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ON THE LINEAR COMPLEMENTARITY PROBLEM

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Several important for the applications mathematical problems can be formulated as the Linear Complementarity problem:

(LCP) Find n-vectors w and z which satisfy the conditions $w=q+Mz$, $w\geq 0$, $z\geq 0$, $w^Tz=0$, where the vector q and the matrix M are given.

The (LCP) has received a remarcable attention. Very often a paper on (LCP) can be found in the publications: Mathematical Programming, Linear Algebra and Its Applications, Journal of Optimization Theory and Applications. But, there is no answer on the questions of existence and uniqueness of solution, and computing solutions for any matrix M and vector q.

For certain classes of matrices, or for special q, there have been developed constructive procedures for solving the (LCP). The most of them evolved from the Complementary Pivot Method of Lemke-Howson [2] and the Principal Pivoting Method of Cottle-Dantzig [1].

Relative to (LCP) Lemke [3] defines the classes of matrices:

$\mathcal{Q}$: $M\in\mathcal{Q} \Leftrightarrow$ a solution exists for all q;

$\mathcal{K}'$: $M \in \mathcal{K}' \iff$ a solution exists for all q for which a nonnegative solution of $w = q + M \cdot z$ exists.

Obviosly, $\mathcal{Q} \subset \mathcal{K}'$. In particular the $\mathcal{P}$-matrices (:positive principal minors) and more general the $\mathcal{E}$-matrices ($\forall o \neq x \geq o \ \exists i : x_i > 0$ and $(Mx)_i > o$) are $\mathcal{Q}$-matrices; the $\mathcal{Z}$-matrices ($(M)_{ij} \leq 0$ for all i and j such that $i \neq j$) and the copositive-plus matrices ($x^T Mx \geq 0$ for all $x \geq o$, $(M+M^T)x = o$ if $x^T Mx = 0$ and $x \geq o$) are $\mathcal{K}'$-matrices.

We consider the (LCP) for M of type

$$\pm((a + 1)P - E) \tag{1}$$

where a is an arbitrary real, P is a given permutation matrix of order n and E is the $n \times n$-matrix all of whose elements are unity.

Though (1) has a very special structure, it need not be a $\mathcal{K}'$-matrix for any a.

For $a + 2 - n > 0$, $M \in \mathcal{K}'$ [4] and in the case when a solution exists, it can be obtained performing one block pivot on a principal sub-matrix M_{II} (card I = $= r \leq n$) of type

$$\pm((a + 1)P_{II} - E_{II}) \tag{2}$$

or

$$\pm\begin{bmatrix} (a + 1)P_{I'J} - E_{I'J} & -e_{I'} \\ -e_J^T & -1 \end{bmatrix} \tag{3}$$

where $I' \cup J = I$, $\text{card}(I' \cap J) = r - 2$ and $e_{I'}$ and e_J are r-1-vectors all of whose elements are unity. As

$$\frac{1}{a+1}\left[P_{II}^T + \frac{1}{a+1-r} E_{II}\right]$$ is the inverse of (2)

and

$$\frac{1}{a+1}\begin{bmatrix} P_{I'J}^T & -e_J \\ -e_{I'}^T & -(a+1-r) \end{bmatrix}$$ is the inverse of (3)

the solution $(\bar{w};\bar{z}) = (o,\bar{w}_{\underline{I}};\bar{z}_I,o)$ $(I \cup \underline{I} = \{1,\dots,n\}$, $I \cap \underline{I} = \emptyset)$ is computed applying the formulas

$$\bar{z}_I = -M_{II}^{-1} q_I, \qquad \bar{w}_I = o$$
$$\bar{w}_{\underline{I}} = q_{\underline{I}} + M_{\underline{I}I}\bar{z}_I, \quad \bar{z}_{\underline{I}} = o. \tag{4}$$

For $M = (a+1)P-E$ $(M=E-(a+1)P)$ let

$$q_r = \max_{1\le j\le n}\{q_j\} \qquad (q_r = \min_{1\le j\le n}\{q_j\})$$

$\ell = \pi^{-1}(r)$, $s = \sum_{j=1}^{n} q_j + (a+1-n)q_r$,

where $\pi(j) = i_j \iff (M)_{i_j j} = \pm a$.

The (LCP) is infeasible if $s < 0$ and $n-2<a\le n-1$ $(s < 0$ and $a \ge n-1)$. The pair of vectors $(\bar{w};\bar{z}) = (0;-M^{-1}q)$ is a solution if $s < 0$ and $a>n-1$, or $s \ge 0$ and $n-2<a<n-1$ (if $s < 0$ and $n-2<a<n-1$, or $s \ge 0$ and $a>n-1$). Otherwise, the set of indices I corresponding to the solution (4) can be obtained applying the following algorithm:

Step 0. Initialize $\nu = 0$, $I^\nu = \{1,\dots,n\}$, $r_\nu = r$, $\ell_\nu = \ell$, $s_\nu = s$ and go to step 1.

Step 1. Set $I^{\nu+1} = I^\nu - \{\ell_\nu\}$ and test $r_\nu \ne \ell_\nu$ (test $r = \ell_\nu$).

1.1. If yes, then $I = I^{\nu+1}$ and M_{II} is of type (3) (or type (2)); stop!

1.2. If no, then go to step 2.

Step 2. Set $\nu = \nu+1$, find $q_{r_\nu} = \max_{j\in I^\nu}\{q_j\}$, $\ell_\nu = \pi^{-1}(r_\nu)$ $(\ell_\nu = \pi^{-1}(\ell_{\nu-1}))$,

$$s_\nu = \sum_{j\in I^\nu} q_j + (a+1-n+\nu)q_{r_\nu} \quad (s_\nu = \sum_{j\in I^\nu} q_j + (a+1-n+\nu)q_r)$$

and test $s_\nu \le 0$:

2.1. If yes, then $I = I^{\nu}$ and M_{II} is of type (2) (of type (3)); stop!

2.2. If no, go to step 1.

A modification of the algorithm can be used for solving:

a) The parametric (LCP) [5], [6]

$w = (q+\alpha p) + Mz, \quad v \geq o, z \geq o, w^T z = 0,$

α-parameter;

b) The dual pair of LP-problems [7]

$\min\{pz \mid q + Mz \geq o, z \geq o\}, \quad \max\{-vq \mid p-vM \geq o, v \geq o\};$

c) The quadratic program [8]

$\min\{x^T M_1^T M_1 x \mid M_2 x \geq c\}.$

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