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AN EXISTENCE THEOREM CONCERNING ORDINARY SHAPE OF CARTESIAN PRODUCTS

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Abstract. The paper is devoted to the question when is the Cartesian product $X \times P$ of a compact metric space X and a polyhedron P a product in the shape category of topological spaces. The question consists of two parts. The existence part, which asks whether, for every topological space Z, every shape morphism $F\colon Z \to X$ and every homotopy class of mappings $[g]\colon Z \to P$, there exists a shape morphism $H\colon Z \to X \times P$, whose compositions with the canonical projections of $X \times P$ equal F and $[g]$, respectively. The uniqueness part asks whether H is unique. It is known that, in general, the uniqueness part does not hold even when Z is a polyhedron. The main result of the paper asserts that the existence part always holds. The proof is based on an analogous result for strong shape.

1. Introduction

In an arbitrary category the (direct) product of two objects is well defined. It may not exist, but if it does, it is unique up to natural isomorphism. It is well known that in the category of topological spaces Top the product of two spaces X and Y exists and consists of the Cartesian product $X \times Y$ and of the canonical projections $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$. Similarly, in the homotopy category of topological spaces H the Cartesian product $X \times Y$ and the homotopy classes $[\pi_X]$, $[\pi_Y]$ of the canonical projections π_X, π_Y form the product of X and Y. Since shape is a modification of homotopy, it is natural to ask the following questions.

Question 1. *Does every pair of topological spaces X, Y admit a product in the shape category* Sh*?*

Question 2. *Is the Cartesian product $X \times Y$ of a pair of spaces X, Y a product in the shape category* Sh*?*

When we say that $X \times Y$ is a product in Sh we mean that, for every topological space Z and shape morphisms $F\colon Z \to X$ and $G\colon Z \to Y$, there exists a unique shape morphism $H\colon Z \to X \times Y$ such that $S[\pi_X]H = F$ and $S[\pi_Y]H = G$. Here

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$S\colon \mathrm{H} \to \mathrm{Sh}$ denotes the shape functor, which keeps objects (spaces) fixed and assigns to every homotopy class of mappings the corresponding shape morphism (see[7], 8.2).

In the case when both spaces X and Y are compact Hausdorff spaces, it is well known that the answer to Question 2 is positive [3]. J. Dydak and S. Mardešić [1] showed that the Cartesian product of a dyadic solenoid and the wedge (pointed sum) of a sequence of copies of the 1-sphere S^1 is not a product in Sh.

In 1972 Y. Kodama proved that the Cartesian product of an FANR and a paracompact space is a product in Sh ([4], Theorem 3′). An open problem of Kodama, raised in 1977 [4], asks whether the Cartesian product of a movable metric compactum X and a metric space Y is a product in Sh. Even in the simple case, when X is the Hawaiian earring and Y is the wedge of a sequence of copies of the 1-sphere S^1, this author does not know if $X \times Y$ is a product in Sh.

Question 1 is wide open. In the present paper we consider Question 2 in the important case when X is a compact metric space and $Y = P$ is a (non-compact) polyhedron (CW-topology). The universal property characterizing products in a category consists of an existence part and a uniqueness part. A positive answer to Question 2 for a compact metric space X and a polyhedron P (CW-topology) is just the assertion that the following statements $(\mathrm{ES})_Z$ and $(\mathrm{US})_Z$ are true, for every topological space Z.

$(\mathrm{ES})_Z$ *For every shape morphism* $F\colon Z \to X$ *and every homotopy class of mappings* $[g]\colon Z \to P$*, there exists a shape morphism* $H\colon Z \to X \times P$ *such that* $S[\pi_X]H = F$ *and* $S[\pi_P]H = S[g]$.

$(\mathrm{US})_Z$ *If* $H_i\colon Z \to X \times P$, $i = 1, 2$*, are two shape morphisms such that* $S[\pi_X]H_1 = S[\pi_X]H_2$ *and* $S[\pi_P]H_1 = S[\pi_P]H_2$*, then* $H_1 = H_2$.

The results of this paper refer to the existence property $(\mathrm{ES})_Z$ (see Theorem 1). Unfortunately, up to now, the author was unable to obtain significant affirmative results concerning the uniqueness property $(\mathrm{US})_Z$. According to [1], if X is the dyadic solenoid and P is the pointed union of a sequence of 1-spheres, $(\mathrm{US})_P$ is false.

The main result of the paper is the following theorem.

Theorem 1. *Let* X *be a compact metric space and let* P *be a polyhedron. Then the existence condition* $(\mathrm{ES})_Z$ *holds, for every topological space* Z.

Denote by $\overline{S}\colon H \to \mathrm{SSh}$ the strong shape functor from the homotopy category H to the category of strong shape of topological spaces SSh. This functor keeps objects (spaces) fixed and associates with every homotopy class of mappings the corresponding strong shape morphism (see[7], 8.2). We will derive Theorem 1 from the following recently proved theorem on strong shape (see [8], Theorem 2).

Theorem 2. *Let* X *be a compact metric space and let* P *be a polyhedron. Then, for every topological space* Z*, the following condition holds.*

(ESS)$_Z$ *For every strong shape morphism* $\overline{F}\colon Z \to X$ *and every homotopy class of mappings* $[g]\colon Z \to P$, *there exists a strong shape morphism* $\overline{H}\colon Z \to X \times P$ *such that* $\overline{S}[\pi_X]H = F$ *and* $\overline{S}[\pi_P]H = S[g]$.

Denote by $\overline{E}\colon \mathrm{SSh} \to \mathrm{Sh}$ the forgetful functor, which keeps spaces fixed and maps strong shape morphisms $\overline{H}\colon Z \to X$ to the corresponding shape morphism $H\colon Z \to X$ by "forgetting" the richer structure of strong shape (see 8.2 in [7]). To prove Theorem 1, we also need the following result.

Theorem 3. *Every shape morphism* $F\colon Z \to X$ *of a topological space* Z *to a compact metric space* X *admits a strong shape morphism* $\overline{F}\colon Z \to X$ *such that* $\overline{E}(\overline{F}) = F$.

Remark 1. If Z is a compact metric space, the assertion of Theorem 3 is contained in [2], Theorem 4.3.

Question 3. *Is Theorem* 3 *valid for compact Hausdorff spaces* X*?*

Proof of Theorem 1. Let X be a compact metric space and P a polyhedron. Let $F\colon Z \to X$ be a shape morphism and let $[g]\colon Z \to P$ be a homotopy class of mappings. We must exhibit a shape morphism $H\colon Z \to X \times P$ such that $S[\boldsymbol{\pi}_X]H = F$ and $S[\boldsymbol{\pi}_P]H = S[g]$. By Theorem 3, there exists a strong shape morphism $\overline{F}\colon Z \to X$ such that $\overline{E}(\overline{F}) = F$. By Theorem 2, there exists a strong shape morphism $\overline{H}\colon Z \to X \times P$ such that

$$\overline{S}[\boldsymbol{\pi}_X]\overline{H} = \overline{F}, \quad \overline{S}[\boldsymbol{\pi}_P]\overline{H} = \overline{S}[g]. \tag{1}$$

Put $H = \overline{E}(\overline{H})\colon Z \to X \times P$. Application of the forgetful functor $\overline{E}$ to (1) yields the desired relations $S[\boldsymbol{\pi}_X]H = F$ and $S[\boldsymbol{\pi}_P]H = S[g]$, because $\overline{E}\overline{S} = S$ (see Theorem 8.9 in [7]). □

In Section 2 we recall the definitions of *homotopy mappings* and coherent homotopy mappings (shorter, *coherent mappigs*) and their relations to shape morphisms and strong shape morphisms, respectively. This will enable us to reduce the proof of Theorem 3 to a result on homotopy mappings and coherent mappings (see Lemma 1). In Section 3 we prove Lemma 1 and thus, complete the proof of Theorem 1.

2. Homotopy mappings and coherent mappings

2.1 Let $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$ be an inverse sequence of compact polyhedra. Recall that a mapping $\boldsymbol{f}\colon \boldsymbol{Z} \to \boldsymbol{X}$ from a topological space Z to $\boldsymbol{X}$ is a sequence of mappings $f_i\colon Z \to X_i$, $i \in \mathbb{N}$, such that

$$f_i = p_{ii'} f_{i'}, \quad i \le i'. \tag{2}$$

If in (2) equality $=$ is replace by homotopy $\simeq$, one obtains the notion of a homotopy mapping, i.e., a homotopy mapping $\boldsymbol{f}\colon Z \to \boldsymbol{X}$ is a sequence of mappings $f_i\colon Z \to X_i$, $i \in \mathbb{N}$, such that

$$f_i \simeq p_{ii'} f_{i'}, \quad i \le i'. \tag{3}$$

Clearly, every mapping $\boldsymbol{f}\colon Z \to \boldsymbol{X}$ is a homotopy mapping.

A coherent mapping $\boldsymbol{f}\colon Z \to \boldsymbol{X}$ consists of a collection of mappings $f_{\boldsymbol{i}} = f_{i_0\dots i_n}\colon Z \times \Delta^n \to X_{i_0}$, where $\Delta^n = [e_0, \dots, e_n] \subseteq \mathbb{R}^{n+1}$ is the standard n-simplex and $\boldsymbol{i} = (i_0, \dots, i_n)$ is a multiindex in $\mathbb{N}$ of length $|\boldsymbol{i}| = n \geq 0$, i.e., $\boldsymbol{i}$ is an increasing sequence $i_0 \leq \dots \leq i_n$ of $n+1$ elements from $\mathbb{N}$. One requires that the following two coherence conditions be fulfilled. The *boundary condition*:

$$f_{\boldsymbol{i}}(z, d_j t) = \begin{cases} p_{i_0 i_1} f_{d^0 \boldsymbol{i}}(z,t), & j = 0 \\ f_{d^j \boldsymbol{i}}(z,t), & 1 \leq j \leq n. \end{cases}, \tag{4}$$

where $d_j\colon \Delta^{n-1} \to \Delta^n$ are the standard boundary operators and d^j is the operator which omits i_j from $\boldsymbol{i} = (i_0, \dots, i_n)$, i.e., $d^j \boldsymbol{i} = (i_0, \dots, \widehat{i_j}, \dots, i_n)$. Condition (4) makes sense only when $n > 0$.

The *degeneracy condition*:

$$f_{\boldsymbol{i}}(z, s_j t) = f_{s^j \boldsymbol{i}}(z,t),\ 0 \leq j \leq n, \tag{5}$$

where $s_j\colon \Delta^{n+1} \to \Delta^n$ are the standard degeneracy operators and s^j is the operator which repeats i_j, i.e., $s^j \boldsymbol{i} = (i_0, \dots, i_j, i_j, \dots, i_n)$.

Coherent mappings can be viewed as generalizations of mappings, because with every mapping $\boldsymbol{f}\colon Z \to \boldsymbol{X}$ one can associate a coherent mapping $C(\boldsymbol{f})\colon Z \to \boldsymbol{X}$ which consists of the mappings $f_{\boldsymbol{i}}\colon Z \times \Delta^n \to X_{i_0}$, where $f_{\boldsymbol{i}}(z,t) = f_{i_0}(z)$.

2.2. Two homotopy mappings $\boldsymbol{f}, \boldsymbol{f}'\colon Z \to \boldsymbol{X}$, given by mappings f_i, f_i', $i \in \mathbb{N}$, are homotopic, $\boldsymbol{f} \simeq \boldsymbol{f}'$, provided for every $i \in \mathbb{N}$,

$$f_i \simeq f_i', \tag{6}$$

Two coherent mappings $\boldsymbol{f}, \boldsymbol{f}'\colon Z \to \boldsymbol{X}$, given by mappings $f_{\boldsymbol{i}}, f_{\boldsymbol{i}}'$ are homotopic, $\boldsymbol{f} \simeq \boldsymbol{f}'$, provided there exists a coherent mapping $\boldsymbol{F}\colon Z \times I \to \boldsymbol{X}$, given by mappings $F_{\boldsymbol{i}}\colon Z \times I \times \Delta^n \to X_{i_0}$, which satisfy the corresponding coherence conditions and

$$F_{\boldsymbol{i}}(z,0,t) = f_{\boldsymbol{i}}(z,t),\ F_{\boldsymbol{i}}(z,1,t) = f_{\boldsymbol{i}}'(z,t). \tag{7}$$

The homotopy relation $\simeq$ for homotopy mappings and coherent mappings $\boldsymbol{f}\colon Z \to \boldsymbol{X}$ are equivalence relations and the corresponding homotopy classes $[\boldsymbol{f}]$ of $\boldsymbol{f}$ are well defined.

Recall that questions concerning shape and strong shape reduce to questions concerning homotopy classes of homotopy mappings and homotopy classes of coherent mappings using the following facts, which are immediate consequences of the definition of shape morphisms and strong shape morphisms. If $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$ is an inverse sequence of compact polyhedra with limit X, then there exists a bijection between the set $\mathrm{Sh}(Z,X)$ of all shape morphisms $F\colon Z \to X$ and the set of all homotopy classes $[\boldsymbol{f}]$ of homotopy mappings $\boldsymbol{f}\colon Z \to \boldsymbol{X}$ (see [9], I,§2, Theorem 5). We say that F and $[\boldsymbol{f}]$ are associated with each other. Analogously, there is a bijection between the set $\mathrm{SSh}(Z,X)$ of all strong shape morphisms $\overline{F}\colon Z \to X$ and the set of all homotopy classes $[\overline{\boldsymbol{f}}]$ of coherent mappings $\overline{\boldsymbol{f}}\colon Z \to \boldsymbol{X}$ (see [7], 8.2). We say that $\overline{F}$ and $[\boldsymbol{f}]$ are associated with each other.

2.3. Denote by E the forgetful functor, which keeps inverse sequences fixed and maps coherent mappings to homotopy mappings by "forgetting" higher homotopies (see [7], 1.4). The following lemma is used in the proof of Theorem 3.

Lemma 1. *Let $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$ be an inverse sequence of compact polyhedra, let Z be a topological space and let $[\boldsymbol{f}]\colon Z \to \boldsymbol{X}$ be a homotopy class of homotopy mappings. Then there exists a homotopy class of coherent mappings $[\overline{\boldsymbol{f}}]\colon Z \to \boldsymbol{X}$ such that $E[\overline{\boldsymbol{f}}] = [\boldsymbol{f}]$.*

Proof of Theorem 3. Let $F\colon Z \to X$ be a shape morphism. Choose an inverse sequence $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$ of compact polyhedra such that $X = \lim \boldsymbol{X}$. With F is associated a homotopy class of homotopy mappings $[\boldsymbol{f}]\colon Z \to \boldsymbol{X}$. By Lemma 1, there exists a homotopy class of coherent mappings $[\overline{\boldsymbol{f}}]\colon Z \to \boldsymbol{X}$ such that $E[\overline{\boldsymbol{f}}] = [\boldsymbol{f}]$. With $[\overline{\boldsymbol{f}}]$ is associated a strong shape morphism $\overline{F}\colon Z \to X$. By the definition of $\overline{E}$ (see [7],page 161), $\overline{E}(\overline{F})$ is the shape morphism associated with $E[\overline{\boldsymbol{f}}]$. Since $E[\overline{\boldsymbol{f}}] = [\boldsymbol{f}]$ and F is also associated with $[\boldsymbol{f}]$, it follows that $\overline{E}(\overline{F}) = F$. Consequently, the strong shape morphism $\overline{F}$ has the property required by Theorem 3. $\square$

3. Proof of Lemma 1

3.1. Recall that the concatenation $F * G\colon Z \times I \to X$ of homotopies $F, G\colon Z \times I \to X$ is defined by the following formula.

$$F * G(z,t) = \begin{cases} F(z,2t), & 0 \le t \le 1/2, \\ G(z,2t-1), & 1/2 \le t \le 1. \end{cases} \tag{8}$$

Recall that $F * G$ is well defined, provided $F(z,1) = G(z,0)$. Also recall that, for homotopies $F, F'\colon Z \times I \to X$ such that $F(z,0) = F'(z,0)$ and $F(z,1) = F'(z,1)$, for $z \in Z$, the expression $F \simeq F'$ $(\mathrm{rel}\,\partial I)$ means that there is a 2-homotopy $H\colon Z \times I \times I \to X$, which connects F to F', i.e., $H(z,t,0) = F(z,t)$, $H(z,t,1) = F'(z,t)$ and $H(z,0,s)$ and $H(z,1,s)$ do not depend on $s \in I$.

To prove Lemma 1, we need three elementary facts on concatenations of homotopies, analogous to facts used in defining the fundamental group. They are stated in the following lemma.

Lemma 2. (i) *(The unit law). If $F\colon Z \times I \to X$ connects mappings $f, f'\colon Z \to X$, then*

$$F \simeq f * F \ (\mathrm{rel}\,\partial I); \ \ F \simeq F * f' \ (\mathrm{rel}\,\partial I). \tag{9}$$

(ii) *(The associativity law). If $F, F', F''\colon Z \times I \to X$ are homotopies, such that $F(z,1) = F'(z,0)$ and $F'(z,1) = F'''(z,0)$, for $z \in Z$, then*

$$(F * F') * F'' \simeq F * (F' * F'') \ (\mathrm{rel}\,\partial I). \tag{10}$$

(*iii*) (*The homotopy invariance*). *If* $F, F', G, G' \colon Z \times I \to X$ *are homotopies such that*

$$F \simeq F' \,(\mathrm{rel}\,\partial I), \quad G \simeq G' \,(\mathrm{rel}\,\partial I), \tag{11}$$

then also

$$F * G \simeq F' * G' \,(\mathrm{rel}\,\partial I). \tag{12}$$

In (*i*) $f * F$ denotes the concatenation of the constant homotopy $(z,t) \mapsto f(z)$ with F; an analogous interpretation applies to $F * f'$.

Proof. (*i*) the 2-order homotopy $H \colon Z \times I \times I \to X$, given by

$$H(z,t,s) = \begin{cases} f(z), & 0 \le t \le \frac{s}{2}, \\ F(z, \frac{2t-s}{2-s}), & \frac{s}{2} \le t \le 1, \end{cases} \tag{13}$$

has the property that $H(z,t,0) = F(z,t)$, $H(z,t,1) = (f * F)(z,t)$, $H(z,0,s) = f(z)$ and $H(z,1,s) = f'(z)$, which shows that the first part of (9) holds. Similarly, the 2-order homotopy $H' \colon Z \times I \times I \to X$, given by

$$H'(z,t,s) = \begin{cases} F(z, \frac{2t}{2-s}), & 0 \le t \le \frac{2-s}{2}, \\ f'(z), & \frac{2-s}{2} \le t \le 1, \end{cases} \tag{14}$$

has the property that $H'(z,t,0) = F(z,t)$, $H'(z,t,1) = (F * f')(z,t)$. Moreover, $H'(z,0,s) = f(z)$ and $H'(z,1,s) = f'(z)$, which shows that also the second part of (9) holds.

(*ii*) The 2-order homotopy $H \colon Z \times I \times I \to X$, given by

$$H(z,t,s) = \begin{cases} F(z, \frac{4t}{s+1}), & 0 \le t \le \frac{s+1}{4}, \\ F'(z, 4t - s - 1), & \frac{s+1}{4} \le t \le \frac{s+2}{4}, \\ F''(z, \frac{4t-s-2}{2-s}), & \frac{s+2}{4} \le t \le 1, \end{cases} \tag{15}$$

has the property that $H(z,t,0) = ((F * F') * F'')(z,t)$, $H(z,t,1) = (F * (F' * F''))(z,t)$, $H(z,0,s) = F(z,0)$ and $H(z,1,s) = F''(z,1)$, which shows that (10) holds.

(*iii*). Let $H, H' \colon Z \times I \times I \to X$ be homotopies rel$\,(\partial I)$, which realize (11), i.e.,

$$\begin{aligned} &H(z,t,0) = F(z,t), \quad H(z,t,1) = F'(z,t), \\ &H(z,0,s) = f(z), \quad\;\; H(z,1,s) = f'(z). \end{aligned} \tag{16}$$

$$\begin{aligned} &H'(z,t,0) = G(z,t), \quad H'(z,t,1) = G'(z,t), \\ &H'(z,0,s) = f'(z), \quad\;\; H'(z,1,s) = f''(z). \end{aligned} \tag{17}$$

Then $K \colon Z \times I \times I \to X$, given by

$$K(z,t,s) = \begin{cases} H(z,2t,s), & 0 \le t \le 1/2, \\ H'(z,2t-1,s), & 1/2 \le t \le 1, \end{cases} \tag{18}$$

is well defined, because $H(z,1,s) = f'(z) = H'(z,0,s)$. Moreover, K realizes (12), because $K(z,t,0) = (F*G)(z,t)$, $K(z,t,1) = (F*G)(z,t)$, $K(z,0,s) = HK(z,0,s)$ and $K(z,1,s) = H'(z,1,s)$. $\square$

3.2. *Proof of Lemma* 1. The coherent category of height 1, here denoted by $\mathrm{CH}^{(1)}$, was first defined by Yu. T. Lisitsa in [5] (see [7], 3.1). It occupies an intermediate position between the categories CH and pro-H. Then in [6] two forgetful functors $E^{(1)}\colon \mathrm{CH}{\to}\mathrm{CH}^{(1)}$ and $E^{(10)}\colon \mathrm{CH}^{(1)}{\to}$pro-H were defined and it was proved that the restriction of $E^{(1)}$ to inverse sequences is an isomorphism of categories ([6], Theorem 1, also see [7], Theorem 3.7). In particular, if $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$ is an inverse sequence of spaces and $[\boldsymbol{f}^{(1)}]\colon Z \to \boldsymbol{X}$ is a morphism of $\mathrm{CH}^{(1)}$, then there exists a morphism $[\overline{\boldsymbol{f}}]\colon Z \to \boldsymbol{X}$ of CH such that $E^{(1)}[\overline{\boldsymbol{f}}] = [\boldsymbol{f}^{(1)}]$. Since $E^{(10)}E^{(1)} = E$, it suffices to find such an $\boldsymbol{f}^{(1)}$ that $E^{(10)}(\boldsymbol{f}^{(1)}) = \boldsymbol{f}$, because then $[\overline{\boldsymbol{f}}]$ will have the desired property that $E[\overline{\boldsymbol{f}}] = E^{(10)}(E^{(1)}[\overline{\boldsymbol{f}}]) = E^{(10)}[\boldsymbol{f}^{(1)}] = [E^{(10)}(\boldsymbol{f}^{(1)})] = [\boldsymbol{f}]$.

Recall that a homotopy mapping $\boldsymbol{f}\colon Z \to \boldsymbol{X}$ consists of a sequence of mappings $f_i\colon Z \to X_i$, $i \in \mathbb{N}$, such that $f_i \simeq p_{ii'}f_{i'}$, for $i \le i'$. By definition and [7], Lemma 3.3, a 1-coherent mapping $\boldsymbol{f}^{(1)}\colon Z \to \boldsymbol{X}$ consists of mappings $f_i\colon Z \to X_i$, $i \in \mathbb{N}$, and homotopies $f_{ii'}\colon Z \times I \to X_i$, $i \le i'$, which connect f_i to $p_{ii'}f_{i'}$. Moreover, one requires that $f_{ii}(z,t) = f_i(z)$ and

$$f_{ii'} * p_{ii'}f_{i'i''} \simeq f_{ii''}\ (\mathrm{rel}\,\partial I), \tag{19}$$

for $i \le i' \le i''$. Note that the left side of (19) is well defined, because $f_{ii'}(z,1) = p_{ii'}f_{i'}(z) = p_{ii'}f_{i'i''}(z,0)$.

By definition, $E^{(10)}(\boldsymbol{f}^{(1)})$ is the homotopy mapping obtained by "forgetting" the homotopies $f_{ii'}$. Consequently, to complete the proof of Lemma 1, it suffices to exhibit homotopies $f_{ii'}\colon Z \times I \to X_i$, $i \le i'$, such that $f_{ii'}(z,0) = f_i(z)$, $f_{ii'}(z,1) = p_{ii'}f_i(z)$, $f_{ii}(z,t) = f_i(z)$ and that (19) holds.

We define the homotopies $f_{ii'}\colon Z{\times}I \to X_i$, $i \le i'$. by induction on $j = i'{-}i \ge 0$. If $j = 0$, i.e., $i = i'$, we put $f_{ii} = f_i$, i.e., $f_{ii}(z,t) = f_i(z)$. If $j = 1$, i.e., $i' = i{+}1$, we choose homotopies $f_{ii'} = f_{i,i+1}\colon Z{\times}I \to X_i$ in such a way that $f_{i,i+1}(z,0) = f_i(z)$ and $f_{i,i+1}(z,1) = p_{i,i+1}f_{i+1}(z)$. If we have already defined $f_{ii'}$, where $i'{-}i = j \ge 1$, we define $f_{i,i'+1}$ as the concatenation $f_{i,i'+1} = f_{ii'} * p_{ii'}f_{i',i'+1}$, i.e., the mapping $f_{i,i'+1}$ is given by

$$f_{i,i'+1}(z,t) = \begin{cases} f_{ii'}(z,2t), & 0 \le t \le 1/2, \\ p_{ii'}f_{i',i'+1}(z,2t-1), & 1/2 \le t \le 1. \end{cases} \tag{20}$$

Note that $f_{i,i'+1}(z,t)$ is well defined by (9), because $f_{ii'}(z,1) = p_{ii'}f_{i'+1}(z) = p_{ii'}f_{i',i'+1}(z,0)$. By (20) and the induction hypothesis, $f_{i,i'+1}(z,0) = f_{ii'}(z,0) = f_i(z)$. Also $f_{i,i'+1}(z,1) = p_{ii'}f_{i',i'+1}(z,1)$ and $f_{i',i'+1}(z,1) = p_{i',i'+1}f_{i'+1}(z)$. Consequently, $f_{i,i'+1}(z,1) = p_{ii'}p_{i',i'+1}f_{i'+1}(z) = p_{i,i'+1}f_{i'+1}(z)$. It remains to verify (19).

If $i = i' = i''$, both sides of (9) equal f_i and the assertion holds. If $i = i' < i''$, then $f_{ii'} * p_{ii'}f_{i'i''} = f_i * f_{ii''}$ and the assertion follows from the first part of (9). If $i < i' = i''$, then $f_{ii'} * p_{ii'}f_{i'i''} = f_{ii'} * p_{ii'}f_{i'}$ and the assertion follows from the second part of (9).

It remains to prove the assertion in the case when $i < i' < i''$. This is done by induction on $l = i'' - i' \ge 1$. If $l = 1$, i.e., $i'' = i' + 1$, then, by definition, $f_{ii''} = f_{i,i'+1} = f_{ii'} * p_{ii'}f_{i'i''}$ and (9) holds. Now assume that we have proved (9), for a given $l \ge 1$. Let us prove it for $l + 1$. By the inductive assumption, $F = f_{i,i'+l} \simeq$

$F' = f_{ii'} * p_{ii'} f_{i',i'+l}$ $(\mathrm{rel}\,\partial I)$. Putting $G = p_{i,i'+l} f_{i'+l,i'+l+1}$ and using (12), we see that $f_{i,i'+l} * p_{i,i'+l} f_{i'+l,i'+l+1} \simeq (f_{ii'} * p_{ii'} f_{i',i'+l}) * p_{i,i'+l} f_{i'+l,i'+l+1}$ $(\mathrm{rel}\,\partial I)$. However, by definition, $f_{ii''} = f_{i,i'+l+1} = f_{i,i'+l} * p_{i,i'+l} f_{i'+l,i'+l+1}$ and thus, $f_{ii''} \simeq (f_{ii'} * p_{ii'} f_{i',i'+l}) * p_{i,i'+l} f_{i'+l,i'+l+1}$ $(\mathrm{rel}\,\partial I)$. Concatenation with G was possible, because $f_{i,i'+l}(z,1) = p_{i,i'+l} f_{i'+l}(z)$ and also $p_{i,i'+l} f_{i'+l,i'+l+1}(z,0) = p_{i,i'+l} f_{i'+l}(z)$. Application of the associativity law (10) shows that $f_{ii''} \simeq f_{ii'} * (p_{ii'} f_{i',i'+l} * p_{i,i'+l} f_{i'+l,i'+l+1})$ $(\mathrm{rel}\,\partial I)$. Since $p_{i,i'+l} = p_{ii'} p_{i',i'+l}$, one readily concludes that $p_{ii'} f_{i',i'+l} * p_{i,i'+l} f_{i'+l,i'+l+1} = p_{ii'}(f_{i',i'+l} * p_{i',i'+l} f_{i'+l,i'+l+1})$ and thus, one obtains the relation $f_{ii''} \simeq f_{ii'} * p_{ii'}(f_{i',i'+l} * p_{i',i'+l} f_{i'+l,i'+l+1})$ $(\mathrm{rel}\,\partial I)$. However, the expression in the parenthesis equals, by definition, $f_{i',i'+l+1} = f_{i'i''}$ and we have obtained the desired relation $f_{ii''} \simeq f_{ii'} * p_{ii'} f_{i'i''}$ $(\mathrm{rel}\,\partial I)$. □

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ЕДНА ТЕОРЕМА ЗА ЕГЗИСТЕНЦИЈА ВО ВРСКА СО СТАНДАРДЕН ОБЛИК НА ПРОИЗВОД

Сибе Мардешиќ

Р е з и м е

Во овој труд се разгледува прашањето кога директниот производ $X \times P$ на компактен метрички простор X и полиедар P е производ во категоријата на облик на тополошки простори.
Прашањето се состои од два дела:
Делот за егзистенција, т.е. дали за секој тополошки простор Z и секој морфизам на облик $F : Z \to X$ и секоја класа на хомотопија на пресликувања $[g] : Z \to P$ постои морфизам на облик $H : Z \to X \times P$ чии што композиции со каноничните проекции на $X \times P$ се еднакви на F и $[g]$, соодветно.
Делот за единственост се однесува на прашањето дали H е единствено.

Познато е дека, во општ случај, делот за единственост не важи дури и кога Z е полиедар.

Главниот резултат во овој труд е доказот дека делот за егзистенција секогаш важи.

Доказот се базира на аналоген резултат за јак облик.

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