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LAST MULTIPLIERS FOR η-RICCI SOLITONS

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Abstract. Last multipliers are determined for the η-Ricci soliton tensor. In the particular case of a perfect fluid spacetime, if the potential vector field of the η-Ricci soliton is of gradient type, we obtain the differential equation satisfied by the last multiplier.

1. Introduction

The notion of *last multiplier* was introduced by Jacobi [10] as a solution of Liouville's transport equation and measures, in a way, the "deviation" of a multivector field from being divergence-free. The theory of last multipliers continued to develop from the general framework of ODE [1] to the Riemannian and Poisson cases, extended from vector fields to multivector fields [6], [8] by means of the curl operator.

Let (M, g) be a Riemannian manifold and denote by $\flat : TM \to T^*M$, $\flat(X) := i_X g$, $\sharp : T^*M \to TM$, $\sharp := \flat^{-1}$. Consider the set $\mathcal{T}^0_{2,s}(M)$ of symmetric $(0,2)$-tensor fields on M and for $Z \in \mathcal{T}^0_{2,s}(M)$, denote by $Z^\sharp : TM \to TM$ and $Z_\sharp : T^*M \to T^*M$ the maps defined by:

$$g(Z^\sharp(X), Y) := Z(X, Y), \quad Z_\sharp(\alpha)(X) := Z(\sharp(\alpha), X).$$

We also identify $Z_\sharp$ by the map also denoted by $Z_\sharp : T^*M \times TM \to C^\infty(M)$:

$$Z_\sharp(\alpha, X) := Z_\sharp(\alpha)(X).$$

The divergence of Z with respect to g is:

$$div(Z) := trace(\nabla Z^\sharp),$$

where ∇ is the Levi-Civita connection of g and by a direct computation, for any smooth function f on M, we obtain:

$$div(fZ) = Z_\sharp(df) + f div(Z). \tag{1}$$

The notion of *last multiplier* for a symmetric $(0,2)$-tensor field on a Riemannian manifold (M, g) was introduced in [5] as a smooth positive

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function f on M such that:

$$div(fZ) = 0 \tag{2}$$

which is equivalent to the *Liouville equation* of Z:

$$Z_\sharp(d(\ln f)) = -div(Z). \tag{3}$$

Remark that if Z is of divergence free, then the last multipliers of Z with respect to g are constant functions.

Let $\mathcal{T}^1_{1,i}(M)$ be the set of invertible $(1,1)$-tensor fields on M i.e. those for which the associated $(0,2)$-tensor field is non-singular and remark that $Z^\sharp$ is non-singular if and only if $Z_\sharp$ is non-singular.

Let Z be a non-degenerate symmetric $(0,2)$-tensor field on the Riemannian manifold (M,g) and denote by $\omega_Z := Z_\sharp^{-1}(div(Z))$ its *Jacobi form*.

Definition 1. [7] *We call* (M,g,Z)*:*

i) exact modular manifold if the ω_Z *is exact and call* $u \in C^\infty(M)$*, with* $\omega_Z = du$*, the potential of* Z*;*

ii) closed modular manifold if ω_Z *is closed and call* $[\omega_Z] \in H^1(M)$ *the modular class of* M*.*

Theorem 1. [7] *Let* $Z \in \mathcal{T}^0_{2,s}(M)$ *be a non-degenerate tensor on the Riemannian manifold* (M,g)*. Then:*

(1) (M,g,Z) *is exact modular manifold if and only if* Z *admits last multipliers; in this case, if* u *is a potential function of* Z*, then a last multiplier* f *for* Z *satisfies* $f = C\cdot\exp(-u)$*,* $C>0$*;*

(2) (M,g,Z) *is closed modular manifold if and only if* $Z^\sharp \in \ker(d \circ Z_\sharp^{-1} \circ trace \circ \nabla)$*.*

2. Last multipliers for η-Ricci soliton tensor

Consider now the following particular cases:

1. $Z := g$

$$div(Z) = 0$$

$$Z^\sharp = I_{\mathfrak{X}(M)}, \quad Z_\sharp = I_{\Omega^1(M)}$$

Z admits as last multipliers all the constant functions since (M,g,g) is exact modular manifold with the Jacobi form

$$\omega_Z = 0.$$

2. $Z := Ric$

$$div(Z) = \frac{1}{2}d(scal)$$

$$Z^\sharp = Q$$

Assume that Z is non-degenerate. Then Z admits last multipliers if and only if (M, g, Ric) is exact modular manifold with the Jacobi form

$$\omega_Z = \frac{1}{2} Ric_{\sharp}^{-1}(d(scal))$$

exact and the potential function u, where $du = \omega_Z$, and a last multiplier f for Z satisfies $f = C \cdot \exp(-u)$, $C > 0$.

3. $Z := Hess(u)$

$$div(Z) = d(\Delta(u)) + i_{Q(grad(u))} g$$

$$Z^{\sharp} = Q$$

Assume that Z is non-degenerate. Then Z admits last multipliers if and only if $(M, g, Hess(u))$ is exact modular manifold with the Jacobi form

$$\omega_Z = Hess(u)_{\sharp}^{-1}(d(\Delta(u)) + i_{Q(grad(u))} g)$$

exact and the potential function u, where $du = \omega_Z$, and a last multiplier f for Z satisfies $f = C \cdot \exp(-u)$, $C > 0$.

4. $Z := \eta \otimes \eta$

$$div(Z) = div(\xi)\eta + \nabla_{\xi}\eta$$

$$Z^{\sharp} = \eta \otimes \xi$$

Then Z admits last multipliers if and only if $(M, g, \eta \otimes \eta)$ is exact modular manifold with the Jacobi form

$$\omega_Z = (\eta \otimes \eta)_{\sharp}^{-1}(div(\xi)\eta + \nabla_{\xi}\eta)$$

exact and the potential function u, where $du = \omega_Z$, and a last multiplier f for Z satisfies $f = C \cdot \exp(-u)$, $C > 0$.

Consider the equation:

$$\pounds_{\xi} g + 2Ric + 2ag + 2b\eta \otimes \eta = 0 \tag{4}$$

where g is a pseudo-Riemannian metric on the n-dimensional manifold M, Ric its Ricci curvature, η a 1-form, ξ a vector field, $\pounds_{\xi}$ the Lie derivative along ξ and a and b are real constants. The data (g, ξ, a, b) which satisfy the equation (4) is said to be an *η-Ricci soliton* on M [4]; in particular, if $b = 0$, (g, ξ, a) is a *Ricci soliton* [9]. If the potential vector field ξ is of gradient type, $\xi = grad(u)$, for u a smooth function on M, then (g, ξ, a, b) is called *gradient η-Ricci soliton*.

Assume now that η is the g-dual of the vector field ξ.

Taking the trace of the equation (4), we obtain:

$$div(\xi) + scal + an + b|\xi|^2 = 0 \tag{5}$$

and taking the divergence of the same equation we get:

$$\frac{1}{2}[d(div(\xi)) + i_{Q\xi}g] + \frac{1}{2}d(scal) + b[div(\xi)\eta + \nabla_\xi \eta] = 0. \tag{6}$$

Let $u \in C^\infty(M)$, $\xi := grad(u)$, $\eta := i_\xi g$. Then η is closed and

$$g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X), \tag{7}$$

for any X, $Y \in \mathfrak{X}(M)$. Also [2]:

$$trace(\eta \otimes \eta) = |\xi|^2 \tag{8}$$

$$div(\eta \otimes \eta) = div(\xi)\eta + \frac{1}{2}d(|\xi|^2) \tag{9}$$

and

$$\nabla_\xi \eta = \frac{1}{2}d(|\xi|^2).$$

For a and b real constants, let $Z^\eta_{a,b} := Hess(u) + Ric + ag + bdu \otimes du$ be the symmetric $(0, 2)$-tensor field called by us *η-Ricci soliton tensor*.

Definition 2. *We say that $f \in C^\infty_+(M)$ is a last multiplier for the η-Ricci soliton tensor $Z^\eta_{a,b}$ if*

$$div(fZ^\eta_{a,b}) = 0. \tag{10}$$

From the above considerations, we obtain:

Theorem 2. *If $f \in C^\infty_+(M)$ is a last multiplier for the η-Ricci soliton tensor $Z^\eta_{a,b}$, then:*

$$f\{d(\Delta(u)) + i_{Q(grad(u))}g + \frac{1}{2}d(scal) + b\Delta(u)du + bi_{\nabla_{grad(u)}grad(u)}g\}+$$
$$+i_{\nabla_{grad(f)}grad(u)}g + i_{Q(grad(f))}g + adf + bdu(grad(f))du = 0. \tag{11}$$

Proof. From a direct computation we obtain:

$$Hess(u)_\sharp(df) = i_{\nabla_{grad(f)}grad(u)}g$$
$$Ric_\sharp(df) = i_{Q(grad(f))}g$$
$$g_\sharp(df) = df$$
$$(du \otimes du)_\sharp(df) = du(grad(f))du,$$

and from the last multiplier condition we obtain:

$$f\{d(\Delta(u)) + i_{Q(grad(u))}g + \frac{1}{2}d(scal) + b\Delta(u)du + bi_{\nabla_{grad(u)}grad(u)}g\}+$$
$$+i_{\nabla_{grad(f)}grad(u)}g + i_{Q(grad(f))}g + adf + bdu(grad(f))du = 0.$$

□

Remark 1. *In the Ricci soliton case, (11) becomes:*

$$f\{d(\Delta(u))+i_{Q(grad(u))}g+\frac{1}{2}d(scal)\}+i_{\nabla_{grad(f)}grad(u)}g+i_{Q(grad(f))}g+adf = 0.$$

3. Applications to perfect fluid spacetime

A perfect fluid can be completely characterized by its rest frame mass density and isotropic pressure. It has no shear stresses, viscosity, nor heat conduction and is characterized by an energy-momentum tensor of the form:

$$T(X,Y) = pg(X,Y) + (\sigma + p)\eta(X)\eta(Y), \tag{12}$$

for any X, $Y \in \mathfrak{X}(M)$, where p is the isotropic pressure, σ is the energy-density, g is the metric tensor of Minkowski spacetime, $\xi := \sharp(\eta)$ is the velocity vector of the fluid and $g(\xi,\xi) = -1$. If $\sigma = -p$, the energy-momentum tensor is Lorentz-invariant ($T = -\sigma g$) and in this case we talk about the *vacuum*. If $\sigma = 3p$, the medium is a *radiation fluid*.

The field equations governing the perfect fluid motion are Einstein's gravitational equations:

$$kT = Ric + (\lambda - \frac{scal}{2})g, \tag{13}$$

where λ is the cosmological constant, k is the gravitational constant (which can be taken $8\pi G$, with G the universal gravitational constant), Ric is the Ricci tensor and $scal$ is the scalar curvature of g.

Replacing T from (12) we obtain:

$$Ric = -(\lambda - \frac{scal}{2} - kp)g + k(\sigma + p)\eta \otimes \eta \tag{14}$$

and the scalar curvature of M is:

$$scal = 4\lambda + k(\sigma - 3p). \tag{15}$$

From [3] we know that for (M,g) a 4-dimensional pseudo-Riemannian manifold and $\eta := du$ the g-dual 1-form of the gradient vector field $\xi := grad(u)$ with $g(\xi,\xi) = -1$, if (4) defines an η-Ricci soliton in M, then:

$$\Delta(u) = -3[b + k(\sigma + p)]. \tag{16}$$

With these considerations, we can state:

Theorem 3. *If $f \in C^{\infty}_{+}(M)$ is a last multiplier for the η-Ricci soliton tensor $Z^{\eta}_{a,b}$ in a perfect fluid spacetime, then:*

$$f\{-\frac{k}{2}(5d\sigma+9dp)+[\lambda-3b^2+\frac{k}{2}(3(1-2b)\sigma+(1-6b)p)]du+bi_{\nabla_{grad(u)}grad(u)}g\}+$$
$$+i_{\nabla_{grad(f)}grad(u)}g+(\lambda+\frac{k}{2}(\sigma-p))i_{grad(f)}g+adf+(b+k(\sigma+p))du(grad(f))du=0. \tag{17}$$

Remark 2. *In a radiation fluid spacetime, a last multiplier f for the η-Ricci soliton tensor satisfies:*

$$f\{-12kdp + [\lambda - 3b^2 + k(5 - 12b)p]du + bi_{\nabla_{grad(u)}grad(u)}g\}+$$
$$+i_{\nabla_{grad(f)}grad(u)}g + (\lambda + kp)i_{grad(f)}g + adf + (b + 4kp)du(grad(f))du = 0$$

and computing the previous relation in ξ we deduce that:

$$\frac{\xi(f)}{f}=\frac{\lambda-3b^2+k(5-12b)p+12k\xi(p)}{\lambda+a-b-3kp}.$$

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