Математички Билтен ISSN 0351-336X
35(LXI)
2011(41-49)
Скопје, Македонија

AN EXTEND OF THE TYPE OF HANH-BANACH FOR SKEW-SYMMETRIC LINEAR FORMS

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Abstract. In the paper, an extend of the type of Han-Banah for the 2-skew-symmetric linear forms will be considered.

The study and work with functionals is present in many mathematical fields. One of the main tasks, when working with functionals, is to extend them, i.e. for a given functional Λ, defined on a set S, subset of a given vector space X and very often its subspace, does there exists a functional Λ', defined on a set S' which contains the set S, and often S' is a vector subspace of X, such that $\Lambda'/S = \Lambda$ and $-p(-x) \leq \Lambda'(x) \leq p(x)$, where p is a mapping, given in advance, and for which the following conditions are true:

a) $p(x+y) \leq p(x) + p(y)$, for $x, y \in X$;
b) $p(tx) = tp(x)$, for $x \in X$ and $t \in \mathbb{R}$, $t > 0$.

Note. It is clear that if p is seminorm, then p satisfies a) and b).

In the paper, it will be considered the classical theorem of Han-Banah for some special cases of 2-skew-symmetric linear forms and for some special cases of S.

Let $S \subseteq X^2$ be a 2-subspace of the 2-space X^2, where X is a vector space over the field of the real numbers $\mathbb{R}$. The mapping $p : X^2 \to \mathbb{R}$ is such that

a) $p(x+y, z) \leq p(x, z) + p(y, z)$, for $x, y, z \in X$;
b) $p(tx, z) = tp(x, z)$, for $x, z \in X$ and $t \in \mathbb{R}$, $t > 0$,

and $\Lambda : S \to \mathbb{R}$ is 2-skew-symmetric linear form, so that $\Lambda(x, y) \leq p(x, y)$, for every $(x, y) \in S$. The problem that will be considered is the following: does there exists 2-skew-symmetric linear form $\Lambda' : X^2 \to \mathbb{R}$ such that $\Lambda'/S = \Lambda$ and $-p(-x, y) \leq \Lambda'(x, y) \leq p(x, y)$ on X^2?

It is natural to start the research with minimal extension i.e. with extension on the smallest subspace S' which contains the set $S \cup \{(u, v)\}$, where $(u, v) \notin S$. We mention that some extensions of 2-skew-symmetric linear form are already considered in some special cases.

Let X denote the vector space over the field Φ (Φ is the field of the real numbers or the field of the complex numbers). $M_2(\Phi)$ denotes the set of quadratic matrixes of order 2. Δ_2 denotes the set of all ordered pairs (x, y), elements of X^2 such that the vectors x, y are linearly dependent.

Definition 1. *Let X denote the vector space over the field Φ. The set X^2 with the operations*

$(x,z)+(y,z)=(x+y,z)$;
$(z,x)+(z,y)=(z,x+y)$;
$A(x,y)=A(x,y)^T$,

where $A\in M_2(\Phi)$, $x,y,z\in X$ is called 2-vector space or, simply, 2-space.

Note. In the definition 1, the first two operations are partial, and the third one is a complete operation.

Definition 2. *Let X denote the vector space over the field Φ . The function $\Lambda: X^2\to\Phi$ which satisfies the conditions:*

a) $\Lambda(x+y,z)=\Lambda(x,z)+\Lambda(y,z)$, *for any* $x,y,z\in X$;
b) $\Lambda(x,y)=-\Lambda(y,x)$, *for any* $x,y\in X$;
c) $\Lambda(\alpha x,y)=\alpha\Lambda(x,y)$, *for any* $x,y\in X$ *and* $\alpha\in\Phi$,

is called 2-skew-symmetric linear form.

The conditions $b)$ and $c)$ are equivalent with the condition $\Lambda(A(x,y))=(\det A)\Lambda(x,y)$, for an arbitrary $A\in M_2(\Phi)$.

Definition 3. *The subset S, $S\subseteq X^2$ which is closed under the operations of the 2-space X^2 is called 2-subspace of X^2 .*

Definition 4. *The subset T of the 2-space X^2 is 2-invariant if $AT\subseteq T$ for every $A\in M_2(\Phi)$ with* $\det A=1$.

Theorem 1. *([2]) Every 2-subspace S of the 2-space X^2 is 2-invariant set.*

Definition 5. *Let X denote the vector space over the field Φ . The function $p: X^2\to\mathbb{R}$ which satisfies the conditions:*

a) $p(A(x,y))=|\det A|p(x,y)$, *for any* $x,y\in X$ *and every* $A\in M_2(\Phi)$;
b) $p(x+y,z)\le p(x,z)+p(y,z)$, *for any* $x,y,z\in X$,

is called 2-seminorm and (X^2,p) is called 2-seminormed space.

In the subsequent part, the set $\{x_1,x_2,....,x_n,...\}$ will be linearly independent set.

We will consider two types of 2-subspaces and extensions on them. In each of them, the vectors from the pair (x,y) will be linearly independent.

I. Let L be a vector subspace of the vector space X , such that $\dim L>1$. The subset $M=L\times L$ is 2-subspace of X^2 which is a kernel 2-subspace.

If $(x,y)\in X^2$ then, for the vectors x and y, we have the following possibilities:

a) $x,y\in L$
b) $x\in L$, $y\notin L$ or $x\notin L$, $y\in L$
c) $x,y\notin L$.

We will consider each of these cases.

Case 1. $x,y\in L$.

We have that $(x,y)\in L\times L$ and the 2-subspace generated with the pair (x,y) is 2-subspace of $L\times L$. Therefore, in this case we do not have an extension of $L\times L$.

Case 2. $x, y \notin L$.

The 2-subspace N generated with the pair (x, y) is

$N = \{(a_{11}x + a_{12}y, a_{21}x + a_{22}y) \,|\, a_{11}, a_{12}, a_{21}, a_{22} \in \Phi\}$ Since $x, y \notin L$ we get that $a_{11}x + a_{12}y, a_{21}x + a_{22}y \notin L$, therefore $M \cap N = \{(0,0)\}$. The 2-subspace generated with $(L \times L)$ and the pair (x, y) is $(L \times L) \cup N$.

Case 3. $x \in L$, $y \notin L$ or $x \notin L$, $y \in L$.

It is enough to consider only one of the two possible sub cases. So, let $x \notin L$, $y \in L$. For an arbitrary $z \in L$, such that $\{x, y, z\}$ is linearly independent set, the 2-subspace generated with the elements of the set $B = \{(y, x), (y, z)\}$ is

$P_B = \bigcup\limits_{\alpha,\beta\in\Phi} L(\alpha x + \beta z, y) \times L(\alpha x + \beta z, y)$.

In this case we will use the denotation $P_B = P_z$. We will consider the set

$$T = \bigcup_{z\in L} P_z = \bigcup_{z\in L} \bigcup_{\alpha,\beta\in\Phi} L(\alpha z + \beta x, y) \times L(\alpha z + \beta x, y). \tag{1}$$

Since $z \in L$ and $\alpha, \beta \in \Phi$ are arbitrary chosen, we get that $\{\alpha z + \beta x \,|\, \alpha, \beta \in \Phi\}$ is a vector subspace of X which is an extension of L with the vector x. Therefore, if Y is a vector subspace of X generated with L and x, (1) can be written in the form $T = \bigcup\limits_{z\in L} P_z = \bigcup\limits_{v\in Y} L(v, y) \times L(v, y)$.

Using the form of the elements of Y, it is not hard to prove that X is 2-subspace of X^2 .

Finally, in this case we get that the generated 2-subspace is

$$\begin{aligned} M \cup T = (L \times L) \cup \bigcup_{z\in L} P_z = (L \times L) \cup \bigcup_{v\in Y} L(v, y) \times L(v, y) = \\ = (L \times L) \cup \bigcup_{\substack{\alpha,\beta\in\Phi \\ z\in L}} L(\alpha z + \beta x, y) \times L(\alpha z + \beta x, y), \end{aligned}$$

where M and T are not disjunctive.

II. Let L and L' be vector subspaces of the vector space X such that $L \cap L' = \{0\}$. With M will be denoted the kernel 2-subspace $L \times L$, and with N will be denoted the kernel 2-subspace $L' \times L'$. From the construction, it is clear that $M \cap N = \{(0, 0)\}$. On the other hand, $M \cup N$ is 2-subspace of X^2 . In this case, we will consider extension of $M \cup N$ with an element $(u, v) \in X^2$, where $u \in L$, $v \in L'$.

The extension of the kernel 2-subspace M with (u, v) is considered in the case 3, and the same is done with the extension of N with the same ordered pair. Let us denote these extensions with M' and N', respectively. According to the case 3, they have the forms: $M' = M \cup \bigcup\limits_{y\in Y} L(y, u) \times L(y, u)$, where Y is a vector subspace of X generated with the vector subspace L and the element v, and $N' = N \cup \bigcup\limits_{x\in Z} L(x, v) \times L(x, v)$, where Z is the subspace of X generated with L' and the element u.

We notice that, for the elements for which adding is possible, the sum of two elements of M is an element of M and the sum of two elements of N is an element of N. There is no element of M which can be added to any element of N and vice versa.

Let $(a,b)\in M$ and $(c,d)\in \bigcup\limits_{x\in Z} L(x,v)\times L(x,v)$ be elements for which adding is possible. The element (c,d) has the form $A(\alpha y+\beta u,v)$, i.e.

$(a_{11}(\alpha y+\beta u)+a_{12}v, a_{21}(\alpha y+\beta u)+a_{22}v)$, where $y\in L'$. Without loss of generality, we may assume that $a_{21}(\alpha y+\beta u)+a_{22}v=b$. Since $L\cap L'=\{0\}$, the last equality is possible if and only if $a_{22}=0$ and $a_{21}\alpha=0$. If $a_{21}=0$, then $b=0$ and the required sum is an element of Δ_2. If $a_{21}\neq 0, \alpha=0$, then $b=a_{21}\beta u$, the required sum is

$(a+a_{11}\beta u+a_{12}v, a_{21}\beta u)\in \bigcup\limits_{y\in Y} L(y,u)\times L(y,u)$.

The case $(e,f)\in N$ and $(g,h)\in \bigcup\limits_{y\in Y} L(y,u)\times L(y,u)$ can be consider analogously.

The sum of an element of $\bigcup\limits_{x\in Z} L(x,v)\times L(x,v)$ and an element of $\bigcup\limits_{y\in Y} L(y,u)\times L(y,u)$ is the same as adding of elements of the branch 2-subspace generated with three elements. The sum belongs to $\bigcup\limits_{x\in Z} L(x,v)\times L(x,v)$ or to $\bigcup\limits_{y\in Y} L(y,u)\times L(y,u)$. Therefore, the extension in this case is

$$M\cup N\cup\left(\bigcup_{y\in Y} L(y,u)\times L(y,u)\right)\cup\left(\bigcup_{x\in Z} L(x,v)\times L(x,v)\right)$$

III. Let $M\subseteq X^2$ be a loop 2-subspace generated with the set $J=\{(x_i,x_1)\,|\,i=2,3,...\}$. The form of this 2-subspace is $M=\bigcup\limits_{u\in L(x_2,x_3,...)} L(u,x_1)\times L(u,x_1)$

We will consider some extensions of this 2-subspace with the an element $(u,v)\in X^2$.

Case 1. $u,v\notin L(x_1,x_2,..)$.

The 2-subspace generated with the set $B=\{(u,v)\}$ is $P_B=\{C(u,v)\,|\,C\in M_2(\Phi)\}=\{(c_{11}u+c_{12}v, c_{21}u+c_{22}v)\,|\,c_{11},c_{12},c_{21},c_{22}\in\Phi\}$. Therefore, for an arbitrary $c_{11},c_{12},c_{21},c_{22}\in\Phi$, $c_{11}u+c_{12}v, c_{21}u+c_{22}v\notin L(x_1,x_2,...,x_n,...)$.

If $(a,b)\in M$, then it has the form $A(x_1,x)$, where $x=\alpha_{i_1}x_{i_1}+\alpha_{i_2}x_{i_2}+...+\alpha_{i_k}x_{i_k}$, for some $x_{i_1},x_{i_2},...,x_{i_k}$ and for some $\alpha_{i_1},\alpha_{i_2},...,\alpha_{i_k}$, and $A=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$. But, then

$$a_{11}x_1+a_{12}x, a_{21}x_1+a_{22}x\in L(x_1,x_2,...,x_n,...).$$

So, $a_{11}x_1+a_{12}x\neq c_{11}u+c_{12}v$, $a_{21}x_1+a_{22}x\neq c_{21}u+c_{22}v$ for any $c_{11},c_{12},c_{21},c_{22}\in\Phi$, and therefore, adding of elements of M with elements of P_B is not possible.

From the above discussion, we get that the extension in this case is $M\cup P_B$.

Case 2. $u\in L(x_1,x_2,\ldots)$, $v\notin L(x_1,x_2,\ldots)$

In this case we have two sub cases.

Sub case 1. $u\in L(x_2,x_3,\ldots,x_n,\ldots)$.

There are $x_{i_1},x_{i_2},...,x_{i_k}$, such that $u=\alpha_{i_1}x_{i_1}+\alpha_{i_2}x_{i_2}+...+\alpha_{i_k}x_{i_k}$ and, because of the assumption, $x_{i_j}\neq x_1$, $j=1,2,...k$. From here, we get that $(u,x_1)\in M$. The 2-subspace generated with this element is $\{A(u,x_1)\,|\,A\in M_2(\Phi)\}$. On the other hand, the 2-subspace P_B for $B=\{(u,v)\}$ is $P_B=\{C(u,v)\,|\,C\in M_2(\Phi)\}$. An arbitrary element from the first 2-subspace is of the form

$$(a_{11}u+a_{12}x_1, a_{21}u+a_{22}x_1) \tag{1}$$

and an arbitrary element of the second 2-subspace is of the form

$$(c_{11}u+c_{12}v, c_{21}u+c_{22}v) \tag{2}$$

Without loss of generality, their adding is possible if and only if $a_{21}u+a_{22}x_1=c_{21}u+c_{22}v$. Since the vectors of the pair (u,v) are linearly independent and $v\notin L(x_1,x_2,..)$, the last equality is possible if and only if $c_{22}=a_{22}=0$ and $a_{21}=c_{21}=t$. Then the elements get the forms $(a_{11}u+a_{12}x_1,tu)$, $(c_{11}u+c_{12}v,tu)$, and their sum is $((a_{11}+c_{11})u+c_{12}v+a_{12}x_1,tu)$. But, because of the 2-invariance of the 2-subspaces, this element belongs to the 2-subspace generated with the element $(c_{12}v+a_{12}x_1,u)$. So, the sum of two elements belongs to the 2-subspace generated with the set $\{(u,x_1),(u,v)\}$ and its form is $\bigcup\limits_{\alpha,\beta\in\Phi} L(\alpha v+\beta x_1,u)\times L(\alpha v+\beta x_1,u)$.

For an arbitrary element of this 2-subspace and an arbitrary element of M that does not belong to this 2-subspace, such that their adding is possible, their sum reduces to adding of elements of the branch 2-subspace generated with the set $\{(v,u),(u,x_1),(x_1,w)\}$. Their sum belongs either to the 2-subspace generated with $\{(v,u),(u,x_1)\}$ or to the 2-subspace generated with $\{(u,x_1),(x_1,w)\}$ and it is an element of M. In this case the extension of M has the form $M\cup\bigcup\limits_{\alpha,\beta\in\Phi} L(\alpha v+\beta x_1,u)\times L(\alpha v+\beta x_1,u)$.

Sub case 2. $u\notin L(x_2,x_3,....,x_n,...)$.

Let $u=\alpha_1x_1+\alpha_{i_1}x_{i_1}+\alpha_{i_2}x_{i_2}+...+\alpha_{i_k}x_{i_k}$, where $\alpha_1\neq 0$. With the denotation, $x=\alpha_{i_1}x_{i_1}+\alpha_{i_2}x_{i_2}+...+\alpha_{i_k}x_{i_k}$ we have $u=\alpha_1x_1+x$. The pair $(u,x_1)=(\alpha_1x_1+x,x_1)$ belongs to M, and because of the 2-invariance of the 2-subspaces, we get that it, also, belongs to the 2-subspace of M generated with (x,x_1) ($(u,x_1)=\begin{bmatrix}1 & \alpha_1\\ 0 & 1\end{bmatrix}(x,x_1)$).

The form of the elements of the 2-subspace generated with (u,v) is

$$B(u,v)=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}(u,v)=(b_{11}u+b_{12}v,b_{21}u+b_{22}v).$$

The general form of the elements of the 2-subspace generated with (x,x_1) is

$$A(x,x_1)=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}(x_1,x)=(a_{11}x_1+a_{12}x,a_{21}x_1+a_{22}x).$$

Without loss of generality, adding of elements of the last two forms, is possible if and only if $b_{21}\alpha_1x_1+b_{21}x+b_{22}v=a_{21}x_1+a_{22}x$.

From the last equality, we have $b_{21}\alpha_1=a_{21}$, $a_{22}=b_{21}$, $b_{22}=0$, and therefore the elements get the form $(a_{11}x_1+a_{12}x,a_{22}u)$ and $(b_{11}u+b_{12}v,a_{22}u)$, and their sum is $((a_{11}+b_{11}\alpha_1)x_1+b_{12}v+(b_{11}+a_{12})x,a_{22}u)$. So, the sum belongs to the loop 2-subspace N generated with the elements $\{(u,x_1),(u,v),(u,x)\}$.

It is not hard to prove that the sum of two elements, one from N, and the other element of M which does not belong to N, either belong to M or to N.

In this case, the extension of M is

$$M \cup N = M \cup \bigcup_{\alpha,\beta,\gamma\in\Phi} L(\alpha x_1 + \beta x + \gamma v, u) \times L(\alpha x_1 + \beta x + \gamma v, u).$$

Case 3. $u \notin L(x_1, x_2, ..)$, $v \in L(x_1, x_2, ..)$

This case is analogue to the case 2.

Case 4. $u \in L(x_1, x_2, ..)$, $v \in L(x_1, x_2, ..)$.

The elements u and v are of the forms $u = \alpha_1 x_1 + \alpha_{i_1} x_{i_1} + \alpha_{i_2} x_{i_2} + ... + \alpha_{i_k} x_{i_k}$ and $v = \beta_1 x_1 + \beta_{j_1} x_{j_1} + \beta_{j_2} x_{j_2} + ... + \beta_{j_s} x_{j_s}$, where $x_{i_1}, x_{i_2}, ..., x_{i_k}, x_{j_1}, x_{j_2}, ..., x_{j_s} \in \{x_2, ..., x_n, ...\}$.

In this case, we have several sub cases.

Sub case 1. $u \in L(x_2, x_3, ..)$, $v \in L(x_2, x_3, ..)$

In this sub case $u = \alpha_{i_1} x_{i_1} + \alpha_{i_2} x_{i_2} + ... + \alpha_{i_k} x_{i_k}$ and $v = \beta_{j_1} x_{j_1} + \beta_{j_2} x_{j_2} + ... + \beta_{j_s} x_{j_s}$. We notice that the elements (u, x_1) and (v, x_1) define 2-subspaces of M. The elements of the set $B = \{(u, x_1), (v, x_1), (u, v)\}$ are elements of the extended 2-subspace, and therefore the kernel 2-subspace generated with them, P_B, is 2-subspace of the extended 2-subspace. Its elements are of the form $(a_{11}x_1 + a_{12}u + a_{13}v, a_{21}x_1 + a_{22}u + a_{23}v)$.

We will consider the possibilities for adding an element (a, b) of M such that $(a, b) \notin P_B$ with an element of P_B.

Without loss of generality, we will assume that the second coordinates are the same.

a) $a_{21} \neq 0$ and $b = a_{21}x_1 + a_{22}u + a_{23}v$

Since $a_{22}u + a_{23}v \in L(x_2, x_3, ..., x_n, ...)$, and according to the forms of the elements of M, we have $a = c_{11}x_1 + c_{12}(a_{22}u + a_{23}v)$. Then

$(a, b) = (c_{11}x_1 + c_{12}(a_{22}u + a_{23}v), a_{21}x_1 + a_{22}u + a_{23}v) \in P_B$, which contradicts the assumption.

b) $a_{21} = 0$ and $b = a_{22}u + a_{23}v$

Since $a_{22}u + a_{23}v \in L(x_2, x_3, ..., x_n, ...)$, and according to the forms of the elements of M, we have $a = c_{11}x_1 + c_{12}(a_{22}u + a_{23}v)$. Then

$(a, b) = (c_{11}x_1 + c_{12}(a_{22}u + a_{23}v), a_{22}u + a_{23}v) \in P_B$, which contradicts the assumption.

So, the extension in this sub case is $M \cup P_B$.

Sub case 2. $u \notin L(x_2, x_3, ..)$, $v \in L(x_2, x_3, ..)$

Sub case 3. $u \in L(x_2, x_3, ..)$, $v \notin L(x_2, x_3, ..)$

Sub case 4. $u \notin L(x_2, x_3, ..)$, $v \notin L(x_2, x_3, ..)$

The sub cases 2, 3 and 4 are analogue to the consideration in the sub case 1.

In the subsequent part, we will give the procedure for extension of 2-skew-symmetric linear form in the sub case 2 of the case 2. In order to get the required extensions in all cases and sub cases in I, II and in III, when the field of scalars is the set of the real numbers $\mathbb{R}$, we have to proceed in the same way as we will show below for this particular sub case 2 of case 2.

Let $p : X^2 \to \mathbb{R}$ be a function which satisfies the following conditions:

a) $p(x + y, z) \leq p(x, z) + p(y, z)$, for $x, y, z \in X$;
b) $p(x, y) = p(y, x)$, for $x, y \in X$;
c) $p(tx, y) = tp(x, y)$, for $x, y \in X$ and $t \in \mathbb{R}$, $t > 0$,

where X is a vector space over the field of the real numbers $\mathbb{R}$. (It is clear that if p is 2-seminorm, then p satisfies $a), b)$ and $c)$)

Theorem 2. *Let $\Lambda : M \to \mathbb{R}$ be a 2-skew-symmetric linear form such that $\Lambda(x,y) \leq p(x,y)$, for every $(x,y) \in M$, where M is a loop 2-subspace of the 2-space X^2 . Let M' be the extension of M as in the sub case 2 of the case 2. Then there exists 2- skew-symmetric linear form $\Lambda' : M' \to \mathbb{R}$ such that*
$\Lambda'/M = \Lambda$
$-p(-x,y) \leq \Lambda(x,y) \leq p(x,y).$ (*)

Proof. Let $(\alpha x_1 + \beta x, u)$ and $(\alpha_1 x_1 + \beta_1 x, u)$ are two elements of M. For the 2-skew-symmetric linear form Λ we have

$$\begin{aligned}\Lambda(\alpha x_1 + \beta x, u) + \Lambda(\alpha_1 x_1 + \beta_1 x, u) &= \Lambda(\alpha x_1 + \beta x + \alpha_1 x_1 + \beta_1 x, u) \leq \\ &\leq p(\alpha x_1 + \beta x + \alpha_1 x_1 + \beta_1 x, u) = p(\alpha x_1 + \beta x - v + \alpha_1 x_1 + \beta_1 x + v, u) \leq \\ &\leq p(\alpha x_1 + \beta x - v, u) + p(\alpha_1 x_1 + \beta_1 x + v, u)\end{aligned}$$

So,

$$\Lambda(\alpha x_1 + \beta x, u) - p(\alpha x_1 + \beta x - v, u) \leq p(\alpha_1 x_1 + \beta_1 x + v, u) - \Lambda(\alpha_1 x_1 + \beta_1 x, u).$$

Since $\alpha,\ \beta \in \mathbb{R}$ and $\alpha_1,\ \beta_1 \in \mathbb{R}$ are arbitrary, we have

$$\begin{aligned}&\sup_{\alpha,\beta\in\Phi} [\Lambda(\alpha x_1 + \beta x, u) - p(\alpha x_1 + \beta x - v, u)] \\ &\qquad = d \leq p(\alpha_1 x_1 + \beta_1 x + v, u) - \Lambda(\alpha_1 x_1 + \beta_1 x, u).\end{aligned}$$

Therefore, for an arbitrary $\alpha, \beta, \alpha_1, \beta_1 \in \mathbb{R}$, it is true that
$\Lambda(\alpha x_1 + \beta x, u) - p(\alpha x_1 + \beta x - v, u) \leq d$, $d \leq p(\alpha_1 x_1 + \beta_1 x + v, u) - \Lambda(\alpha_1 x_1 + \beta_1 x, u)$,
i.e.
$\Lambda(\alpha x_1 + \beta x, u) - d \leq p(\alpha x_1 + \beta x - v, u),\ \alpha, \beta \in \mathbb{R}$ (3)
$\Lambda(\alpha_1 x_1 + \beta_1 x, u) + d \leq p(\alpha_1 x_1 + \beta_1 x + v, u),\ \alpha_1, \beta_1 \in \mathbb{R}$ (4)
Let $\Lambda' : M' \to \mathbb{R}$ be defined by

$$\Lambda'[A(\alpha x_1 + \beta x + \gamma v, u)] = (\det A)[\Lambda(\alpha x_1 + \beta x, u) + \gamma d],\ \gamma \in \mathbb{R},$$

$$\Lambda'(x,y) = \Lambda(x,y),\ (x,y) \in M$$

We have $\Lambda'/M = \Lambda$.

On the other hand, if in (3), instead of α and β, we choose $\frac{\alpha}{t}$ and $\frac{\beta}{t}$, $t > 0$ respectively, using the properties of Λ and p, we get
$\Lambda(\alpha x_1 + \beta x, u) - td \leq p(\alpha x_1 + \beta x - tv, u).$ (5)
If in (4) instead of α_1 and β_1, we choose $\frac{\alpha_1}{t}$ and $\frac{\beta_1}{t}$, $t > 0$ respectively, we get
$\Lambda(\alpha_1 x_1 + \beta_1 x, u) + td \leq p(\alpha_1 x_1 + \beta_1 x + tv, u).$ (6)
From (3) and (4), we have
$\Lambda'(\alpha_1 x_1 + \beta_1 x + \gamma v, u) \leq p(\alpha_1 x_1 + \beta_1 x + \gamma v, u)$, and it is clear that $\Lambda' \leq p$ on M' and $(*)$ is satisfied.

For the extensions of 2-skew-symmetric linear forms defined on 2-subspace $S \neq X^2$ to 2-skew-symmetric linear forms defined on X^2 , if it is possible, it is needed to consider the extension of branch 2-subspace, the extension of cycle

2-subspace and the extension of an arbitray 2-subspace, S. Having many possibilities of combining the main types of 2-subspaces of X^2, such as the kernel, loop, branch and cycle 2-subspaces, this problem in the general case is not that easy. $\square$

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ПРОШИРУВАЊЕ ОД ТИПОТ НА ХАН–БАНАХ ЗА КОСО СИМЕТРИЧНИ ЛИНЕАРНИ ФОРМИ

Алекса Малчески, Весна Манова-Ераковиќ

Р е з и м е

Во трудот ќе биде разгледано пордолжување од типот на Хан-Банах за 2-косо симетрични линеарни форми.

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