begin{cases} (a^{n-i} - s)/a^{i-1}, & i=j \\ a^{n-i-j+1}, & i \neq j \end{cases}$

for a given real $a \neq 0, -1$. The inverse of A is

$$A^{-1} = E - \operatorname{diag}(a^0, a^1, \ldots, a^{n-1});$$

$\operatorname{diag}(a^0, a^1, \ldots, a^{n-1})$ denotes the diagonal matrix with diagonal elements a^{i-1}, $i=1,\ldots,n$. In the case when

$$0 < |a| < 1,$$

we have $A^{-1} > 0$. Let again $\beta_o = b^T e$, $\gamma_o = c^T e$. If $\gamma_o \geq \max_{j\in N}\{c_j a^{j-1}\}$, then, (P) and (D) are solvable. Moreover, if $\beta_o \geq \max_{j\in N}\{b_j a^{j-1}\}$ then $\hat{x} = [\hat{x}_j]_{n\times 1}$, where $\hat{x}_j = \beta_o + b_j a^{j-1}$, is an optimal solution to (P), $\hat{y} = [\hat{y}_j]_{n\times 1}$, where $\hat{y}_j = \gamma_o + c_j a^{j-1}$, is an optimal solution to (D), and $\beta_o\gamma_o + \sum_{j\in N} b_j c_j a^{j-1}$ is the optimal value of the objective functions.

3. Description od the convex polyhedral cones C_1, C_2, C_3

As we know, the cone $C=\{u \mid u^T A^{-1} \geq o\}$ for (P) and (D) represents the set of vectors c for which (P) and (D) are solvable. In the case of symmetric matrix A the cone C also contains the vectors b for which $A^{-1}b$ is an optimal solution to (P). Therefore, it is of interest to have an explicit form of C as a sum of its edges,

$$C = \{u = \sum_{j\in J} \mu_j q_j,\ \mu_j \geq 0,\ j\in J\}$$

for some subset J of integers.

In the case $C_1 = \{u \mid (E+aI)u \geq o\}$ $0 < |a| \leq 1$, we get J=N, and q_j, $j\in N$, defined as follows:

$$(q_j)_i = \begin{cases} -\frac{1}{a+n-1}, & i\neq j \\ 1, & i=j \end{cases},\quad j\in N$$

In the case $C_2 = \left\{ \begin{bmatrix} u_{J_1} \\ u_{J_2} \end{bmatrix} \middle| \begin{bmatrix} (E-I)_{J_1J_1} & E_{J_1J_2} \\ E_{J_2J_1} & (E+I)_{J_2J_2} \end{bmatrix} \begin{bmatrix} u_{J_1} \\ u_{J_2} \end{bmatrix} \geq \begin{bmatrix} o_{J_1} \\ o_{J_2} \end{bmatrix} \right\}$

where $J_1=\{2j-1, j=1,\dots,m\}$, $J_2=\{2j, j=1,\dots,m\}$, n=2m, for the vectors $q_j, j\in N=J_1\cup J_2$ we have

$$(q_{2j-1})_i = \begin{cases} -1, & i\in J_1-\{2j-1\} \\ -2, & i=2j-1 \\ 1, & i\in J_2 \end{cases},\qquad (q_{2j})_i = \begin{cases} 1, & i\in J_1 \\ -1, & i\in J_2-\{2j\} \\ 0, & i=2j \end{cases}$$

If $n=2m+1$, $J_1=\{2j-1, j=1,\dots,m+1\}$, $J_2=\{2j, j=1,\dots,m\}$, then there are $2m+1$ edges of C_2 corresponding to the vectors

$$q_{ij} = \begin{bmatrix} q_{J_1} \\ q_{J_2} \end{bmatrix}, \quad i \in J_1,\ j \in J_2$$

where

$$(q_{J_1})_s = \begin{cases} -1, & s=i \\ 0, & s \neq i \end{cases}, \qquad (q_{J_2})_s = \begin{cases} 1, & s=j \\ 0, & s \neq j \end{cases}$$

For

$$C_3 = \{u \mid (E-\mathrm{diag}(a^0, a^1, \dots, a^{n-1}))\, u \geq o\}$$

we can get the explicit form without additional assumptions in the case of a such that $0 < a < 1$. Then we have

$$(q_j)_i = \begin{cases} -1, & i=j \\ \dfrac{a^{n-i}}{\sum\limits_{k=o,k\neq n-j}^{n-2} a^k}, & i \neq j,\ j \in N. \end{cases}$$

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ЗА ДУАЛНИОТ ПАР ЛП-ПРОБЛЕМИ ВО КАНОНИЧНА ФОРМА СО НЕНЕГАТИВНА ИНВЕРЗНА МАТРИЦА

Д.Л. Карчицка

Р е з и м е

Дуалниот пар ЛП-проблеми

$$\min\{z=c^T x \mid Ax \geq b,\ x \geq o\},\quad \max\{w=b^T y \mid A^T y \leq c,\ y \geq o\}$$

каде што A е $n \times n$ матрица со инверзна $A^{-1} \geq 0$, е решлив ако и само ако $c^T A^{-1} \geq o$. Во спротивно, функцијата на целта z на примарната задача е неограничена на допустливата област. Разгледани се неколку посебни случаи на матрица A.