LAST MULTIPLIERS FOR $\eta$-RICCI SOLITONS

ADARA M. BLAGA

Abstract. Last multipliers are determined for the $\eta$-Ricci soliton tensor. In the particular case of a perfect fluid spacetime, if the potential vector field of the $\eta$-Ricci soliton is of gradient type, we obtain the differential equation satisfied by the last multiplier.

1. Introduction

The notion of last multiplier was introduced by Jacobi [10] as a solution of Liouville's transport equation and measures, in a way, the "deviation" of a multivector field from being divergence-free. The theory of last multipliers continued to develop from the general framework of ODE [1] to the Riemannian and Poisson cases, extended from vector fields to multivector fields [6], [8] by means of the curl operator.

Let $(M, g)$ be a Riemannian manifold and denote by $\flat : TM \to T^*M$, $\flat(X) := i_X g$, $\sharp : T^*M \to TM$, $\sharp := \flat^{-1}$. Consider the set $T_{2, s}^0(M)$ of symmetric $(0, 2)$-tensor fields on $M$ and for $Z \in T_{2, s}^0(M)$, denote by $Z^\sharp : TM \to TM$ and $Z^\sharp : T^*M \to T^*M$ the maps defined by:

$$g(Z^\sharp(X), Y) := Z(X, Y), \quad Z^\sharp(\alpha)(X) := Z(\sharp(\alpha), X).$$

We also identify $Z_\sharp$ by the map also denoted by $Z_\sharp : T^*M \times TM \to C^\infty(M)$:

$$Z_\sharp(\alpha, X) := Z_\sharp(\alpha)(X).$$

The divergence of $Z$ with respect to $g$ is:

$$\text{div}(Z) := \text{trace}(\nabla Z^\sharp),$$

where $\nabla$ is the Levi-Civita connection of $g$ and by a direct computation, for any smooth function $f$ on $M$, we obtain:

$$\text{div}(fZ) = Z_\sharp(df) + f \text{div}(Z).$$

(1)

The notion of last multiplier for a symmetric $(0, 2)$-tensor field on a Riemannian manifold $(M, g)$ was introduced in [5] as a smooth positive
function \( f \) on \( M \) such that:
\[
div(fZ) = 0
\]  
which is equivalent to the *Liouville equation* of \( Z \):
\[
Z_\sharp(d(ln f)) = -div(Z). \tag{3}
\]

Remark that if \( Z \) is of divergence free, then the last multipliers of \( Z \) with respect to \( g \) are constant functions.

Let \( T^1_{1,i}(M) \) be the set of invertible \((1,1)\)-tensor fields on \( M \) i.e. those for which the associated \((0,2)\)-tensor field is non-singular and remark that \( Z^\sharp \) is non-singular if and only if \( Z_\sharp \) is non-singular.

Let \( Z \) be a non-degenerate symmetric \((0,2)\)-tensor field on the Riemannian manifold \((M, g)\) and denote by \( \omega_Z := Z^\sharp_\sharp^{-1}(div(Z)) \) its *Jacobi form*.

**Definition 1.** [7] We call \((M, g, Z)\):

i) exact modular manifold if the \( \omega_Z \) is exact and call \( u \in C^\infty(M) \), with \( \omega_Z = du \), the potential of \( Z \);

ii) closed modular manifold if \( \omega_Z \) is closed and call \( [\omega_Z] \in H^1(M) \) the modular class of \( M \).

**Theorem 1.** [7] Let \( Z \in T^0_{2,s}(M) \) be a non-degenerate tensor on the Riemannian manifold \((M, g)\). Then:

(1) \((M, g, Z)\) is exact modular manifold if and only if \( Z \) admits last multipliers; in this case, if \( u \) is a potential function of \( Z \), then a last multiplier \( f \) for \( Z \) satisfies \( f = C \cdot \exp(-u), \) \( C > 0 \);

(2) \((M, g, Z)\) is closed modular manifold if and only if \( Z^\sharp \in \ker(d \circ Z^\sharp_\sharp^{-1} \circ \text{trace} \circ \nabla) \).

2. Last multipliers for \( \eta \)-Ricci soliton tensor

Consider now the following particular cases:

1. \( Z := g \)

\[
div(Z) = 0
\]

\[
Z^\sharp = I_\chi(M), \quad Z_\sharp = I_{\Omega^1(M)}
\]

\( Z \) admits as last multipliers all the constant functions since \((M, g, g)\) is exact modular manifold with the Jacobi form

\[
\omega_Z = 0.
\]

2. \( Z := Ric \)

\[
div(Z) = \frac{1}{2}d(\text{scal})
\]

\[
Z^\sharp = Q
\]
Assume that $Z$ is non-degenerate. Then $Z$ admits last multipliers if and only if $(M, g, \text{Ric})$ is exact modular manifold with the Jacobi form

$$\omega_Z = \frac{1}{2} \text{Ric}_g^{-1}(d(\text{scal}))$$

exact and the potential function $u$, where $du = \omega_Z$, and a last multiplier $f$ for $Z$ satisfies $f = C \cdot \exp(-u)$, $C > 0$.

3. $Z := \text{Hess}(u)$

$$\text{div}(Z) = d(\Delta(u)) + i_Q(\text{grad}(u))g$$

$$Z^\sharp = Q$$

Assume that $Z$ is non-degenerate. Then $Z$ admits last multipliers if and only if $(M, g, \text{Hess}(u))$ is exact modular manifold with the Jacobi form

$$\omega_Z = \text{Hess}(u)^{-1}_g(d(\Delta(u)) + i_Q(\text{grad}(u))g)$$

exact and the potential function $u$, where $du = \omega_Z$, and a last multiplier $f$ for $Z$ satisfies $f = C \cdot \exp(-u)$, $C > 0$.

4. $Z := \eta \otimes \eta$

$$\text{div}(Z) = \text{div}(\xi)\eta + \nabla_\xi \eta$$

$$Z^\sharp = \eta \otimes \xi$$

Then $Z$ admits last multipliers if and only if $(M, g, \eta \otimes \eta)$ is exact modular manifold with the Jacobi form

$$\omega_Z = (\eta \otimes \eta)^{-1}_g(d(\text{div}(\xi)\eta + \nabla_\xi \eta))$$

exact and the potential function $u$, where $du = \omega_Z$, and a last multiplier $f$ for $Z$ satisfies $f = C \cdot \exp(-u)$, $C > 0$.

Consider the equation:

$$\mathcal{L}_\xi g + 2\text{Ric} + 2ag + 2b\eta \otimes \eta = 0 \quad (4)$$

where $g$ is a pseudo-Riemannian metric on the $n$-dimensional manifold $M$, $\text{Ric}$ its Ricci curvature, $\eta$ a 1-form, $\xi$ a vector field, $\mathcal{L}_\xi$ the Lie derivative along $\xi$ and $a$ and $b$ are real constants. The data $(g, \xi, a, b)$ which satisfy the equation (4) is said to be an $\eta$-Ricci soliton on $M$ [4]; in particular, if $b = 0$, $(g, \xi, a)$ is a Ricci soliton [9]. If the potential vector field $\xi$ is of gradient type, $\xi = \text{grad}(u)$, for $u$ a smooth function on $M$, then $(g, \xi, a, b)$ is called gradient $\eta$-Ricci soliton.

Assume now that $\eta$ is the $g$-dual of the vector field $\xi$.

Taking the trace of the equation (4), we obtain:

$$\text{div}(\xi) + \text{scal} + an + b|\xi|^2 = 0 \quad (5)$$
and taking the divergence of the same equation we get:
\[
\frac{1}{2} [d(div(\xi)) + i\eta g] + \frac{1}{2} d(scal) + b[div(\xi)\eta + \nabla_\xi \eta] = 0. \tag{6}
\]

Let \( u \in C^\infty(M), \xi := \text{grad}(u), \eta := i\xi \). Then \( \eta \) is closed and
\[
g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X), \tag{7}
\]
for any \( X, Y \in \mathfrak{X}(M) \). Also [2]:
\[
\text{trace}(\eta \otimes \eta) = |\xi|^2 \tag{8}
\]
\[
div(\eta \otimes \eta) = div(\xi)\eta + \frac{1}{2} d(|\xi|^2) \tag{9}
\]
and
\[
\nabla_\xi \eta = \frac{1}{2} d(|\xi|^2). \tag{10}
\]

For \( a \) and \( b \) real constants, let \( Z^a_{a,b} := \text{Hess}(u) + \text{Ric} + ag + bdu \otimes du \) be the symmetric \((0, 2)\)-tensor field called by us \( \eta\)-Ricci soliton tensor.

**Definition 2.** We say that \( f \in C^\infty_+(M) \) is a last multiplier for the \( \eta\)-Ricci soliton tensor \( Z^a_{a,b} \) if
\[
div(f Z^a_{a,b}) = 0. \tag{11}
\]

From the above considerations, we obtain:

**Theorem 2.** If \( f \in C^\infty_+(M) \) is a last multiplier for the \( \eta\)-Ricci soliton tensor \( Z^a_{a,b} \), then:
\[
f\{d(\Delta(u)) + i\eta g + \frac{1}{2} d(scal) + b\Delta(u)du + b_\nabla\eta(\text{grad}(u))g\} + 
+ i\nabla_\text{grad}(f)\text{grad}(u)g + i\eta \text{grad}(f)g + adf + bdu(\text{grad}(f))du = 0. \tag{11}
\]

**Proof.** From a direct computation we obtain:
\[
\text{Hess}(u)_2(df) = i\nabla_\text{grad}(f)\text{grad}(u)g \\
\text{Ric}_2(df) = i\eta \text{grad}(f)g \\
g_2(df) = df \\
(du \otimes du)_2(df) = du(\text{grad}(f))du,
\]
and from the last multiplier condition we obtain:
\[
f\{d(\Delta(u)) + i\eta \text{grad}(u)g + \frac{1}{2} d(scal) + b\Delta(u)du + b_\nabla\text{grad}(u)\text{grad}(u)g\} + 
+ i\nabla_\text{grad}(f)\text{grad}(u)g + i\eta \text{grad}(f)g + adf + bdu(\text{grad}(f))du = 0.
\]

**Remark 1.** In the Ricci soliton case, (11) becomes:
\[
f\{d(\Delta(u)) + i\eta \text{grad}(u)g + \frac{1}{2} d(scal)\} + i\nabla_\text{grad}(f)\text{grad}(u)g + i\eta \text{grad}(f)g + adf = 0.
\]
3. Applications to Perfect Fluid Spacetime

A perfect fluid can be completely characterized by its rest frame mass density and isotropic pressure. It has no shear stresses, viscosity, nor heat conduction and is characterized by an energy-momentum tensor of the form:

\[ T(X,Y) = \rho g(X,Y) + (\sigma + p)\eta(X)\eta(Y), \quad (12) \]

for any \( X, Y \in \mathcal{X}(M) \), where \( \rho \) is the isotropic pressure, \( \sigma \) is the energy-density, \( \rho g \) is the metric tensor of Minkowski spacetime, \( \xi := \sharp(\eta) \) is the velocity vector of the fluid and \( g(\xi, \xi) = -1 \). If \( \sigma = -p \), the energy-momentum tensor is Lorentz-invariant \( (T = -\sigma g) \) and in this case we talk about the vacuum. If \( \sigma = 3p \), the medium is a radiation fluid.

The field equations governing the perfect fluid motion are Einstein's gravitational equations:

\[ kT = \nabla + (\lambda - \frac{scal}{2})g, \quad (13) \]

where \( \lambda \) is the cosmological constant, \( k \) is the gravitational constant (which can be taken \( 8\pi G \)), with \( G \) the universal gravitational constant), \( \nabla \) is the Ricci tensor and \( scal \) is the scalar curvature of \( g \).

Replacing \( T \) from (12) we obtain:

\[ \nabla = -\left(\lambda - \frac{scal}{2} - kp\right)g + k(\sigma + p)\eta \otimes \eta \quad (14) \]

and the scalar curvature of \( M \) is:

\[ scal = 4\lambda + k(\sigma - 3p). \quad (15) \]

From [3] we know that for \( (M, g) \) a 4-dimensional pseudo-Riemannian manifold and \( \eta := du \) the \( g \)-dual 1-form of the gradient vector field \( \xi := \nabla(u) \) with \( g(\xi, \xi) = -1 \), if (4) defines an \( \eta \)-Ricci soliton in \( M \), then:

\[ \Delta(u) = -3[b + k(\sigma + p)]. \quad (16) \]

With these considerations, we can state:

**Theorem 3.** If \( f \in C^\infty_+(M) \) is a last multiplier for the \( \eta \)-Ricci soliton tensor \( Z_{a,b}^\eta \) in a perfect fluid spacetime, then:

\[ f\left\{-\frac{k}{2}(5\sigma + 9dp) + \left[\lambda - 3b^2 + \frac{k}{2}(3(1 - 2b)\sigma + (1 - 6b)p)\right]du + bi\nabla_{\nabla(u)}\nabla(u)g\right\} + \
+ i\nabla_{\nabla(f)}\nabla(u)g + (\lambda + \frac{k}{2}(\sigma - p))i_{\nabla(f)}g + adf + (b + k(\sigma + p))du(\nabla(f))du = 0. \quad (17) \]

**Remark 2.** In a radiation fluid spacetime, a last multiplier \( f \) for the \( \eta \)-Ricci soliton tensor satisfies:

\[ f\left\{-12kp + \left[\lambda - 3b^2 + k(5 - 12b)p\right]du + bi\nabla_{\nabla(u)}\nabla(u)g\right\} + \
+ i\nabla_{\nabla(f)}\nabla(u)g + (\lambda + kp)i_{\nabla(f)}g + adf + (b + 4kp)du(\nabla(f))du = 0 \]
and computing the previous relation in $\xi$ we deduce that:

$$
\frac{\xi(f)}{f} = \frac{\lambda - 3b^2 + k(5 - 12b)p + 12k\xi(p)}{\lambda + a - b - 3kp}.
$$

**REFERENCES**


**DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE**

**WEST UNIVERSITY OF TIMIȘOARA**

**Bld. V. Pârvan nr. 4, 300223, Timișoara, România**

**E-mail address:** adarablaga@yahoo.com