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REITERATIVE (m_n)-DISTRIBUTIONAL CHAOS FOR BINARY RELATIONS OVER METRIC SPACES

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Abstract. This paper intends to be a heuristical study. Let (m_n) be an increasing sequence in $[1,\infty)$ satisfying $\liminf_{n\to\infty}\frac{m_n}{n} > 0$, $\lambda \in (0,1]$, $s \in \{0, 1-, 1+, 2-\}$ and $i \in \mathbb{N} \cap [1,20]$. We introduce and analyze the concepts of reiterative $(m_n, \tilde{X})$-distributional chaos of type s, reiterative $(\lambda, \tilde{X})$-distributional chaos of type s, reiterative $[\tilde{X}, m_n, i]$-distributional chaos and reiterative $[\tilde{X}, \lambda, i]$-distributional chaos for general sequences of binary relations over metric spaces.

1. Introduction and Preliminaries

Suppose, for the time being, that X is a separable Fréchet space. A linear operator T on X is said to be hypercyclic iff there exists an element $x \in D_\infty(T) \equiv \bigcap_{n\in\mathbb{N}} D(T^n)$ whose orbit $\{T^n x : n \in \mathbb{N}_0\}$ is dense in X; T is said to be topologically transitive, resp. topologically mixing, iff for every pair of open non-empty subsets U, V of X, there exists $n_0 \in \mathbb{N}$ such that $T^{n_0}(U) \cap V \neq \emptyset$, resp. there exists $n_0 \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n \geq n_0$, $T^n(U) \cap V \neq \emptyset$. A linear operator T on X is said to be chaotic iff it is topologically transitive and the set of periodic points of T, defined by $\{x \in D_\infty(T) : (\exists n \in \mathbb{N})\, T^n x = x\}$, is dense in X. The basic facts about topological dynamics of linear continuous operators in Banach and Fréchet spaces can be obtained by consulting the monographs [1] by F. Bayart, E. Matheron and [11] by K.-G. Grosse-Erdmann, A. Peris.

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In a joint research studies with J. A. Conejero, C.-C. Chen and M. Murillo-Arcila [8]-[9], the first named author has recently investigated a great deal of topologically dynamical properties for multivalued linear operators between Fréchet spaces and general binary relations between topological spaces. Concerning similar researches within the field of topological dynamics, one may refer e.g. to the recent papers [14]-[16]. Mention should be also made of the papers [20] by R. A. Martínez-Avendano, [21] by P. Namayanja, and [12]-[13], written in collaboration of D. Goncalves with D. Royer and B. B. Uggioni.

On the other hand, the notions of (m_n)-distributional chaos and λ-distributional chaos have been recently analyzed in [19], for linear continuous operators and their sequences in Fréchet spaces, while the notion of distributional chaos of type s for orbits of a linear continuous operator acting on a Banach space, where $s \in \{1, 2, 2\frac{1}{2}, 3\}$, has been analyzed by N. C. Bernardes Jr. et al in [4]. Concerning multivalued non-linear setting, it should be noted that various notions of reiterative distributional chaos and Li-Yorke chaos have been recently examined in [16], for general sequences of binary relations in metric spaces.

The main aim of this paper is to propose a great deal of new unification concepts extending the notions considered in the above-mentioned papers. Let $\tilde{X}$ be a non-empty subset of the pivot space X. We analyze the notions of reiterative $(m_n, \tilde{X})$-distributional chaos of type s, reiterative $(\lambda, \tilde{X})$-distributional chaos of type s, reiterative $[\tilde{X}, m_n, i]$-distributional chaos and reiterative $[\tilde{X}, \lambda, i]$-distributional chaos for general sequences of binary relations in metric spaces; here, (m_n) is an increasing sequence in $[1, \infty)$ satisfying $\liminf_{n\to\infty} \frac{m_n}{n} > 0$, $\lambda \in (0, 1]$, $s \in \{0, 1-, 1+, 2-\}$ and $i \in \mathbb{N} \cap [1, 20]$.

The organization of material is briefly described as follows. In Subsection 1.1 and Subsection 1.2, we remind ourselves of the basic definitions and results about binary relations, multivalued linear operators and various types of lower and upper (Banach) m_n-densities. The main aim of Section 2 is to introduce and analyze reiterative $(m_n, \tilde{X})$-distributional chaos of type s and reiterative $(\lambda, \tilde{X})$-distributional chaos of type s for binary relations in metric spaces; in Subsection 1.1, we analyze the corresponding notion for sequences of multivalued linear operators in Fréchet spaces. Section 3 is devoted to the study of reiterative $[\tilde{X}, m_n, i]$-distributional chaos, where $i \in \mathbb{N} \cap [1, 20]$. It is very important to stress that any notion of distributional chaos considered in the papers [2]-[4], [6], [7], [16] and [19] is a particular case of reiterative $[\tilde{X}, m_n, i]$-distributional chaos for some $i \in \mathbb{N} \cap [1, 20]$ or $(m_n, \tilde{X})$-distributional chaos of type s for some $s \in \{0, 1-, 1+, 2-\}$ (because of a great number of the notions of (reiterative) distribitional chaos recently examined, we have been forced to slightly exchanged the terminology). We state several simple statements, mostly without giving corresponding proofs, and propose an open problem. Conclusions and final remarks section is included at the end of paper.

Here and herafter, it will be always assumed that (X, d) and (Y, d_Y) are metric spaces. For any set $D = \{d_n : n \in \mathbb{N}\}$, where $(d_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of positive integers, we define its complement $D^c := \mathbb{N} \setminus D$ and difference set $\{e_n := d_{n+1} - d_n | \, n \in \mathbb{N}\}$. Let us recall that an infinite subset A of $\mathbb{N}$ is said to be syndetic, or relatively dense, iff its difference set is bounded. For any $s \in \mathbb{R}$, we define $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \geq l\}$.

1.1. **Binary relations and multivalued linear operators.** Let X, Y, Z and T be given non-empty sets. A binary relation between X into Y is any subset $\rho \subseteq X \times Y$. If $\rho \subseteq X \times Y$ and $\sigma \subseteq Z \times T$ with $Y \cap Z \neq \emptyset$, then we define $\rho^{-1} \subseteq Y \times X$ and $\sigma \circ \rho \subseteq X \times T$ by $\rho^{-1} := \{(y, x) \in Y \times X : (x, y) \in \rho\}$ and
$\sigma \circ \rho := \big\{(x, t) \in X \times T : \exists y \in Y \cap Z \text{ such that } (x, y) \in \rho \text{ and } (y, t) \in \sigma\big\}$,
respectively. Domain and range of ρ are introduced by $D(\rho) := \{x \in X : \exists y \in Y \text{ such that } (x, y) \in \rho\}$ and $R(\rho) := \{y \in Y : \exists x \in X \text{ such that } (x, y) \in \rho\}$, respectively; $\rho(x) := \{y \in Y : (x, y) \in \rho\}$ $(x \in X)$, $x \; \rho \; y \Leftrightarrow (x, y) \in \rho$. If ρ is a binary relation on X and $n \in \mathbb{N}$, then we define ρ^n inductively; $\rho^{-n} := (\rho^n)^{-1}$ and $\rho^0 := \{(x, x) : x \in X\}$. Put $D_\infty(\rho) := \bigcap_{n\in\mathbb{N}} D(\rho^n)$, $\rho(X') := \{y : y \in \rho(x) \text{ for some } x \in X'\}$ $(X' \subseteq X)$.

Let X and Y be two Fréchet spaces over the same field of scalars $\mathbb{K}$. For any mapping $\mathcal{A} : X \to P(Y)$ we define $\check{\mathcal{A}} := \{(x, y) : x \in D(\mathcal{A}), \; y \in \mathcal{A}x\}$. Then $\mathcal{A}$ is a multivalued linear operator (MLO) iff the associated binary relation $\check{\mathcal{A}}$ is a linear relation in $X \times Y$, i.e., iff $\check{\mathcal{A}}$ is a linear subspace of $X \times Y$. In our work, we will identify $\mathcal{A}$ and its associated linear relation $\check{\mathcal{A}}$, so that the notion of $D(\mathcal{A})$, which is a linear subspace of X, as well as the sets $R(\mathcal{A})$ and $D_\infty(\mathcal{A})$ are clear. For more details about multivalued linear operators, we refer the reader to the monograph [10] by R. Cross.

1.2. **Lower and upper densities.** In this subsection, we recall the basic things about lower and upper densities that will be necessary for our further work. For more details about the subject, the reader may consult [17] and references cited therein.

Let $A \subseteq \mathbb{N}$ be non-empty. The lower density of A, denoted by $\underline{d}(A)$, is defined by

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n},$$

and the upper density of A, denoted by $\overline{d}(A)$, is defined by

$$\overline{d}(A) := \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

Further on, the lower Banach density of A, denoted by $\underline{Bd}(A)$, is defined by

$$\underline{Bd}(A) := \lim_{s\to+\infty} \liminf_{n\to\infty} \frac{|A \cap [n+1,n+s]|}{s}$$

and the (upper) Banach density of A, denoted by $\overline{Bd}(A)$, is defined by

$$\overline{Bd}(A) := \lim_{s\to+\infty} \limsup_{n\to\infty} \frac{|A \cap [n+1,n+s]|}{s}.$$

It is well known that the limits appearing in definitions of $\underline{Bd}(A)$ and $\overline{Bd}(A)$ exist as s tends to $+\infty$, as well as that

$$0 \leq \underline{Bd}(A) \leq \underline{d}(A) \leq \overline{d}(A) \leq \overline{Bd}(A) \leq 1, \tag{1.1}$$

$$\underline{d}(A) + \overline{d}(A^c) = 1 \text{ and } \underline{Bd}(A) + \overline{Bd}(A^c) = 1. \tag{1.2}$$

We will use the following notions of lower and upper densities for a subset $A \subseteq \mathbb{N}$:

Definition 1.1. Let $q \in [1,\infty)$, and let (m_n) be an increasing sequence in $[1,\infty)$. Then:

(i) The lower q-density of A, denoted by $\underline{d}_q(A)$, is defined through:

$$\underline{d}_q(A) := \liminf_{n\to\infty} \frac{|A \cap [1,n^q]|}{n}.$$

(ii) The upper q-density of A, denoted by $\overline{d}_q(A)$, is defined through:

$$\overline{d}_q(A) := \limsup_{n\to\infty} \frac{|A \cap [1,n^q]|}{n}.$$

(iii) The lower (m_n)-density of A, denoted by $\underline{d}_{m_n}(A)$, is defined through:

$$\underline{d}_{m_n}(A) := \liminf_{n\to\infty} \frac{|A \cap [1, m_n]|}{n}.$$

(iv) The upper (m_n)-density of A, denoted by $\overline{d}_{m_n}(A)$, is defined through:

$$\overline{d}_{m_n}(A) := \limsup_{n\to\infty} \frac{|A \cap [1, m_n]|}{n}.$$

Then we know the following:

Lemma 1. *Suppose that $q \geq 1$, $A = \{n_1,\ n_2, \cdots,\ n_k, \cdots\}$, where (n_k) is a strictly increasing sequence of positive integers.*
Then $\underline{d}_q(A) = \liminf_{k\to\infty} \frac{k}{n_k^{1/q}}$ and $\underline{d}_q(A) > 0$ iff there exists a finite constant $L > 0$ such that $n_k \leq Lk^q$, $k \in \mathbb{N}$.

We also need the following notions of lower and upper densities:

Definition 1.2. Suppose $q \in [1, \infty)$, (m_n) is an increasing sequence in $[1, \infty)$ and $A \subseteq \mathbb{N}$. Then we define:

(i) The lower $l;q$-Banach density of A, denoted shortly by $\underline{Bd}_{l;q}(A)$, as follows

$$\underline{Bd}_{l;q}(A) := \liminf_{s\to+\infty} \liminf_{n\to\infty} \frac{|A \cap [n+1, n+s^q]|}{s}.$$

(ii) The lower $u;q$-Banach density of A, denoted shortly by $\underline{Bd}_{u;q}(A)$, as follows

$$\underline{Bd}_{u;q}(A) := \limsup_{s\to+\infty} \liminf_{n\to\infty} \frac{|A \cap [n+1, n+s^q]|}{s}.$$

(iii) The $l;q$-Banach density of A, denoted shortly by $\overline{Bd}_{l;q}(A)$, as follows

$$\overline{Bd}_{l;q}(A) := \liminf_{s\to+\infty} \limsup_{n\to\infty} \frac{|A \cap [n+1, n+s^q]|}{s}.$$

(iv) The $u;q$-Banach density of A, denoted shortly by $\overline{Bd}_{u;q}(A)$, as follows

$$\overline{Bd}_{u;q}(A) := \limsup_{s\to+\infty} \limsup_{n\to\infty} \frac{|A \cap [n+1, n+s^q]|}{s}.$$

(v) The lower $l;(m_n)$-Banach density of A, denoted shortly by $\underline{Bd}_{l;m_n}(A)$, as follows

$$\underline{Bd}_{l;m_n}(A) := \liminf_{s\to+\infty} \liminf_{n\to\infty} \frac{|A \cap [n+1, n+m_s]|}{s}.$$

(vi) The lower $u;(m_n)$-Banach density of A, denoted shortly by $\underline{Bd}_{u;m_n}(A)$, as follows

$$\underline{Bd}_{u;m_n}(A) := \limsup_{s\to+\infty} \liminf_{n\to\infty} \frac{|A\cap[n+1,n+m_s]|}{s}.$$

(vii) The (upper) $l;(m_n)$-Banach density of A, denoted shortly by $\overline{Bd}_{l;m_n}(A)$, as follows

$$\overline{Bd}_{l;m_n}(A) := \liminf_{s\to+\infty} \limsup_{n\to\infty} \frac{|A\cap[n+1,n+m_s]|}{s}.$$

(viii) The (upper) $u;(m_n)$-Banach density of A, denoted shortly by $\overline{Bd}_{u;m_n}(A)$, as follows

$$\overline{Bd}_{u;m_n}(A) := \limsup_{s\to+\infty} \limsup_{n\to\infty} \frac{|A\cap[n+1,n+m_s]|}{s}.$$

The condition $\liminf_{n\to\infty} \frac{m_n}{n} > 0$ is equivalent to saying that there exists a finite constant $L \geq 1$ such that $n \leq Lm_n$, $n \in \mathbb{N}$. Keeping in mind this observation, it can be simply seen that the following auxiliary result holds true (cf. also [17]):

Lemma 2. *Let* $A \subseteq \mathbb{N}$.

(i) *Suppose that* $\liminf_{n\to\infty} \frac{m_n}{n} > 0$. *Then* $\underline{Bd}_{l;m_n}(A) = 0$ *iff* $\underline{Bd}_{u;m_n}(A) = 0$ *iff* A *is finite or* A *is infinite non-syndetic.*

(ii) *Suppose that* $\liminf_{n\to\infty} \frac{m_n}{n} > 0$. *Then* $\underline{d}_{m_n}(A) > 0$ *provided that* A *is syndetic.*

2. Reiterative (m_n)-distributional chaos of type s and reiterative λ-distributional chaos of type s for binary relations

In the remaining part of paper, we always assume that (m_n) is an increasing sequence in $[1,\infty)$ satisfying $\liminf_{n\to\infty} \frac{m_n}{n} > 0$, i.e. there exists a finite constant $L \geq 1$ such that $n \leq Lm_n$, $n \in \mathbb{N}$.

We will use the following auxiliary lemma:

Lemma 3. *Suppose that* $A \subseteq \mathbb{N}$ *and* $\underline{d}_{m_n}(A) = 0$. *Then* $\overline{d}_{m_n}(A^c) \geq \frac{1}{L}$.

Proof. It is clear that

$$\frac{|A^c\cap[1,m_n]|}{n} \geq \frac{m_n-1}{n} - \frac{|A\cap[1,m_n]|}{n} \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.1}$$

Let (n_k) be a strictly increasing sequence of positive integers such that

$$\lim_{k\to+\infty}\frac{|A\cap[1,m_{n_k}]|}{n_k}=0. \tag{2.2}$$

Due to (2.1)-(2.2), we have

$$\overline{d}_{m_n}(A^c)\geq\limsup_{k\to+\infty}\frac{m_{n_k}-1}{n_k}-\lim_{k\to+\infty}\frac{|A\cap[1,m_{n_k}]|}{n_k}\geq\frac{1}{L},$$

as claimed. □

Suppose that $\sigma>0$, $\epsilon>0$ and $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ are two given sequences in X. Consider the following conditions:

$$\begin{aligned}&\underline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}: d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)=0,\\&\underline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}: d_Y\bigl(x_k,y_k\bigr)\geq\epsilon\bigr\}\Bigr)=0,\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\underline{Bd}_{l;m_n}\Bigl(\bigl\{k\in\mathbb{N}: d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)=0,\\&\underline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}: d_Y\bigl(x_k,y_k\bigr)\geq\epsilon\bigr\}\Bigr)=0,\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\underline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}: d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)=0,\\&\underline{Bd}_{l;m_n}\Bigl(\bigl\{k\in\mathbb{N}: d_Y\bigl(x_k,y_k\bigr)\geq\epsilon\bigr\}\Bigr)=0,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\underline{Bd}_{l;m_n}\Bigl(\bigl\{k\in\mathbb{N}: d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)=0,\\&\underline{Bd}_{l;m_n}\Bigl(\bigl\{k\in\mathbb{N}: d_Y\bigl(x_k,y_k\bigr)\geq\epsilon\bigr\}\Bigr)=0.\end{aligned} \tag{2.6}$$

In the following definition, we introduce the notion of an $(m_n,\tilde{X})$-reiteratively distributionally chaotic sequence $(\rho_k)_{k\in\mathbb{N}}$ of binary relations of type $s\in\{0,1-,1+,2-\}$ following the ideas of J. C. Xiong, H. M. Fu and H. Y. Wang [22] for continuous mappings between compact metric spaces ($m_n\equiv n^{1/\lambda}$) and A. Bonilla, M. Kostić [6] for linear continuous mappings between Banach spaces.

Definition 2.1. Suppose that, for every $k\in\mathbb{N}$, $\rho_k: D(\rho_k)\subseteq X\to Y$ is a binary relation and $\tilde{X}$ is a non-empty subset of X. If there exist an uncountable set $S\subseteq\bigcap_{k=1}^{\infty}D(\rho_k)\cap\tilde{X}$ and $\sigma>0$ such that for each $\epsilon>0$ and for each pair $x,\ y\in S$ of distinct points we have that for each $k\in\mathbb{N}$ there exist elements $x_k\in\rho_k x$ and $y_k\in\rho_k y$ such that (2.3) [(2.4),(2.5),(2.6)] holds, then we say that the sequence $(\rho_k)_{k\in\mathbb{N}}$ is $(m_n, \tilde{X})$-reiteratively distributionally chaotic of type $1-$ $[1+,\ 2-,\ 0]$ ($\tilde{X}$-reiteratively distributionally chaotic of type $1-$ $[1+,\ 2-,\ 0]$, if $m_n \equiv n$).

Let $s \in \{0, 1-, 1+, 2-\}$. The sequence $(\rho_k)_{k\in\mathbb{N}}$ is said to be densely $(m_n, \tilde{X})$-reiteratively distributionally chaotic of type s (densely $\tilde{X}$-reite–ratively distributionally chaotic of type s, if $m_n \equiv n$) iff S can be chosen to be dense in $\tilde{X}$. A binary relation $\rho : D(\rho) \subseteq X \to X$ is said to be (densely) $(m_n, \tilde{X})$-reiteratively distributionally chaotic of type s ((densely) $\tilde{X}$)-reiteratively distributionally chaotic of type s, if $m_n \equiv n$) iff the sequence $(\rho_k \equiv \rho^k)_{k\in\mathbb{N}}$ is. The set S is said to be $(m_n, \sigma_{\tilde{X}})$-reiteratively scrambled set of type s ((m_n, σ)-reiteratively scrambled set of type s in the case that $\tilde{X} = X$) of the sequence $(\rho_k)_{k\in\mathbb{N}}$ (the binary relation ρ); in the case that $\tilde{X} = X$, then we also say that the sequence $(\rho_k)_{k\in\mathbb{N}}$ (the binary relation ρ) is m_n-reiteratively distributionally chaotic of type s (reiteratively distributionally chaotic of type k, if $m_n \equiv n$). If $m_n \equiv n$, then it is said that the set S is $\sigma_{\tilde{X}}$-reiteratively scrambled set of type s (σ-reiteratively scrambled set of type s in the case that $\tilde{X} = X$).

Let $\lambda \in (0, 1)$ and $m_n \equiv n^{1/\lambda}$. Then the (dense) $(m_n, \tilde{X})$-reiterative distributional chaos of type s is also called (dense) $(\lambda, \tilde{X})$-reiterative distributional chaos of type s, the (dense) m_n-reiterative distributional chaos of type s is also called (dense) λ-reiterative distributional chaos of type s and the $(m_n, \sigma_{\tilde{X}})$-reiteratively scrambled set S of type s is also called $(\lambda, \sigma_{\tilde{X}})$-reiteratively scrambled set of type s.

The usually examined notion of ($\tilde{X}$-)distributional chaos is obtained simply by plugging $m_n \equiv n$ in (2.3): in our framework, ($\tilde{X}$-)distributional chaos is nothing else but ($\tilde{X}$-)reiterative distributional chaos of type $1-$. Since for each sequence (m_n) under our consideration there exists a finite constant $L > 0$ such that $\underline{d}_{m_n}(A) \geq L^{-1}\underline{d}(A)$ for any subset $A \subseteq \mathbb{N}$, it readily follows that any $(m_n, \tilde{X})$-distributionally chaotic sequence of type $1-$ is $\tilde{X}$-distributionally chaotic. Furthermore, for any infinite set $A \subseteq \mathbb{N}$, having a positive Banach density and being syndetic are the same things. Therefore, if $(\rho_k)_{k\in\mathbb{N}}$ is an $(m_n, \tilde{X})$-distributionally chaotic sequence of type $s \in \{1+, 2-, 0\}$, then $(\rho_k)_{k\in\mathbb{N}}$ is a $\tilde{X}$-reiteratively distributionally chaotic sequence of type s, as well (see also [16, Definition 2.1], where the notion of $\tilde{X}$-reiterative distributional chaos of type s has been introduced for the first time, under a slightly different designation).

Due to Lemma 2(i), we immediately get that the notion of $(m_n, \tilde{X})$-reiterative distributional chaos of type 0 does not depend on the particular choice of sequence (m_n) because our standing assumption is the existence a finite constant $L \geq 1$ such that $n \leq Lm_n$, $n \in \mathbb{N}$, so that (2.6) holds iff the both sequences $\{k \in \mathbb{N} : d_Y(x_k, y_k) < \sigma\}$ and $\{k \in \mathbb{N} : d_Y(x_k, y_k) \geq \epsilon\}$ are infinite and thick (i.e., not syndetic). Therefore, we have the following:

Proposition 2.1. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of binary relations between the spaces X and Y. Then $(\rho_k)_{k\in\mathbb{N}}$ is $(m_n, \tilde{X})$-reiteratively distributionally chaotic of type 0 iff $(\rho_k)_{k\in\mathbb{N}}$ is $\tilde{X}$-reiteratively distributionally chaotic of type 0.*

Furthermore, almost immediately from the given definitions we have the following:

Proposition 2.2. *Suppose that $s \in \{0, 1-, 1+, 2-\}$. Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of binary relations between the spaces X and Y, and let (m_n') be another increasing sequence in $[1, \infty)$ satisfying that $\liminf_{n\to\infty} \frac{m_n'}{n} > 0$. Then $(\rho_k)_{k\in\mathbb{N}}$ is $(m_n, \tilde{X})$-reiteratively distributionally chaotic of type s provided that $(\rho_k)_{k\in\mathbb{N}}$ is $(m_n', \tilde{X})$-reiteratively distributionally chaotic of type s.*

Corollary 2.1. *Suppose that $s \in \{0, 1-, 1+, 2-\}$, $0 < \lambda_1 \leq \lambda_2 \leq 1$ and $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of binary relations between the spaces X and Y. Then the sequence $(\rho_k)_{k\in\mathbb{N}}$ is $(\lambda_2, \tilde{X})$-reiteratively distributionally chaotic of type s provided that the sequence $(\rho_k)_{k\in\mathbb{N}}$ is $(\lambda_1, \tilde{X})$-reiteratively distributionally chaotic of type s.*

Keeping in mind Lemma 2(i)-(ii) and an elementary reasoning, we can simply clarify the following:

Proposition 2.3. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of binary relations between the spaces X and Y. Then the following holds:*

(i) *Suppose that $(\rho_k)_{k\in\mathbb{N}}$ is $(m_n, \tilde{X})$-reiteratively distributionally chaotic of type $1-$. Then $(\rho_k)_{k\in\mathbb{N}}$ is $(m_n, \tilde{X})$-reiteratively distributionally chaotic of type $s \in \{0, 1+, 2-\}$.*

(ii) *Suppose that $(\rho_k)_{k\in\mathbb{N}}$ is $(m_n, \tilde{X})$-reiteratively distributionally chaotic of type $1+$ or $2-$. Then $(\rho_k)_{k\in\mathbb{N}}$ is $(m_n, \tilde{X})$-reiteratively distributionally chaotic of type 0.*

The existence of sequences of single-valued linear continuous operators on finite dimensional spaces which are λ-reiteratively distributionally chaotic of type $1-$ for each number $\lambda \in (0,1)$ can be simply proved:

Example 1. *([19])* Due to Proposition 1, if $A = \{n_1,\ n_2, \cdots,\ n_k, \cdots\}$, where (n_k) is a strictly increasing sequence of positive integers, then $\underline{d}_{1/\lambda}(A) = 0$ iff for any finite constant $L > 0$ there exists $k \in \mathbb{N}$ such that $n_k > Lk^{1/\lambda}$. Therefore, it is very simple to construct two disjoint subsets A and B of $\mathbb{N}$ such that $\mathbb{N} = A \cup B$ and $\underline{d}_{1/\lambda}(A) = \underline{d}_{1/\lambda}(B) = 0$ for each number $\lambda \in (0,1)$; for example, set $a_n := \sum_{i=1}^{n} 2^{2^{i^2}}$ $(n \in \mathbb{N})$, $A := \bigcup_{n \in 2\mathbb{N}} [a_n, a_{n+1}]$ and $B := \mathbb{N} \setminus A$. After that, set $X := \mathbb{K}$, $T_k := kI$ $(k \in A)$ and $T_k := 0$ $(k \in B)$. Then it can be simply checked that the sequence $(T_k)_{k \in \mathbb{N}}$ is densely λ-distributionally chaotic for each number $\lambda \in (0,1)$, and that the corresponding scrambled set S can be chosen to be the whole space X.

Furthermore, any two notions introduced above do not coincide for general sequences of linear continuous operators on finite dimensional spaces; this can be inspected as in Example 1 above. Concerning the orbits of linear continuous operators on Fréchet spaces, it should be recalled that there is no Li-Yorke chaotic (distributionally chaotic, therefore) operator on a finite-dimensional Fréchet space ([5]). Mention should be also made of paper [19], where we have constructed a weighted forward shift operator on the Banach space l^2 that is λ-distributionally chaotic for any number $\lambda \in (0,1]$.

If $(\rho_k)_{k \in \mathbb{N}}$ and $\tilde{X}$ are given in advance, then we define the binary relations $\rho_k' : D(\rho_k') \subseteq X \to Y$ by $D(\rho_k') := D(\rho_k) \cap \tilde{X}$ and $\rho_k' x := \rho_k x$, $x \in D(\rho_k')$ $(k \in \mathbb{N})$. Then we have the following simple result, showing that the case in which $\tilde{X} = X$ can be assumed in a certain sense (our notion can be introduced without $\tilde{X}$, but it is important to know somehow the minimality of $\tilde{X}$ in Definition 2.1):

Proposition 2.4. *Suppose that $s \in \{0, 1-, 1+, 2-\}$. Then the sequence $(\rho_k)_{k \in \mathbb{N}}$ is $(m_n, \tilde{X})$-distributionally chaotic of type s iff $(\rho_k')_{k \in \mathbb{N}}$ is m_n-distributionally chaotic of type s.*

If the sequence $(\rho_k)_{k \in \mathbb{N}}$ is $(m_n, \tilde{X})$-reiteratively distributionally chaotic of type $s \in \{0, 1-, 1+, 2-\}$, then it has to be $\tilde{X}$-Li-Yorke chaotic in the sense of [16, Definition 2.2]. In this paper, we have also analyzed the notions of

$(\tilde{X}, i)$-mixed chaoticity, where $i \in \{1, 2, 3, 4\}$. Keeping in mind our previous analyses, it could be of some importance to further specify the notions of $(\tilde{X}, 2)$-mixed chaoticity and $(\tilde{X}, 4)$-mixed chaoticity for general sequences of binary relations (the notions of $(\tilde{X}, 1)$-mixed chaoticity and $(\tilde{X}, 3)$-mixed chaoticity are not interesting here because the use of lower (Banach) m_n-densities leads to the same notions). Consider the following conditions:

$$\underline{d}_{m_n}\Bigl(\bigl\{k \in \mathbb{N} : d_Y\bigl(x_k, y_k\bigr) < \sigma\bigr\}\Bigr) = 0 \text{ and } \liminf_{k\to\infty} d_Y\bigl(x_k, y_k\bigr) = 0, \quad (2.7)$$

$$\limsup_{k\to\infty} d_Y\bigl(x_k, y_k\bigr) > 0 \text{ and } \underline{d}_{m_n}\Bigl(\bigl\{k \in \mathbb{N} : d_Y\bigl(x_k, y_k\bigr) \geq \epsilon\bigr\}\Bigr) = 0. \quad (2.8)$$

Then the following notions can be introduced:

Definition 2.2. Suppose that, for every $k \in \mathbb{N}$, $\rho_k \subseteq X \times Y$ is a binary relation and $\tilde{X}$ is a non-empty subset of X. If there exist an uncountable set $S \subseteq \bigcap_{k=1}^{\infty} D(\rho_k) \cap \tilde{X}$ and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that for each $k \in \mathbb{N}$ there exist elements $x_k \in \rho_k x$ and $y_k \in \rho_k y$ such that (2.7) holds, resp. (2.8) holds, then we say that the sequence $(\rho_k)_{k\in\mathbb{N}}$ is $(\tilde{X}, m_n, 2)$-mixed chaotic, resp. $(\tilde{X}, m_n, 4)$-mixed chaotic.

Let $i \in \{2, 4\}$. The notion of densely $(\tilde{X}, m_n, i)$-mixed chaotic sequence $(\rho_k)_{k\in\mathbb{N}}$ (the binary relation ρ), the corresponding $(\sigma_{\tilde{X}}, m_n, i)$-mixed scrambled set ($(\sigma, m_n, i)$-mixed scrambled set, in the case that $\tilde{X} = X$), where $i = 2$, the corresponding $(\tilde{X}, m_n, i)$-mixed scrambled set ((m_n, i)-mixed scrambled set, in the case that $\tilde{X} = X$), where $i = 4$, of the sequence $(\rho_k)_{k\in\mathbb{N}}$ (the binary relation ρ) is introduced as above; in the case that $\tilde{X} = X$, then we also say that the sequence $(\rho_k)_{k\in\mathbb{N}}$ (the binary relation ρ) is (m_n, i)-mixed chaotic.

It is clear that any $(\tilde{X}, m_n, 2)$-mixed chaotic, resp. $(\tilde{X}, m_n, 4)$-mixed chaotic, sequence of binary relations needs to be $(\tilde{X}, 2)$-mixed chaotic (i.e., $(\tilde{X}, n, 2)$-mixed chaotic), resp. $(\tilde{X}, 4)$-mixed chaotic (i.e., $(\tilde{X}, n, 4)$-mixed chaotic). The converse statement is not true, however.

The previous definition covers, in particular, the case in which $m_n \equiv n^{1/\lambda}$ for some number $\lambda \in (0, 1)$. Then $(\tilde{X}, m_n, 2)$-mixed chaoticity and $(\tilde{X}, m_n, 4)$-mixed chaoticity is called $(\tilde{X}, \lambda, 2)$-mixed chaoticity and $(\tilde{X}, \lambda, 4)$-mixed chaoticity, respectively. Similar terminology is accepted for all other terms introduced above.

For some notions of distributional chaos, it is an essential thing that the pivot spaces X and Y are equipped with linear vector structures. For example, the notion of $\langle \tilde{X}, m_n, 2\rangle$-mixed chaoticity can be also introduced for general sequences of binary relations between the spaces X and Y, provided that the space Y is a Fréchet space (see also [16, Definition 4.1]).

2.1. **Reiterative (m_n)-distributional chaos of type s and reiterative λ-distributional chaos of type s in Fréchet spaces.** For the sake of brevity and better exposition, in this subsection we will always assume that the both spaces, X and Y, are Fréchet spaces over the same field of scalars. And, more to the point, we will consider only multivalued linear operators (although general definitions can be adapted for binary relations in a not satisfactory way for further investigations).

The notion of a reiteratively m_n-distributionally irregular vector of type s for general sequence of MLOs $(\mathcal{A}_k)_{k\in\mathbb{N}}$ is introduced as follows (see [19, Definition 2.5] for single-valued case, and [16, Subsection 4.1] for the case in which $m_n \equiv n$):

Definition 2.3. Suppose that $(\mathcal{A}_k)_{k\in\mathbb{N}}$ is a sequence of MLOs between the spaces X and Y, $m \in \mathbb{N}$ and $x \in \bigcap_{k\in\mathbb{N}} D(\mathcal{A}_k)$. Then we say that:

(i) x is (reiteratively) m_n-distributionally near to zero for $(\mathcal{A}_k)_{k\in\mathbb{N}}$ iff there exists a set $B \subseteq \mathbb{N}$ such that $(\underline{Bd}_{l;m_n}(B^c) = 0)$ $\underline{d}_{m_n}(B^c) = 0$ and for each $k \in \mathbb{N}$ there exists $x_k \in \mathcal{A}_k x$ such that $\lim_{k\in B} x_k = 0$;

(ii) x is (reiteratively) m_n-distributionally m-unbounded for $(\mathcal{A}_k)_{k\in\mathbb{N}}$ iff there exists a set $B \subseteq \mathbb{N}$ such that $(\underline{Bd}_{l;m_n}(B^c) = 0)$ $\underline{d}_{m_n}(B^c) = 0$ and for each $k \in \mathbb{N}$ there exists $x_k \in \mathcal{A}_k x$ such that $\lim_{k\in B} p_m^Y(x_k) = +\infty$; x is (reiteratively) m_n-distributionally unbounded for $(\mathcal{A}_k)_{k\in\mathbb{N}}$ iff there exists an integer $m \in \mathbb{N}$ such that x is (reiteratively) m_n-distributionally m-unbounded for $(\mathcal{A}_k)_{k\in\mathbb{N}}$;

(iii) x is reiteratively m_n-distributionally irregular vector of type $1-$ for $(\mathcal{A}_k)_{k\in\mathbb{N}}$ iff x is m_n-distributionally near to zero for $(\mathcal{A}_k)_{k\in\mathbb{N}}$ and x is m_n-distributionally unbounded for $(\mathcal{A}_k)_{k\in\mathbb{N}}$;

(iv) x is reiteratively m_n-distributionally irregular vector of type $1+$ for $(\mathcal{A}_k)_{k\in\mathbb{N}}$ iff x is m_n-distributionally near to zero for $(\mathcal{A}_k)_{k\in\mathbb{N}}$ and x is reiteratively m_n-distributionally unbounded for $(\mathcal{A}_k)_{k\in\mathbb{N}}$;

(v) x is reiteratively m_n-distributionally irregular vector of type $2-$ for $(\mathcal{A}_k)_{k\in\mathbb{N}}$ iff x is reiteratively m_n-distributionally near to zero for $(\mathcal{A}_k)_{k\in\mathbb{N}}$ and x is m_n-distributionally unbounded for $(\mathcal{A}_k)_{k\in\mathbb{N}}$;

(vi) x is reiteratively m_n-distributionally irregular vector of type 0 for $(\mathcal{A}_k)_{k\in\mathbb{N}}$ iff x is reiteratively m_n-distributionally near to zero for $(\mathcal{A}_k)_{k\in\mathbb{N}}$ and x is reiteratively m_n-distributionally unbounded for $(\mathcal{A}_k)_{k\in\mathbb{N}}$.

If $m_n \equiv n^{1/\lambda}$ for some $\lambda \in (0,1]$, then we obtain the notion of a reiteratively λ-distributionally irregular vector of type s for sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$.

Definition 2.4. Suppose that $s \in \{0, 1-, 1+, 2-\}$, $m \in \mathbb{N}$ and $(\mathcal{A}_k)_{k\in\mathbb{N}}$ is a sequence of MLOs between the spaces X and Y. Then we say that X' is reiteratively $\tilde{X}_{m_n}$-distributionally irregular manifold for $(\mathcal{A}_k)_{k\in\mathbb{N}}$ of type s (reiteratively m_n-distributionally irregular manifold of type s in the case that $\tilde{X} = X$) iff any element $x \in (X' \cap \bigcap_{k=1}^{\infty} D(\mathcal{A}_k)) \setminus \{0\}$ is reiteratively $\tilde{X}_{m_n}$-distributionally irregular vector of type s for $(\mathcal{A}_k)_{k\in\mathbb{N}}$. Moreover, it is said that X' is a uniformly reiteratively $\tilde{X}_{m_n}$-distributionally irregular manifold of type s for the sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$ (uniformly reiteratively m_n-distributionally irregular manifold of type s in the case that $\tilde{X} = X$) iff there exists $m \in \mathbb{N}$ such that the orbit of each vector $x \in (X' \cap \bigcap_{k=1}^{\infty} D(\mathcal{A}_k)) \setminus \{0\}$ under $(\mathcal{A}_k)_{k\in\mathbb{N}}$ is both reiteratively m_n-distributionally m-unbounded of type s and reiteratively m_n-distributionally near to 0 of type s, with the meaning clear. Finally, if X' is dense in $\tilde{X}$, then we say that X' is dense (uniformly) reiteratively $\tilde{X}_{m_n}-$distributionally irregular manifold of type s for $(\mathcal{A}_k)_{k\in\mathbb{N}}$.

Again, if $m_n \equiv n^{1/\lambda}$ for some number $\lambda \in (0,1]$, then we obtain the notion of a (dense, uniformly) reiteratively $\tilde{X}_{\lambda}$-distributionally irregular manifold of type s for $(\mathcal{A}_k)_{k\in\mathbb{N}}$.

The notions from the previous two definitions are introduced for an MLO $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ similarly as before, by considering the sequence $(\mathcal{A}_k \equiv \mathcal{A}^k)_{k\in\mathbb{N}}$.

If the set $A \subseteq \mathbb{N}$ has the lower (Banach) m_n-density equal to zero, then the same holds for the set $A \cup B$, where B is any finite subset of $\mathbb{N}$. Using this fact, we can almost immediately clarify the following:

(i) If X' is a uniformly reiteratively $\tilde{X}_{m_n}$-distributionally irregular manifold of type s for the sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$, then X' is reiteratively

$\tilde{X}_{2^{-m-1}}$-scrambled set of type s for the sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$ (here $m \in \mathbb{N}$ has the same value as in Definition 2.4).

(ii) If $x \in \tilde{X} \cap \bigcap_{k=1}^{\infty} D(\mathcal{A}_k)$ is a reiteratively $\tilde{X}_{m_n}$-distributionally irregular vector of type s for the sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$, then $X' \equiv span\{x\}$ is a uniformly reiteratively $\tilde{X}_{m_n}$-distributionally irregular manifold of type s for the sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$.

For the sake of brevity, we will not consider here the notions of $(\tilde{X}, m_n, i)$-mixed chaotic irregular vectors and $(\tilde{X}, m_n, i)$-mixed chaotic irregular manifolds for sequences of MLOs ($i \in \{2, 4\}$). Finally, based on our recent results [7, Theorem 3.8] and [19, Corollary 3.6], we would like to propose the following problem that is very similar to Problem 1 in [16]:

Problem 1. Suppose that Ω is an open connected subset of $\mathbb{K} = \mathbb{C}$ satisfying $\Omega \cap \{z \in \mathbb{C} : |z| = 1\} \neq \emptyset$. Let $f : \Omega \to X \setminus \{0\}$ be an analytic mapping such that $\lambda f(\lambda) \in \mathcal{A} f(\lambda)$ for all $\lambda \in \Omega$. Set $\tilde{X} := \overline{span\{f(\lambda) : \lambda \in \Omega\}}$. Is it true that the operator $\mathcal{A}_{|\tilde{X}}$ is densely m_n-distributionally chaotic in the space $\tilde{X}$, for any increasing sequence (m_n) in $[1, \infty)$ satisfying $\liminf_{n\to\infty} \frac{m_n}{n} > 0$?

3. Further extensions of reiterative (m_n)-distributional chaos of type s and reiterative λ-distributional chaos of type s

In [4], N. C. Bernardes Jr. et al. have considered the notion of distributional chaos of type $s \in \{1, 2, 2\frac{1}{2}, 3\}$ for linear continuous operators acting on Banach spaces. The notion from this paper has been recently analyzed and extended in [16] for general sequences of binary relations in metric spaces by using the lower and upper Banach densities. More precisely, in [16, Definition 3.1], the author has introduced the notion of reiterative $\tilde{X}$-distributional chaos of type $i; s$, where $i \in \{0, 1, 2\}$ and $s \in \{1, 2, 2\frac{1}{2}, 3\}$. There are several different ways how one can further specify and generalize this notion but, for the sake of brevity, we will analyze here just a few possible ways for doing so.

Suppose that σ, $\sigma' > 0$, $\epsilon > 0$ and $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ are two given sequences in X. Consider the following conditions:

$$\overline{d}_{m_n}\Big(\big\{k \in \mathbb{N} : d_Y\big(x_k, y_k\big) \geq \sigma\big\}\Big) > 0, \tag{3.1}$$

$$\underline{d}_{m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)<\sigma'\big\}\Big)<+\infty,$$
$$\underline{d}_{m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\epsilon\big\}\Big)=0;$$

$$\begin{aligned}&\overline{Bd}_{l;m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\sigma\big\}\Big)>0,\\&\underline{d}_{m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)<\sigma'\big\}\Big)<+\infty,\\&\underline{d}_{m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\epsilon\big\}\Big)=0;\end{aligned}\tag{3.2}$$

$$\begin{aligned}&\overline{d}_{m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\sigma\big\}\Big)>0,\\&\underline{Bd}_{l;m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)<\sigma'\big\}\Big)<+\infty,\\&\underline{Bd}_{l;m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\epsilon\big\}\Big)=0;\end{aligned}\tag{3.3}$$

$$\begin{aligned}&\overline{Bd}_{l;m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\sigma\big\}\Big)>0,\\&\underline{Bd}_{l;m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)<\sigma'\big\}\Big)<+\infty,\\&\underline{Bd}_{l;m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\epsilon\big\}\Big)=0;\end{aligned}\tag{3.4}$$

$$\begin{aligned}&\overline{d}_{m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\sigma\big\}\Big)>0,\\&\underline{d}_{m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\epsilon\big\}\Big)=0;\end{aligned}\tag{3.5}$$

$$\begin{aligned}&\overline{d}_{m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\sigma\big\}\Big)>0,\\&\underline{Bd}_{l;m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\epsilon\big\}\Big)=0;\end{aligned}\tag{3.6}$$

$$\begin{aligned}&\overline{Bd}_{l;m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\sigma\big\}\Big)>0,\\&\underline{d}_{m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\epsilon\big\}\Big)=0;\end{aligned}\tag{3.7}$$

$$\begin{aligned}&\overline{Bd}_{l;m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\sigma\big\}\Big)>0,\\&\underline{Bd}_{l;m_n}\Big(\big\{k\in\mathbb{N}: d_Y\big(x_k,y_k\big)\geq\epsilon\big\}\Big)=0;\end{aligned}\tag{3.8}$$

there exist $c>0$ and $r>0$ such that

$$\underline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)<c<\overline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr) \tag{3.9}$$

for $0<\sigma<r$;
there exist $c>0$ and $r>0$ such that

$$\underline{Bd}_{l;m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)<c<\overline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr) \tag{3.10}$$

for $0<\sigma<r$;
there exist $c>0$ and $r>0$ such that

$$\underline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)<c<\overline{Bd}_{l;m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr) \tag{3.11}$$

for $0<\sigma<r$;
there exist $c>0$ and $r>0$ such that

$$\begin{aligned}&\underline{Bd}_{l;m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)<c<\\&<\overline{Bd}_{l;m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)\end{aligned} \tag{3.12}$$

for $0<\sigma<r$;
there exist $c>0$ and $r>0$ such that

$$\begin{aligned}0=&\underline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)<c<\\&<\overline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)\end{aligned} \tag{3.13}$$

for $0<\sigma<r$;
there exist $c>0$ and $r>0$ such that

$$\begin{aligned}0=&\underline{Bd}_{l;m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)<c<\\&<\overline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)\end{aligned} \tag{3.14}$$

for $0<\sigma<r$;
there exist $c>0$ and $r>0$ such that

$$\begin{aligned}0=&\underline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)<c<\\&<\overline{Bd}_{l;m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_k,y_k\bigr)<\sigma\bigr\}\Bigr)\end{aligned} \tag{3.15}$$

for $0 < \sigma < r$;
there exist $c > 0$ and $r > 0$ such that

$$0 = \underline{Bd}_{l;m_n}\Bigl(\bigl\{k \in \mathbb{N} : d_Y\bigl(x_k, y_k\bigr) < \sigma\bigr\}\Bigr) < c <$$

$$< \overline{Bd}_{l;m_n}\Bigl(\bigl\{k \in \mathbb{N} : d_Y\bigl(x_k, y_k\bigr) < \sigma\bigr\}\Bigr) \tag{3.16}$$

for $0 < \sigma < r$;

$$\text{there exist } a,\ b,\ c > 0 \text{ such that (3.9) holds for all } \sigma \in [a,b]; \tag{3.17}$$

$$\text{there exist } a,\ b,\ c > 0 \text{ such that (3.10) holds for all } \sigma \in [a,b]; \tag{3.18}$$

$$\text{there exist } a,\ b,\ c > 0 \text{ such that (3.11) holds for all } \sigma \in [a,b]; \tag{3.19}$$

$$\text{there exist } a,\ b,\ c > 0 \text{ such that (3.12) holds for all } \sigma \in [a,b]. \tag{3.20}$$

Now we are ready to introduce the notion of $[\tilde{X}, m_n, i]$-distributional chaos for any integer $i \in [1, 20]$:

Definition 3.1. Suppose that, for every $k \in \mathbb{N}$, $\rho_k : D(\rho_k) \subseteq X \to Y$ is a binary relation and $\tilde{X}$ is a non-empty subset of X.

- (i) $1 \leq i \leq 4$: If there exist an uncountable set $S \subseteq \bigcap_{k=1}^{\infty} D(\rho_k) \cap \tilde{X}$ and $\sigma,\ \sigma' > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that for each $k \in \mathbb{N}$ there exist elements $x_k \in \rho_k x$ and $y_k \in \rho_k y$ such that (3.1), resp. <(3.2), (3.3), (3.4)> holds, then we say that the sequence $(\rho_k)_{k\in\mathbb{N}}$ is reiteratively $[\tilde{X}, m_n, 1]$-distributionally chaotic, resp. <reiteratively $[\tilde{X}, m_n, 2]$-distributionally chaotic, reiteratively $[\tilde{X}, m_n, 3]$-distributionally chaotic, reiteratively $[\tilde{X}, m_n, 4]$-distributionally chaotic>;
- (ii) $5 \leq i \leq 8$: If there exist an uncountable set $S \subseteq \bigcap_{k=1}^{\infty} D(\rho_k) \cap \tilde{X}$ and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that for each $k \in \mathbb{N}$ there exist elements $x_k \in \rho_k x$ and $y_k \in \rho_k y$ such that (3.7) holds, resp. <(3.6), (3.5), (3.8)> holds, then we say that the sequence $(\rho_k)_{k\in\mathbb{N}}$ is reiteratively $[\tilde{X}, m_n, 5]$-distributionally chaotic, resp. <reiteratively $[\tilde{X}, m_n, 6]$-distributionally chaotic, reiteratively $[\tilde{X}, m_n, 7]$-distributionally chaotic, reiteratively $[\tilde{X}, m_n, 8]$-distributionally chaotic>;

(iii) $9 \leq i \leq 16$: If there exist an uncountable set $S \subseteq \bigcap_{k=1}^{\infty} D(\rho_k) \cap \tilde{X}$ and $c,\ r > 0$ such that for each pair $x,\ y \in S$ of distinct points we have that for each $k \in \mathbb{N}$ there exist elements $x_k \in \rho_k x$ and $y_k \in \rho_k y$ such that (3.9) holds for $0 < \sigma < r$, resp. <(3.9), ..., (3.16)> holds for $0 < \sigma < r$, then we say that the sequence $(\rho_k)_{k\in\mathbb{N}}$ is reiteratively $[\tilde{X}, m_n, 9]$-distributionally chaotic, resp. <reiteratively $[\tilde{X}, m_n, 10]$-distributionally chaotic, ..., reiteratively $[\tilde{X}, m_n, 16]$-distributionally chaotic>;

(iv) $17 \leq i \leq 20$: If there exist an uncountable set $S \subseteq \bigcap_{k=1}^{\infty} D(\rho_k) \cap \tilde{X}$ and $a,\ b,\ c > 0$ such that for each pair $x,\ y \in S$ of distinct points we have that for each $k \in \mathbb{N}$ there exist elements $x_k \in \rho_k x$ and $y_k \in \rho_k y$ such that (3.17) holds for $\sigma \in [a, b]$, resp. <(3.18) holds for $\sigma \in [a, b]$, (3.19) holds for $\sigma \in [a, b]$, resp. (3.20) holds for $\sigma \in [a, b]$ >, then we say that the sequence $(\rho_k)_{k\in\mathbb{N}}$ is reiteratively $[\tilde{X}, m_n, 17]$-distributionally chaotic, resp. <reiteratively $[\tilde{X}, m_n, 18]$-distributionally chaotic, reiteratively $[\tilde{X}, m_n, 19]$-distri–butionally chaotic, reiteratively $[\tilde{X}, m_n, 20]$-distributionally chaotic>.

Let $i \in \mathbb{N} \cap [1, 20]$. Then we say that the sequence $(\rho_k)_{k\in\mathbb{N}}$ is densely reiteratively $[\tilde{X}, m_n, i]$-distributionally chaotic iff S can be chosen to be dense in $\tilde{X}$. A binary relation $\rho : D(\rho) \subseteq X \to X$ is said to be (densely) reiteratively $[\tilde{X}, m_n, i]$-distributionally chaotic iff the sequence $(\rho_k \equiv \rho^k)_{k\in\mathbb{N}}$ is. The set S is said to be reiteratively $[\tilde{X}, m_n, i]$-scrambled set, resp. reiteratively $[m_n, i]$-scrambled set in the case that $\tilde{X} = X$, of the sequence $(\rho_k)_{k\in\mathbb{N}}$ (the binary relation ρ); in the case that $\tilde{X} = X$, then we also say that the sequence $(\rho_k)_{k\in\mathbb{N}}$ (the binary relation ρ) is reiteratively $[m_n, i]$-distributionally chaotic.

The case in which $m_n \equiv n^{1/\lambda}$ for some number $\lambda \in (0, 1]$ is most intriguing and, in this case, the (dense) reiterative $[\tilde{X}, m_n, i]$-distributional chaos is also called reiterative $[\tilde{X}, \lambda, i]$-distributional chaos (reiterative $[\tilde{X}, i]$-distri–butional chaos, if $\lambda = 1$), the set S is also called reiteratively $[\tilde{X}, \lambda, i]$-scrambled set (reiteratively $[\lambda, i]$-scrambled set in the case that $\tilde{X} = X$)/reiteratively $[\tilde{X}, i]$-scrambled set, if $\lambda = 1$ (reiteratively $[i]$-scram–bled set in the case that $\tilde{X} = X$ and $\lambda = 1$), and the reiterative $[m_n, i]$-distributional chaos is also called reiterative $[\lambda, i]$-distributional chaos ($[i]$-distributional chaos, if $\lambda = 1$).

It is clear that we can formulate a great deal of statements concerning $[\tilde{X}, m_n, i]$-distributional chaos that are straightforward consequences of the introduced definition. On the other hand, some questions are not so easy and trivial for consideration; for example, the following holds (see also Proposition 2.3(i)):

Proposition 3.1. *Suppose that $\tilde{X}$ is a non-empty subset of X and $(\rho_k)_{k\in\mathbb{N}}$ is a given sequence of binary relations. If $(\rho_k)_{k\in\mathbb{N}}$ is $(m_n, \tilde{X})$-distributionally chaotic of type $1-$, then $(\rho_k)_{k\in\mathbb{N}}$ is $[\tilde{X}, m_n, i]$-distributionally chaotic for any $i \in \{1, 5, 17\}$. Furthermore, if $|\rho_k x| = 1$ for each $k \in \mathbb{N}$ and $x \in \bigcap_{k=1}^{\infty} D(\rho_k) \cap \tilde{X}$, then $(\rho_k)_{k\in\mathbb{N}}$ is $[\tilde{X}, m_n, i]$-distributionally chaotic for $i \in \{9, 13\}$.*

Proof. Suppose that $(\rho_k)_{k\in\mathbb{N}}$ is $(m_n, \tilde{X})$-distributional chaotic of type $1-$. Let $\epsilon \in (0, \liminf_{n\to\infty} \frac{m_n}{2n})$, let the uncountable set $S \subseteq \bigcap_{k=1}^{\infty} D(\rho_k) \cap \tilde{X}$, the number $\sigma > 0$ and the elements $x_k \in \rho_k x$ and $y_k \in \rho_k y$ ($k \in \mathbb{N}$) satisfy the requirements prescribed in Definition 2.1. Since the second inequality in (3.1) holds with $\sigma = \sigma'$, in order to prove that $(\rho_k)_{k\in\mathbb{N}}$ is $[\tilde{X}, m_n, 1]$-distributionally chaotic, we only need to show that $\overline{d}_{m_n}(\{k \in \mathbb{N} : d_Y(x_k, y_k) \geq \sigma\}) > 0$. Suppose the contrary. Then there exists an integer $n_0(\epsilon) \in \mathbb{N}$ such that the segment $[1, m_n]$ contains at least $\lfloor m_n - \epsilon n - 2 \rfloor$ elements of the set $\{k \in \mathbb{N} : d_Y(x_k, y_k) < \sigma\}$ ($n \in \mathbb{N}$, $n \geq n_0(\epsilon)$). This yields $\underline{d}_{m_n}(\{k \in \mathbb{N} : d_Y(x_k, y_k) < \sigma\}) > \liminf_{n\to\infty} \frac{m_n - \epsilon n - 3}{n} = \liminf_{n\to\infty} [\frac{m_n}{n} - \epsilon] > 0$, which is a contradiction. Using Lemma 3, it readily follows that the first inequality in (3.5) holds, so that $(\rho_k)_{k\in\mathbb{N}}$ is $[\tilde{X}, m_n, 5]$-distributionally chaotic. It is almost immediate from definitions that $(m_n, \tilde{X})$-distributional chaos of type $1-$ implies $[\tilde{X}, m_n, 17]$-distributionally chaos. Suppose now that $|\rho_k x| = 1$ for each $k \in \mathbb{N}$ and $x \in \bigcap_{k=1}^{\infty} D(\rho_k) \cap \tilde{X}$. Then we have $\underline{d}_{m_n}(\{k \in \mathbb{N} : d_Y(x_k, y_k) \geq \epsilon\}) = 0$ for any $\epsilon > 0$, so that $(\rho_k)_{k\in\mathbb{N}}$ is $[\tilde{X}, m_n, i]$-distributionally chaotic for $i \in \{9, 13\}$ due to Lemma 3, with $c = 1/L$. □

In general case, we can have $\overline{Bd}_{l;m_n}(A) = 0$ but $\overline{d}_{m_n}(A) = +\infty$ for a set $A \subseteq \mathbb{N}$ (see e.g. [17, Example 2.1(ii)]), so that it is not expected that the $[\tilde{X}, m_n, 1]$-distributional chaos implies $[\tilde{X}, m_n, i]$-distributional chaos for $i \in \{2, 3, 4\}$, as in the case that $m_n \equiv n$. Any further analysis of reiterative $[\tilde{X}, m_n, i]$-distributional chaos is without scope of this paper and, from the sake of brevity, we will also skip all related details regarding reiteratively

$[\tilde{X}, m_n, i]$-distributionally irregular manifolds for MLOs in Fréchet spaces (see also [19]). Speaking-matter-of-factly, we would like to make only one question more to complete this section. Let $i \in \mathbb{N} \cap [1, 20]$, and let the sequence $(\rho_k)_{k\in\mathbb{N}}$ be $[\tilde{X}, m_n, i]$-distributionally chaotic. Suppose that (m'_n) is another increasing sequence in $[1, \infty)$ satisfying $\liminf_{n\to\infty} \frac{m'_n}{n} > 0$ and $m'_n \leq m_n$, $n \in \mathbb{N}$. Then it is not clear from definition whether the sequence $(\rho_k)_{k\in\mathbb{N}}$ will be $[\tilde{X}, m'_n, i]$-distributionally chaotic.

4. Conclusions and final remarks

Suppose that (m_n) is an increasing sequence in $[1, \infty)$ satisfying $\liminf_{n\to\infty} \frac{m_n}{n} > 0$, $\lambda \in (0, 1]$, $s \in \{0, 1-, 1+, 2-\}$ and $i \in \mathbb{N} \cap [1, 20]$. In this heuristical study, we have introduced and analyzed the concepts of reiterative $(m_n, \tilde{X})$-distributional chaos of type s, reiterative $(\lambda, \tilde{X})$-distributional chaos of type s, reiterative $[\tilde{X}, m_n, i]$-distributional chaos and reiterative $[\tilde{X}, \lambda, i]$-distribu-tional chaos. Our definitions and results are given for general sequences of binary relations over metric spaces, while special attention is paid to multivalued linear operators in Fréchet spaces. The main aim of study is, actually, to fix the notion necessary for further examinations of distributional chaos in multi-valued setting.

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