A VARIANT OF THE PRINCIPAL PIVOT METHOD

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0. Introduction

This study centres on the task of finding a solution to the Linear Complementarity Problem:

$$w = q + Mz$$

$$w,\ z \geqslant 0$$

$$w^{\mathrm{T}} z = 0,$$

for a matrix $M \in \pm$ (Leontief $\cap Z$), which has the property: there exists a positive vector a such that $a^{\mathrm{T}} M = 0$.

The approach is by Principal Pivot Method [1].

1. Notations

The Linear Complementarity Problem (LCP) is stated as follows: Find real column n-vectors $w = [w_i]$ and $z = [z_i]$ satisfying the conditions:

$$w = q + M\,z \tag{1}$$

$$w, z \geqslant 0 \tag{2}$$

$$w^{\mathrm{T}} z = 0, \tag{3}$$

where $M = [m_{ij}]$ is a given nxn matrix and $q = [q_i]$ is a given n-vector.

A pair of vectors $(w;\ z)$ which satisfies (1) and (2) is called a feasible solution; $(w;\ z)$ satisfying (1) and (3) is a complementary solution. Any solution to (1), (2), (3) is known as a complementary feasible solution. We notice that for any complementary feasible solution $w^{\mathrm{T}} z = 0 \Leftrightarrow w_i z_i = 0$ $(i = 1, \ldots, n)$.

Set $N = \{1, \ldots, n\}$. Let $\varnothing \neq I \subseteq N$ and $\underline{I} = N - I$. Let M_{II} denote the matrix obtained from a matrix M by deleting all rows and columns corresponding to indices in $\underline{I} \cdot M_{II}$ is a principal submatrix of M; it's determinant is a principal minor. We may partition M as:

$$M = \begin{bmatrix} M_{II}, & M_{I\underline{I}} \\ M_{\underline{I}I}, & M_{\underline{I}\underline{I}} \end{bmatrix} \tag{4}$$

When M_{II} is nonsingular, a principal transform with block pivot M_{II} is the matrix:

$$\begin{bmatrix} -M_{II}, & 0 \\ -M_{\underline{I}I}, & I \end{bmatrix}^{-1} \begin{bmatrix} -I, & M_{I\underline{I}} \\ 0, & M_{\underline{I}\underline{I}} \end{bmatrix} = \begin{bmatrix} M_{II}^{-1} & -M_{II}^{-1}M_{I\underline{I}} \\ M_{\underline{I}I}M_{II}^{-1}, & M_{\underline{I}\underline{I}}-M_{\underline{I}I}M_{II}^{-1}M_{I\underline{I}} \end{bmatrix}$$

The pivot is scalar if I consists of one element.

Associate with system (1) the scheme:

	1	z	
w	q	M .	(5)

The components w_i $(i \in N)$ of w are basic and the components z_i $(i \in N)$ of z are nonbasic. Associate with (5) the basic solution to (1) obtained by setting nonbasic variables to zero: $w = q$; $z = 0$. If M is partitioned as (4) we can write scheme (5) in partitioned form:

	1	z_I	$z_{\underline{I}}$	
w_I	q_I	M_{II}	$M_{I\underline{I}}$	(5′)
$w_{\underline{I}}$	$q_{\underline{I}}$	$M_{\underline{I}I}$	$M_{\underline{I}\underline{I}}$	

A principal transform by nonsingular M_{II} yields:

	1	w_I	$z_{\underline{I}}$	
z_I	$-M_{II}^{-1}q_I$	M_{II}^{-1}	$-M_{II}^{-1}M_{I\underline{I}}$	(6)
$w_{\underline{I}}$	$q_{\underline{I}}-M_{\underline{I}I}M_{II}^{-1}q_I$	$M_{\underline{I}I}M_{II}^{-1}$	$M_{\underline{I}\underline{I}}-M_{\underline{I}I}M_{II}^{-1}M_{I\underline{I}}$	

In (6) the components of z_I and $w_{\underline{I}}$ are basic. Setting $w_I = 0$; $z_{\underline{I}} = 0$ we obtain a new complementary solution to the associated linear complementarity problem:

$$(w_I, w_{\underline{I}};\ z_I, z_{\underline{I}}) = (0,\ q_{\underline{I}}-M_{\underline{I}I}M_{II}^{-1}q_I;\ -M_{II}^{-1}q_I,\ 0).$$

Following Lemke [5] define a class K of matrices relative to linear complementarity problem:

$M \in K \Leftrightarrow$ a complementary feasible solution exists for all feasible vectors q.

Denote Z the class of matrices M such that

$$m_{ij} \leqslant 0 \ \ (i, j \in N,\ i \neq j)$$

A matrix M is said to be Leontief if each row and column has exactly one positive element.

If $-M$ is the negative of a matrix M and S is a given class, we write $M \in -S$ to mean $-M \in S$.

2. *A Variant of the Principal Pivot Method*

Consider the LCP (1)—(3) for a given vector q and matrix $M = [m_{ij}]$ which has the properties:

$$M \in \pm(\text{Leontief} \cap Z), \tag{7}$$

$$m_{ij} \neq 0 \quad (i, j \in N), \tag{8}$$

there exists á positive vector $a = [a_i]$ such that $a^T M = 0$. (9)

In both cases of matrix M the considerations may by viewed simultaneously. The expressions in parenthesis will be associated with the case of sign „—".

Theorem 1. For a matrix M which satisfies (7), (8), (9), the LCP has no solutions if $a^T q < 0$.

Proof. It is a consequence of Farkas' Theorem [4] that (1), (2) has no feasible solution if there exists a vector $u = a$ such that $uM \leqslant 0$, $uq < 0$, $u \geqslant 0$.

Corollary. $q \nleqslant 0$ is a nacessary condition for a feasibility of LCP.

For a feasibility of LCP let

$$a^T q \geqslant 0. \tag{10}$$

That implies (by $a > 0$) $q \nleqslant 0$. If $q \geqslant 0$, then $(w; z) = (q; 0)$ is a complementary feasible solution. Therefore, suppose $q \ngeqslant 0$ and let

$$q_1 < (>) \ 0,$$

without loss of generality.

The scalar pivot on m_{11} in scheme (5) associated to LCP yields the scheme (6) with $I = \{1\}$. It's scalar form is:

	1	w_1	z_2	....	z_n
z_1	$q_1^{(1)}$	$m_{11}^{(1)}$	$m_{12}^{(1)}$	....	$m_{1n}^{(1)}$
w_2	$q_2^{(1)}$	$m_{21}^{(1)}$	$m_{22}^{(1)}$	....	$m_{2n}^{(1)}$
...	...	...	...	...	...
w_n	$q_n^{(1)}$	$m_{n1}^{(1)}$	$m_{n2}^{(1)}$	....	$m_{nn}^{(1)}$

(6')

where:

$$q_1^{(1)} = -q_1/m_{11}, \; m_{11}^{(1)} = 1/m_{11}, \; m_{1i}^{(1)} = -m_{1i}/m_{11} \quad (i \in \underline{I}) \tag{11}$$

$$m_{ij}^{(1)} = m_{ij} - m_{i1} m_{1j}/m_{11} \quad (i, j \in \underline{I}) \tag{12}$$

$$q_i^{(1)} = q_i - m_{i1}/m_{11}, \qquad m_{i1}^{(1)} = m_{i1}/m_{11} \quad (i \in \underline{I}).$$

As $q_I^{(1)}>0$, if $q_I^{(1)}\geqslant 0$, then $(w;\ z)=(0,q_2^{(1)},\ldots,q_n^{(1)};\ q_1^{(1)},0\ \ldots,0)$ is a complementary feasible solution.

Suppose $q_I \ngeqslant 0$. (11) indicates that the pair $(w_1,\ z_1)$ is feasible and complementary for every $z_I\geqslant 0$ if w_1 remains nonbasic. Setting $w_1=0$ we delete terms associated with w_1 in (6′). Then, omitting the first row in (6′), associated with the basic z_1, we get the subscheme:

$$\begin{array}{c|cc} & 1 & z_I \\ \hline w_I & q_I^{(1)} & M_{II}^{(1)} \end{array} \tag{13}$$

Obviously, the nondiagonal elements of $M_{II}^{(1)}$ are negative (positive).

Lemma. 1) For $i\in I$, $m_{ii}^{(1)}>(<)\,0$; 2) $a_I^T M_{II}^{(1)}=0$; 3) $a_I^T\, q_I^{(1)}\geqslant 0$

Proof. (9) implies that for $i=2,\ldots,n$ there exists a positive solution $x_1=a_1,\ x_i\ =\ a_i$ to the subsystem:

$$m_{11}x_1+m_{i1}x_i\ =\ -\sum_{j\neq 1,i} m_{j1}\,a_j$$

$$m_{1i}x_1+m_{ii}\,x_i\ =\ -\sum_{j\neq 1,i} m_{ji}\,a_j.$$

That is possible if, and only if, the determinant $m_{11}m_{ii}-m_{1i}m_{i1}$ is positive or, since $m_{11}>(<)\,0$, $m_{ii}-m_{i1}\,m_{1i}/m_{11}>(<)\,0$. It is easy to see that

$$a_I^T M_{II}^{(1)}=a^T\,[m_2,\ldots,m_n\,],$$

where m_j denotes the j-th column of M $(j=2,\ldots n)$. So, by (9). $a_I^T M_{II}^{(1)}=0$. The assumtion $a_I^T q_I^{(1)}<0$ implies $a^T q<0$, which is a contradiction. Therefore it must be $a_I^T q_I^{(1)}\geqslant 0$.

The above Lemma implies that subscheme (13) has all the properties of the initial scheme (5). We may therefore repeat the procedure for (13), and so on. On each step k the size of the subscheme we have to deal with is decreasing and $q_I^{(k)}$ has at least one positive component if $q_I^{(k)}\ngeqslant 0$. So, at least for $k=n-1$ we get $q_I^{(k)}\geqslant 0$. On the last step k setting $w_I=q_I^{(k)}$, $z_I=0$, and successively going back we may compute the new (obviously positive) value of each basic variable z from the row, which was omitted on the associated step. We thus have a complementary feasible solution to the given LCP.

We can summarize the above results as follows·

Theorem 2. There is a complementary feasible solution to LCP for a matrix M, which has the properties (7), (8), (9), if $a^T q \geqslant 0$.

Corollary. A matrix M, such that conditions (7), (8), (9) are satisfied, is of the class K.

3. The algorithm of computations.

Step 0. Set $N = \{1, \ldots, n\}$, $k = 0$, $I_k = \varnothing$, $\underline{I}_k = N - I_k$

$q^{(k)}_{I_k} = q$, $M^{(k)}_{\underline{I}_k I_k} = M$.

Test $q^{(k)}_{\underline{I}_k} \geqslant 0$:

1° If yes, $(w; z) = (q^{(k)}_{\underline{I}_k};\ 0)$ is a compl. feas. solution

2° If no, go to Step 1.

Step 1. Set $k = k+1$. Determine $i_k \in \underline{I}_{k-1}$ such that $q^{(k-1)}_{i_k} < (>)\, 0$.

Set $I_k = I_{k-1} \cup \{i_k\}$, $\underline{I}_k = N - I_k$. Compute:

$$m^{(k)}_{i_k i_k} = 1/m^{(k-1)}_{i_k i_k}$$

$$m^{(k)}_{i_k j} = -m^{(k-1)}_{i_k j}\, m^{(k)}_{i_k i_k} \quad (j \in \underline{I}_k)$$

$$M^{(k)}_{\underline{I}_k I_k} : m^{(k)}_{ij} = m^{(k-1)}_{ij} - m^{(k-1)}_{i_k j} m^{(k-1)}_{i i_k} m^{(k)}_{i_k i_k} \qquad (i, j \in \underline{I}_k)$$

$$q^{(k)}_{i_k} = -q^{(k-1)}_{i_k} m^{(k)}_{i_k i_k},$$

$$q^{(k)}_{\underline{I}_k} : q^{(k)}_i = q^{(k-1)}_i + m^{(k-1)}_{i i_k} q^{(k)}_{i_k}. \qquad (i \in \underline{I}_k)$$

Test $q^{(k)}_{\underline{I}_k} \geqslant 0$:

1° If no, go to Step 1.

2° If yes, go to Step 2.

Step 2. Set $w_{\underline{I}_k} = q^{(k)}_{\underline{I}_k}$; $z_{\underline{I}_k} = 0$; $w_{I_k} = 0$

and successively compute the components of z_{I_k}

$$z_{i_{k-r}} = q^{(k-r)}_{i_{k-r}} + \sum_{j \in I_k - I_{k-r}} m^{(k-r)}_{i_{k-r} j} z_j \quad (r = 0, 1, \ldots, k-1)$$

The pair $(w_{I_k}, w_{\underline{I}_k};\ z_{I_k}, z_{\underline{I}_k})$

is a complementary feasible solution. Stop!

4. *A note*

The matrix $M \in (\text{Leontief} \cap Z)$ such that (8) and (9) are satisfied, has the following properties:

1) The rank of M is $n-1$;

2) M is adequate.

Indeed, suppose that the vector a is normalized, say $a_i = 1$. Then $a^T M = 0$ may be written as:

$$\sum_{k \neq i} m_{kj} a_k = -m_{ij} \quad (j \neq i) \tag{14}$$

$$-\sum_{k \neq i} m_{ki} a_k = m_{ii}.$$

(14) implies that there exist $n-1$ positive values $x_k = a_k$ $(k \neq i)$ such that

$$\sum_{k \neq i} m_{kj} x_k > 0 \quad (j \neq i).$$

This statement is equivalent to the following:

The matrix
$$\begin{bmatrix} m_{11} \dots m_{1,i-1} & m_{1,i+1} \dots & m_{1n} \\ \dots\dots\dots\dots\dots & & \\ m_{i-1,1} \dots m_{i-1,i-1} & m_{i-1,i+1} \dots & m_{i-1,n} \\ m_{i+1,1} \dots m_{i+1,i-1} & m_{i+1,i+1} \dots & m_{i+1,n} \\ \dots\dots\dots\dots\dots & & \\ m_{n1} \dots m_{n,i-1} & m_{n,i+1} \dots & m_{nn} \end{bmatrix}$$

is nonsingular; its inverse is nonnegative; all its principal minors are positive. The arbitrary choise of the index i implies that for M all the principal minors of order $\leqslant n-1$ are positive. Because M is singular, its rows, as well as the columns, are linearly dependent. Then, by definition, M is adequate. In this case the solution for w in the linear complementarity problem (1) — (3) is unique ([3]). If $a^T q > 0$, then the solution for z is minimal. The proof is similar as in [2].