Математички Билтен
Vol. 39(LXV) No. 1
2015 (5-11)
Скопје, Македонија

ISSN 0351-336X
UDC: 528.247.026:517.956.224

REAL EARTH BASED SPLINE FOR GRAVITATIONAL POTENTIAL DETERMINATION

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Abstract. For computational reasons, the spline interpolation of gravitational potential is usually done in a spherical framework [3]. However, the increasing observational accuracy requires mathematical methods for geophysically more relevant surfaces. We propose a spline method with respect to the real Earth. The spline formulation reflects the specific geometry of a given regular surface. This is due to the representation of the reproducing kernel as a Newton integral over the inner space of a regular surface. The approximating potential functions have the same domain of harmonicity as the actual Earth's gravitational potential. Moreover, this approach is a generalization to spherical kernels.

1. INTRODUCTION

The Earth's gravity field is one of the most fundamental forces. Although invisible, gravity is a complex force of nature that has an immeasurable impact on our everyday lives. It is often assumed that the force of gravity on the Earth's surface has a constant value, and gravity is considered acting in straight downward direction, but in fact its value varies subtly from place to place and its direction known as the plumb line is actually slightly curved. If the Earth had a perfectly spherical shape and if the mass inside the Earth were distributed homogeneously or rotationally symmetric, these considerations would be true and the line along which Newton's apple fell would indeed be a straight one. The gravitational field obtained in this way would be perfectly spherically symmetric. In reality, however, the situation is much more complex. Gravitational force deviates from one place to the other from that of a homogeneous sphere, due to a number of factors, such as the rotation of the Earth, the topographic features (the position of mountains, valleys or ocean trenches) and variations in density of the Earth's interior. As a consequence the precise knowledge of the Earth's gravitational potential and equipotential surfaces is crucial for all sciences that contribute to the study of the Earth, such as seismology, topography, solid geophysics or oceanography. With the growing awareness with respect to environmental problems like pollution and climate changes, this problem becomes every day a more and more important issue.

2010 *Mathematics Subject Classification.* Primary: 31B20 Secondary: 31B10, 31B05
Key words and phrases. potential , regular surface, reproducing kernel, harmonic function

So, how is gravitational potential calculated nowadays?

The obvious complexity of the mathematical representation of equipotential surfaces, was the reason why geoscientists were choosing more suitable surfaces for the (approximate) construction of the geoid. It is known that the geoid to a first approximation is a sphere with radius about 6371 km, so first of all, approaches to deal with a spherical Earth have been considered. The traditional way to model the gravitational field is to use (Fourier) expansions with spherical harmonics as basis functions (which is a technique developed by Gauss in the nineteenth century). The spherical harmonics and their continuation to the inner and outer space as solution to Laplace equation are well-known. Much research concerning these functions has been done so far. As a result, a great number of mathematical theories concerning gravitational field determination were developed in the spherical framework, and corresponding numerical methods are known to give good accuracy. The developed theory of spherical harmonic splines and wavelets in [2]-[5] showed that spline functions can be viewed as canonical generalizations of the outer harmonics, having desirable properties such as interpolating, smoothing, and best approximation functions, while harmonic wavelets are giving possibility of multiscale analysis as constituting `building blocks' in the approximation of the gravitational potential. Even the latest gravitational potential model EGM2008 and it's predecessor EGM96 are providing spherical harmonics coefficients for the geoid. The spherical framework however, was sufficient for modelling the gravitational field until recently. The available data in the recent past reflected gravitational field changes at the long to medium length scales, and the approximations in the spherical framework could have been considered satisfactory. But today due to the newest satellite techniques we are able to get much more detailed picture of the local changes of the geopotential. On relatively short length scales (a few km to a few hundred km) the geoid is closely related to topography and we know that todays accuracies of satellite data gives us the possibility to reconstruct the geoid on very short length scales (e.g., the GOCE data). Also, the surface of the Earth become measurable with greatest precision, so today we are in position to discuss various developments and generalization of mathematical methods for integrals over regular regions, such as for example Newton integral. This situation offers new challenges to the geomathematicians in developing a new mathematical framework for the determination of the geoid. Today we are interested in non-spherical boundaries when solving potential theory problems, such as ellipsoids, or the real Earth's surface. [1] introduces the reproducing kernel Hilbert space of Newton potentials on and outside a given regular surface with reproducing kernel defined as a Newton integral over it's interior. Under this framework, a real Earth oriented strategy and method for the Earth's gravitational potential determination was proposed. This paper presents some of the results of the non-spherical theory presented in [1].

2. REPRODUCING KERNEL HILBERT SPACE OF NEWTONIAN POTENTIALS

In Newtonian nomenclature, the gravitational potential V of the Earth generated by a mass-distribution F inside the Earth is given by the volume integral (Newton integral)

$$V(x) = G\int_{Earth} \frac{F(y)}{|x-y|}dy, \quad x \in \mathbb{R}^3, \tag{0.1}$$

where G is the gravitational constant ($G = 6.67422 \cdot 10^{-11} m^3 / (\mathrm{kgs}^2)$) and dy is the volume element. The gravitational potential of the Earth corresponding to an integrable and bounded density function F, satisfies the Laplace equation $\Delta V = 0$ in the outer space and the Poisson equation $\Delta V = -4\pi F$ in the interior space.

The Newton integral (1.1) and its first derivatives are continuous everywhere on $\mathbb{R}^3$, i.e., $V \in C^{(1)}(\mathbb{R}^3)$. The second derivatives are analytic everywhere outside the real Earth surface, but they have a discontinuity when passing across the surface. Moreover, the gravitational potential V of the Earth, shows at infinity the following behavior:

(i) $|V(x)| = \mathcal{O}(\frac{1}{|x|}),\ x \to \infty$,

(ii) $|\nabla V(x)| = \mathcal{O}(\frac{1}{|x|^2}),\ x \to \infty$,

i.e., it is regular at infinity.

It was shown in [1] that associating the density function to the class of distributionally harmonic functions in $L^2(\Sigma^{int})$ ensures the appropriate RKHS structure of the space of Newton potentials in the Earth's exterior. The Newton integral given by

$$V(x) = \int_{\Sigma^{int}} \frac{F(y)}{|x-y|}dy, \quad x \in \overline{\Sigma^{ext}}, F \in L^2(\Sigma^{int}) \tag{0.2}$$

defines a linear operator $\mathcal{P}: L^2(\Sigma^{int}) \to \mathcal{P}(L^2(\Sigma^{int}))$, with $\mathcal{P}: F \mapsto V$, such that for every density function $F \in L^2(\Sigma^{int})$, $\mathcal{P}F = \int_{\Sigma^{int}} \frac{F(y)}{|\cdot-y|}dy$ is a Newtonian potential in the free space $\overline{\Sigma^{ext}}$.

We denote by $\mathcal{H}$ the space $\mathcal{P}(L^2(\Sigma^{int}))$ of potentials in $\overline{\Sigma^{ext}}$, i.e., we say that a function V is an element in $\mathcal{H}$, if we can write V in the form (1.2).

Theorem 1. *The space $\mathcal{H}$ of Newton integrals in $\overline{\Sigma^{ext}}$ corresponding to harmonic density functions, is a reproducing kernel Hilbert space with the reproducing kernel*

$$\mathcal{K}(x,\cdot) = \int_{\Sigma^{int}} \frac{dz}{|x-z||\cdot-z|}, \quad x \in \overline{\Sigma^{ext}}. \tag{0.3}$$

It is clear that for a fixed $x \in \overline{\Sigma^{ext}}$, the reproducing kernel $\mathcal{K}(x,\cdot)$ is a Newtonian potential corresponding to the harmonic density function $\frac{1}{|x-\cdot|}$ from $L^2(\Sigma^{int})$.

Moreover, for a fixed $x \in \overline{\Sigma^{ext}}$, the potential $\mathcal{K}(x,\cdot)$ to the density $\frac{1}{|x-\cdot|}$ is an element of $L^1(\Sigma^{int})$.This fact assures that $\mathcal{K}(x,\cdot)$ satisfies the Laplace equation in Σ^{ext}. Moreover, the potentials corresponding to densities in $L^2(\Sigma^{int})$ are elements in $C^{(0)}(\mathbb{R}^3)$. This is an extraordinary fact, since it means that now in interpolation methods we will be able to use potentials of the same nature as the Earth's gravitational potential, i.e., functions that are harmonic in the free space and continuous on the boundary instead of using outer harmonic expressions which are harmonic down to the Runge sphere completely situated in the Earth's enterior. The reproducing kernel is available in integral form for any geophisically relevant geometry (like ellipsoid, geoid, actual Earth's surface).

3. REAL EARTH BASED SPLINE

It was shown in [1] that the Dirichlet functional of the gravitational potential for points on the surface Σ, is bounded on the reproducing kernel Hilbert space $\mathcal{H}$ as defined before. Let $\{\alpha_1,...,\alpha_N\}$ be a given data set of Dirichlet functionals for the unknown potential U, corresponding to the discrete set $X_N = \{x_1,...,x_N\}$ of pairwise disjoint points on Σ, i.e., for $i = 1,...,N$

$$\mathcal{D}_i U = U(x_i) = \alpha_i .$$

Our aim is to find the smoothest $\mathcal{H}$ - interpolant corresponding to data set $\{\alpha_1,...,\alpha_N\}$ where by `smoothest' we mean that the norm is minimized in $\mathcal{H}$. In other words, the problem is to find a function $S^U_{\mathcal{D}_1,...,\mathcal{D}_N}$ in the set

$$\mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N} = \{P \in \mathcal{H} \mid \mathcal{D}_i P = \alpha_i, \, i = 1,...,N\},$$

such that

$$\| S^U_{\mathcal{D}_1,...,\mathcal{D}_N} \|_{\mathcal{H}} = \inf_{P \in \mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N}} \| P \|_{\mathcal{H}} .$$

The corresponding representer of the functional $\mathcal{D}_i$ can be written as

$$\mathcal{D}_i \mathcal{K}(\cdot,\cdot) = \mathcal{K}(x_i,\cdot) ,$$

where $\mathcal{K}$ is the reproducing kernel of $\mathcal{H}$. Then, for a given set $\{\mathcal{D}_1,...,\mathcal{D}_N\}$ of N Dirichlet functionals on $\mathcal{H}$, corresponding to the set $X_N = \{x_1, ..., x_N\}$ of points on Σ, we have the set of representers

$$\{\mathcal{D}_1 \mathcal{K}(\cdot,\cdot), ..., \mathcal{D}_N \mathcal{K}(\cdot,\cdot)\} .$$

The reproducing property of $\mathcal{K}$ yields, for $i = 1,...,N$, and $P \in \mathcal{H}$

$$\mathcal{D}_i P = (\mathcal{D}_i \mathcal{K}(\cdot,\cdot), P)_{\mathcal{H}} .$$

Having in mind that the reproducing kernel is given as a Newton integral (1.3), so are the representers of the functionals $\mathcal{D}_i$, i.e.

$$\mathcal{D}_i\mathcal{K}(\cdot,\cdot) = \int_{\Sigma^{int}} \frac{dz}{|x_i - z||\cdot - z|}.$$

Definition 1. *A system X_N of points $x_1,...,x_N$ on the surface Σ is called fundamental system on Σ, if the corresponding representers $\mathcal{L}_1\mathcal{K}(\cdot,\cdot),...,\mathcal{L}_N\mathcal{K}(\cdot,\cdot)$ of a given linear functional $\mathcal{L}$ are linearly independent.*

The interpolating spline is defined as follows:

Definition 2. *Let $X_N = \{x_1, ..., x_N\}$ be a given fundamental system of points on Σ and let $\{\mathcal{D}_1,...,\mathcal{D}_N\}$ be the set of the corresponding bounded linear Dirichlet functionals. Then, any function of the form*

$$S(x) = \sum_{i=1}^{N} a_i \mathcal{D}_i \mathcal{K}(\cdot, x) = \sum_{i=1}^{N} a_i \int_{\Sigma^{int}} \frac{dz}{|x_i - z||x - z|}, \quad x \in \overline{\Sigma^{ext}},$$

with arbitrarily given (real) coefficients $a_1,...,a_N$ is called a $\mathcal{H}-$ spline relative to $\{\mathcal{D}_1,...,\mathcal{D}_N\}$.

Obviously the space $S_{\mathcal{H}}(\mathcal{D}_1,...,\mathcal{D}_N) = span\{\mathcal{D}_1\mathcal{K}(\cdot,\cdot),...,\mathcal{D}_N\mathcal{K}(\cdot,\cdot)\}$, of all $\mathcal{H}-$splines relative to $\{\mathcal{D}_1,...,\mathcal{D}_N\}$ is an N-dimensional subspace of $\mathcal{H}$.

As an immediate consequence of the reproducing property ,viz. the $\mathcal{H}-$spline formula we get the following

Lemma 1: *Let S be a function of class $S_{\mathcal{H}}(\mathcal{D}_1,...,\mathcal{D}_N)$. Then for each $P \in \mathcal{H}$, the following identity is valid*

$$(S, P)_{\mathcal{H}} = \sum_{i=1}^{N} a_i \mathcal{D}_i P.$$

Now the problem of determining the smoothest function in the set of all $\mathcal{H}$-interpolants is related to a system of linear equations which needs to be solved to obtain the spline coefficients. Indeed, the application of the linear functionals $\{\mathcal{D}_1,...,\mathcal{D}_N\}$ to the $\mathcal{H}$ spline, yields N linear equations in the coefficients $a_1,...,a_N$

$$\sum_{j=1}^{N} a_j^N \mathcal{D}_i \mathcal{D}_j \mathcal{K}(\cdot,\cdot) = \mathcal{D}_i U, \quad i = 1,..., N.$$

The elements of the coefficients matrix $(\mathcal{D}_i\mathcal{D}_j\mathcal{K}(\cdot,\cdot))_{i,j=1,...,N}$ are given by

$$\mathcal{D}_i\mathcal{D}_j\mathcal{K}(\cdot,\cdot) = \int_{\Sigma^{int}} \frac{1}{|x_i - z||x_j - z|} dz.$$

Since the coefficient matrix as Gram matrix of the N linearly independent functions is non-singular, the linear system is uniquely solvable. Together with the set of linear bounded functionals and the reproducing kernel Hilbert space $\mathcal{H}$, the coefficients $a_1, ..., a_N$ define the unique interpolating spline we are looking for. Thus we can state

Lemma 2 (Uniqueness of interpolation). *For given $U \in \mathcal{H}$ there exist a unique element in $S_{\mathcal{H}}(\mathcal{D}_1, ..., \mathcal{D}_N) \cap \mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N}$.*

We denote this element by $S^U_{\mathcal{D}_1,...,\mathcal{D}_N}$. Moreover, we have the following

Lemma 3. *The interpolating $\mathcal{H}$-spline (relative to $\{\mathcal{D}_1, ..., \mathcal{D}_N\}$) is the $\mathcal{H}$-orthogonal projection of U onto the space $S_{\mathcal{H}}(\mathcal{D}_1, ..., \mathcal{D}_N)$.*

The upcoming lemmata give several properties, namely the minimum norm properties which also justify the use of the name `spline' for such interpolants.

Lemma 4 (First minimum property). If $P \in \mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N}$, then

$$\| P \|^2_{\mathcal{H}} = \| S^U_{\mathcal{D}_1,...,\mathcal{D}_N} \|^2_{\mathcal{H}} + \| S^U_{\mathcal{D}_1,...,\mathcal{D}_N} - P \|^2_{\mathcal{H}} .$$

Lemma 5 (Second minimum property). If $S \in S_{\mathcal{H}}(\mathcal{D}_1, ..., \mathcal{D}_N)$ and $P \in \mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N}$, then

$$\| S - P \|^2_{\mathcal{H}} = \| S^U_{\mathcal{D}_1,...,\mathcal{D}_N} - P \|^2_{\mathcal{H}} + \| S - S^U_{\mathcal{D}_1,...,\mathcal{D}_N} \|^2_{\mathcal{H}} .$$

Summarizing our results we finally find

Theorem 2: *The interpolation problem*

$$\| S^U_{\mathcal{D}_1,...,\mathcal{D}_N} \|_{\mathcal{H}} = \inf_{P \in \mathcal{I}^U_{\mathcal{D}_1,...,\mathcal{D}_N}} \| P \|_{\mathcal{H}},$$

is well-posed in the sense that its solution exists, is unique, and depends continuously on the data $\alpha_1, ..., \alpha_N$. The uniquely determined solution $S^U_{\mathcal{D}_1,...,\mathcal{D}_N}$ is given in the explicit form

$$S^{U}_{\mathcal{D}_1,\ldots,\mathcal{D}_N}(x) = \sum_{i=1}^{N} a_i^N \int_{\Sigma^{int}} \frac{1}{|x_i - z||x - z|} dz, \quad x \in \overline{\Sigma^{ext}},$$

where the coefficients $a_1^N, \ldots, a_N^N$ *satisfy the linear equations*

$$\sum_{i=1}^{N} a_i^N \int_{\Sigma^{int}} \frac{1}{|x_i - z||x_j - z|} dz = \alpha_j, \quad j = 1, \ldots, N.$$

Remark. It should be noted that the requirement for the linear independence of the given bounded linear functionals is not necessary from the theoretical point of view, but essential for numerical computations. It guarantees that the $\mathcal{H}$-spline coefficients are uniquely determined, i.e., that the linear equation system is uniquely solvable. Without linear independence of the functionals, the dimension of the spline space is smaller than N, and the coefficients of the interpolating $\mathcal{H}$-spline of U relative to $\{\mathcal{D}_1, \ldots, \mathcal{D}_N\}$ are no longer uniquely determined. Nevertheless, the interpolating $\mathcal{H}$-spline is the uniquely determined orthogonal projection of U onto the spline space and all the spline properties are still valid.

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