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FACTORIZATION OF POLYNOMIAL DIFFERENTIAL OPERATORS

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1. Let a differential operator be given

$$P(D)=D^n+B_{n-1}D^{n-1}+\cdots+B_1D+B_0,\ D\equiv\frac{d}{dx},\ n=2,3,\ldots, \tag{1.1}$$

where $B_i=B_i(x)\,(i=0,1,\ldots,n-1)$ are arbitrary differentiable functions on some interval. Let us suppose that $P(D)$ can be written in the form

$$P(D)=(D+\xi)(D^{n-1}+\alpha_{n-2}D^{n-2}+\cdots+\alpha_1 D\cdot\alpha_0). \tag{1.2}$$

From the condition of equality of (1.1) and (1.2), we find the following system of equations:

$$\begin{aligned}&\xi+\alpha_{n-2}=B_{n-1}\\ &\xi\alpha_j+\alpha_j'+\alpha_{j-1}=B_j,\ j=1,2,\ldots,n-2\\ &\xi\alpha_0+\alpha_0'=B_0\left(\alpha'=\frac{d\alpha}{dx}\right).\end{aligned} \tag{1.3}$$

From the first $(n-1)$ equations of this system we find the following expressions for the coefficients $\alpha_j\,(j=0,1,\ldots,n-2)$

$$\alpha_{j-1}=B_j+\sum_{k=1}^{n-1-j}(-1)^k A_{j+k}^{[k]}+(-1)^{n-j}\psi^{[n-j-1]}{}_{j=1,2,\ldots,n-1}, \tag{1.4}$$

where following notations are introduced:

$$\begin{aligned}&A_{j+k}^{[k]}=\xi A_{j+k}^{[k-1]}+(A_{j+k}^{[k-1]})'\\ &A_{j+k}^{[1]}=\xi B_{j+k}+B_{j+k}',\quad A_{j+k}^{[0]}=B_{j+k}\\ &\psi^{[k]}=\xi\psi^{[k-1]}+(\psi^{[k-1]})',\quad \psi^{[0]}=\xi;\quad \left(\psi'=\frac{d\psi}{dx}\right).\end{aligned} \tag{1.5}$$

On the basis of (1.4) and (1.5), from the last equation of the system (1.3) we find

$$\xi\left(B_1+\sum_{k=1}^{n-2}(-1)^k A_{k+1}^{[k]}+(-1)^{n-1}\psi^{[n-2]}\right)+ \tag{1.6}$$
$$+\left(B_1+\sum_{k=1}^{n-2}(-1)^k A_{k+1}^{[k]}+(-1)^{n-1}\psi^{[n-2]}\right)'=B_0.$$

2. We shall inspect the case when the coefficients $B_i(x)$ are polynomials in x respectively with degrees $b_i\,(i=0, 1, \ldots, n-1)$ and we shall try to give an answer to the question: Can the operator (1.1) be factorized in the form (1.2), where $\xi(x)$, $\alpha_i(x)\,(i=0, 1, \ldots, n-2)$ are also polynomials in x? Taking into consideration that the relations (1.4) determine the coefficients $\alpha_j(x)$ $(j=0, 1, \ldots, n-2)$ as a function of $\xi(x)$, the following result is valid.

Lemma 1. *The differential operator* (1.1) *is factorized in the form* (1.2), *where* $\xi(x)$, $\alpha_i(x)\,(i=0, 1, \ldots, n-2)$ *are polynomials if and only if the equation* (1.6) *has at least one polynomial solution for* $\xi(x)$.

If we suppose that the polynomial

$$\xi(x)=H_m(x)=C_m x^m+C_{m-1}x^{m-1}+\cdots+C_0, \tag{2.1}$$

satisfies the equation (1.6) *then, on the basis of* (1.5) *it is clear that* $\psi^{[k]}$ *is also a polynomial in* x *for which the following result holds.*

Lemma 2. *If we have* (2.1) *and if we determine the polynomial* $R_k(x)$, *so that*

$$\psi^{[k]}=\xi^{k+1}+R_k, \qquad k=1, 2, \ldots, n-2,$$

then

$$d(R_k)<d(\xi^k). \tag{2.2}$$

Proof: For $k=1$ we have

$$\psi^{[1]}=\xi\,\psi^{[0]}+(\psi^{[0]})'=\xi^2+\xi'=\xi^2+R_1;$$

so $d(R_1)=d(\xi')<d(\xi)$. Let (2.2) be correct for some k. Since

$$\psi^{[k+1]}=\xi\,\psi^{[k]}+(\psi^{[k]})'=\xi\,(\xi^{k+1}+R_k)+(\xi^{k+1}+R_k)'=$$
$$\xi^{k+2}+\{\xi R_k+(k+1)\,\xi^k\,\xi'+R_k'\},$$

then

$$R_{k+1}=\xi R_k+(k+1)\,\xi^k\,\xi'+R_k'$$

i.e.

$$d(R_{k+1})=\max\{d(\xi R_k),\ d(\xi'\xi^k),\ d(R_k')\}.$$

According to the supposition, $d(\xi R_k)=d(\xi)+d(R_k)<d(\xi^{k+1})$ and $d(\xi'\xi^k)$ $<d(\xi^{k+1})$, it follows $d(R_{k+1})<d(\xi^{k+1})$.

The proof is completed.

In the same way we can prove

Lemma 3. *If we have* (2.1) *and if we determine the polynomial so that*

$$A^{[k]}_{k+1}=B_{k+1}\,\xi^k+P_k,$$

then

(2.3) $$d(P_k)<d(B_{k+1}\,\xi^{k-1}).$$

If (2.1) holds then on bases of lemmas 2. and 3., the equations (1.6) can be written as

(2.4) $$(-1)^{n-1}\xi^n+\sum_{k=1}^{n-1}(-1)^{k-1}B_k\,\xi^k+\sum_{k=0}^{n-2}(-1)^k P^*_k+(-1)^{n-1}R^*_{n-2}=B_0,$$

where

$$R^*_{n-2}=\xi\,R_{n-2}+(n-1)\,\xi'\,\xi^{n-2}+R'_{n-2}$$

$$P^*_k=\xi\,P_k+B'_{k+1}\,\xi^k+k\,B_{k+1}\,\xi'\xi^{k-1}+P'_k.$$

On the basis of (2.2) and (2.3) it is clear that

(2.5) $$d(P^*_k)<d(B_{k+1}\,\xi^k);\quad d(R^*_{n-2})<d(\xi^{n-1}).$$

Lemma 4. *Let the numbers* $b_i\,(i=0,\ 1,\ \ldots,\ n-1)$ *satisfy following expressions*

(i) $b_0=nm;\quad m=1,\ 2,\ \ldots$

(ii) $b_i<\dfrac{n-(i+1)}{n}\,b_0,\quad i=1,\ 2,\ \ldots,\ n-2;\quad b_{n-1}<\dfrac{1}{2}\,\dfrac{b_0}{n}$

(iii) $b_{i+1}<\dfrac{n-(i+1)}{n-i}\,b_i,\quad i=1,\ 2,\ \ldots,\ n-2.$

Then the equations (1.6) *can be satisfied only by polynomials of degree* $m=b_0/n$. *If* b_0 *is not a multiple from* n, *and the supposition* (ii) *and* (iii) *are fulfilled, then the same equation has no polynomial solutions.*

Proof: If the equation (1.6) is satisfied by some polynomial (2.1), then the same equation can be written in the form (2.4). According to (2.5), the terms with the highest degrees in the equation (2.4) are always determined by the expressions in which

$P^*_k\,(k=0,\ 1,\ \ldots,\ n-2)$ and R^*_{n-2} do not figure. It means that the degrees of the polynomials which satisfy the equation (2.4), are equal to the corresponding degrees of the polynomial solutions of the differential equation $y'=y^n+B_{n-1}y^{n-1}+\cdots+B_1y+B_0$. Bhargava and Kaufman [1] have shown the last equation, under the supposition that (i), (ii) and (iii) are valid, can have polynomial solutions only with degree $m=b_0/n$. According to what we have said before, it is the same case with the equation (2.4) and accordingly with the equation (1.6).

If the conditions of lemma 4. are fulfilled and the polynomial (2.1) $(m=b_0/n)$ satisfies the equation (1.6), then on the bases of (2.5), the equation (2.4) can be written in the form

$$(-1)^{n-1}\xi^n+(-1)^{n-2}B_{n-1}\xi^{n-1}+\cdots+B_1\xi+T=B_0 \tag{2.6}$$

$$T=T(\xi,\ \xi',\ \ldots,\ \xi^{(n-1)},\ B_1,\ \ldots,\ B_{n-1}),$$

and for the polynomials $T(x)$ and $B_j(x)\{\xi(x)\}^j$ we have

$$d(T)<(n-1)(b_0/n) \tag{2.7}$$

$$d(B_j\xi^j)<(n-1)(b_0/n),\ j=1,\ 2,\ \ldots,\ n-2.$$

Definition. If $G_{mn}(x)$ is a polynomial of degree $mn\,(n=2,\ 3,\ \ldots;\ m=1,\ 2,\ \ldots)$; then we use $S=[\sqrt[n]{G_{mn}(x)}]$ to denote the polynomial part of the development of $\sqrt[n]{G_{mn}(x)}$ on whole degrees on x.

So, for example $[\sqrt{x^4-2x^3+x-6}\,]=x^2-x-\dfrac{1}{2}.$

If we determine the polynomial Q as

$$G=S^n+Q,\ S=[\sqrt[n]{G}\,].$$

then [1] we have

$$d(Q)<d(S^{n-1}). \tag{2.8}$$

Let us determine now the polynomials S and Q as

$$(-1)^{n-1}B_0=S^n+Q,\ S=[\sqrt[n]{(-1)^{n-1}B_0}\,]. \tag{2.9}$$

Since

$$[\sqrt[n]{H^n-B_{n-1}H^{n-1}}\,]=$$

$$=\text{polynomial part of } H\{1-(B_{n-1}/H)\}^{1/n}=$$

$$=\text{polynomial part of } H\left\{1-\frac{B_{n-1}}{nH}+\binom{1/n}{2}\frac{B_{n-1}^2}{H^2}+\cdots\right\}=H-\frac{1}{n}B_{n-1},$$

then we can determine the polynomials L and M as

$$H^n-B_{n-1}H^{n-1}=L^n+M,\ L=H-\frac{1}{n}B_{n-1}. \tag{2.10}$$

On the basis of (2.8), for the polynomials Q and M we have

$$d(Q)<d(S^{n-1});\quad d(M)<d(L^{n-1})=d(S^{n-1}). \tag{2.11}$$

If we put (2.1), (2.9) and (2.10) into the equation (2.6), then we get

$$L^n + M + \sum_{k=1}^{n-2} (-1)^{k+n-2} B_k H^k + (-1)^{n-1} T = S^n + Q. \tag{2.12}$$

On the basis of (2.7) and (2.11), all terms (except L^n and S^n) of the equation (2.12) are polynomials of degree $<(n-1)m=(n-1)(b_0/n)$.

Setting

$$L = l_m x^m + l_{m-1} x^{m-1} + \cdots + l_0 \quad \text{and} \quad S = s_m x^m + s_{m-1} x^{m-1} + \cdots + s_0$$

in (2.12), and equating the coefficients of degrees x^i $(i=nm, mn-1, \ldots, nm-m)$ from the both sides of the same equation, we find $l_j = \omega_\nu s_j$

$$(j=0, 1, \ldots, m) \text{ that is } L=\omega_\nu S.$$

Here ω_ν $(\nu=1, 2, \ldots, n)$ are roots of the equation $\omega^n=1$.

So the equation (1.6) for which the conditions (i), (ii) and (iii) are fulfilled, can be satisfied only by the polynomials

$$\xi(x) = \frac{1}{n} B_{n-1} + \omega_\nu S, \quad S = [\sqrt[n]{(-1)^{n-1} B_0}]. \tag{2.13}$$

At the end we can formulate the following result:

Theorem. *The differential operator* (1.1) *where* $B_i(x)$ $(i=0, 1, \ldots, n-1)$ *are polynomials for whose degrees the conditions* (i), (ii) *and* (iii) *are fulfilled, is factorized in the form* (1.2), *where* $\xi(x)$, $\alpha_j(x)$ $(j=0, 1, \ldots, n-2)$ *are polynomials, if and only if exists a number* $1 \leqslant \nu \leqslant n$ *so that for* (2.13) *the relation* (1.6) *is valid. The polynomials* $\alpha_j(x)$ $(j=0, 1, \ldots, n-2)$ *are determined with* (1.4) *that is* (1.5), *while* ω_ν $(\nu=1, 2, \ldots, n)$ *are roots of the equation* $\omega^n=1$. *If* b_0 *is not a multiple of* n, *and the conditions* (ii) *and* (iii) *are valid, then the operator* (1.1) *cannot be factorized in the form* (1.2).

We note that for the operator (1.1) n factorizations of the form (1.2) are possible.

3. When $B_1 = B_2 = \cdots = B_{n-1} = \text{const.}$, then the operator (1.1) can be factorized in the form (1.2) only if b_0 is a multiple of n, that is $b_0 = mn$; $m=1, 2, \ldots$.

When the last condition is fulfilled, the above mentioned theorem is valid, only the coefficients $A_{j+k}^{[1]}$ are given now as

$$A_{j+k}^{[1]} = \xi B_{j+k}; \quad (B_{j+k} = \text{const.}).$$

We shall take as an example the differential operator

$$P(D) = D^n + B(x); \quad B(x) - \text{ polynomial of degree } nm. \tag{3.1}$$

Here

$$A_{j+k}^{[k]} = 0, \ \alpha_{j-1} = (-1)^{n-j} \psi^{[n-j-1]},$$

and for the operator (3.1) we have the following factorizations of the form (1.2):

$$D^n + B(x) = (D + \omega_\nu [\sqrt[n]{(-1)^{n-1} B}]) \times$$
$$\times \left(D^{n-1} + \sum_{j=1}^{n-1} ((-1)^{n-j} \psi^{[n-j-1]}) D^{j-1} \right)$$
$$\psi^{[k]} = \xi \psi^{[k-1]} + (\psi^{[k-1]})'; \quad \psi^{[0]} = \xi = \omega_\nu [\sqrt[n]{(-1)^{n-1} B}]$$
$$\nu = 1, 2, \ldots, n; \quad \omega_\nu^n = 1.$$

REFERENCE

[1] M. Bhargava and H. Kaufman, *Existence of polynomial solutions of a class of Riccati — type differential equations*, Collectanea Mathematica, 17 (1965), 135—143.

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