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ON THE MINKOWSKI DIMENSION OF CERTAIN KAKEYA SETS

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Abstract. The Kakeya conjecture states that all compact subsets of $\mathbb{R}^n$ containing a unit line segment in every direction have full Hausdorff dimension. The analogue of the Kakeya conjecture with $\mathbb{R}^n$ replaced by $\mathbb{Q}_p^n$ was recently proved in [1]. An earlier draft of that article proved a special case concerning Minkowski dimension by using a more specialized combinatorial argument. The referees for [1] suggested that this is of independent interest, and that it should be published as a separate article. Thus, this article documents that argument.

1. Introduction

In 1917, Sōichi Kakeya posed the Kakeya needle problem, asking about the minimum area of a region in the plane in which a needle of unit length can be rotated around by 360°. Besicovitch [2] proved that in a certain sense the answer is “arbitrarily small”, by constructing such a region of Lebesgue measure zero. On the other hand, Davies [3] proved that such a region must be large in a different sense: it must have Minkowski dimension 2. Subsequently, regions in Euclidean space containing a unit line segment in every direction were dubbed Kakeya sets. The construction of [2] immediately extends to higher dimensions, showing that any finite-dimensional Euclidean space contains a Kakeya set of Lebesgue measure zero. Much more difficult is the analogue of the result of [3] in higher dimensions: it is the Minkowski dimension version of the Kakeya conjecture, which is one of the most important open problems in geometric measure theory, and analysis in general.

Conjecture A (Kakeya, Minkowski version)**:** Let n be a positive integer. All Kakeya sets in $\mathbb{R}^n$ have Minkowski dimension n.

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The Kakeya conjecure has deep connections with harmonic analysis among other fields, and it is open for $n \geqslant 3$: the state of the art is the result of Katz–Tao [8] that all Kakeya sets in $\mathbb{R}^n$ have Minkowski dimension at least $(2-\sqrt{2})(n-4)+3$. As a possible approach to the Euclidean Kakeya conjecture, Wolff [9] suggested the analogous question over finite fields, and this finite field Kakeya conjecture was proved by Dvir [5]. As noted by Ellenberg–Oberlin–Tao [6], the analogy between the Euclidean and the finite field Kakeya conjectures breaks down in that there is no non-trivial natural notion of distance in finite vector spaces. Therefore, the question arises whether there is a version of the Kakeya conjecture over rings that have multiple scales, such as the ring of p-adic integers $\mathbb{Z}_p$ for a prime number p, which is topologically much more similar to $\mathbb{R}$ than finite fields are (this was originally asked by James Wright). Our main result is a proof of this version of the Kakeya conjecture.

Theorem 1. *Let p be a prime number and n a positive integer. All Kakeya sets in $\mathbb{Z}_p^n$ have Minkowski dimension n.*

We obtain this result as the limit of the following theorem.

Theorem 2. *Let p be a prime number and n and k positive integers. All Kakeya sets in $(\mathbb{Z}/p^k\mathbb{Z})^n$ have size at least $(kn)^{-n}p^{kn}$.*

The proof involves a generalization of a recent idea of Dhar–Dvir [4], and a tensor product trick over local rings which we suspect may be applicable to other similar questions. The proof is surprisingly simple and elegant, and by virtue of this we keep the article fully self-contained; in particular, we do not rely on any results from [1] or [4].

2. Proof

Let p be a prime number, n and k be positive integers, and $q = p^k$. Let $\mathbb{F} = \mathbb{F}_p$, and $R = \mathbb{Z}/q\mathbb{Z}$. Let $\mathbb{Q}_p$ denote the p-adic numbers, and $\mathbb{Z}_p$ denote the p-adic integers.

Definition 2.1. *A Kakeya set in R^n is a subset $S \subseteq R^n$ such that, for all $x \in R^n$, there is a $b_x \in R^n$ such that $b_x + \lambda x \in S$ for all $\lambda \in R$.*

A Kakeya set in $\mathbb{Z}_p^n$ is a subset $S \subseteq \mathbb{Z}_p^n$ such that, for all $x \in \mathbb{Z}_p^n$, there is a $b_x \in \mathbb{Z}_p^n$ such that $b_x + \lambda x \in S$ for all $\lambda \in \mathbb{Z}_p$.

The Minkowski dimension of a subset $S \subseteq R^n$ is $\dim_{\mathrm{Min}} S = \frac{\log_p|S|}{\log_p|R|}$.

Let $S \subseteq \mathbb{Z}_p^n$, and, for all positive integers l, let S_l be the image of S under the projection $\mathbb{Z}_p^n \to (\mathbb{Z}/p^l\mathbb{Z})^n$. The Minkowski dimension of S is the limit

$$\dim_{\mathrm{Min}} S = \lim_{l\to\infty} \dim_{\mathrm{Min}} S_l,$$

if that limit exists.

The definitions in [4, 6, 7] are slightly different (they only consider directions in $\mathbb{P}^{n-1}(R)$), but they are equivalent. It is clear that theorem 2 implies theorem 1: if $S \subseteq \mathbb{Z}_p^n$ is a Kakeya set, then so is each S_l, so, assuming the bound in theorem 2,

$$n \geqslant \dim_{\mathrm{Min}} S_l \geqslant n\left(1 - \frac{\log_p(ln)}{l}\right) \text{ for all positive integers } l$$
$$\implies n \geqslant \lim_{l\to\infty} \dim_{\mathrm{Min}} S_l \geqslant n \implies \lim_{l\to\infty} \dim_{\mathrm{Min}} S_l = n.$$

Thus our effort for the remainder of this article is dedicated to proving theorem 2. Let $\zeta \in \overline{\mathbb{Q}}_p$ be a primitive qth root of unity. Let

$$T = \mathbb{Z}[z] \text{ and } \overline{T} = \mathbb{F}[z]/(z^q - 1) = T/(p, z^q - 1).$$

The element $t = z - 1 \in \overline{T}$ is such that $t^q = (z-1)^q = z^q - 1 = 0$, so $\overline{T} = \mathbb{F}[t]/(t^q)$. Let us define the $\mathbb{F}$-rank of a matrix M over $\overline{T}$ as the maximum number of $\mathbb{F}$-linearly independent columns of M, and let us denote it by $\operatorname{rank}_{\mathbb{F}} M$. For a positive integer m, let M_m be the $q^m \times q^m$ matrix over $\overline{T}$ defined by

$$M_m = \left(z^{\langle u,v\rangle}\right)_{u,v\in R^m}.$$

So the rows of M_m are indexed by $u = (u_1, \ldots, u_m) \in R^m$, the columns are indexed by $v = (v_1, \ldots, v_m) \in R^m$, and the entry in row u and column v is

$$z^{u_1v_1+\cdots+u_mv_m} = (1+t)^{u_1v_1+\cdots+u_mv_m} \in \overline{T}.$$

This entry is well-defined since $z^q = 1$. The following proposition is a generalization of a result of Dhar–Dvir [4].

Proposition 2.1. *All Kakeya sets in R^n have size at least* $\operatorname{rank}_{\mathbb{F}} M_n$.

Proof. Proof. Let $S \subseteq R^n$ be a Kakeya set. Let U_S be the $|S|\times q^n$ matrix over

$$\mathbb{Q}_p(\zeta)[z]/(z^q - 1),$$

with rows indexed by $s \in S$ and columns indexed by $v \in R^n$, with the entry in row s and column v equal to

$$(U_S)_{s,v} = \zeta^{\langle s,v\rangle} \in \mathbb{Q}_p(\zeta) \subset \mathbb{Q}_p(\zeta)[z]/(z^q - 1).$$

Let r_S be the maximum number of $\mathbb{Z}_p[\zeta]$-linearly independent columns of U_S. As all entries of U_S belong to $\mathbb{Q}_p(\zeta)$, r_S is equal to the $\mathbb{Q}_p(\zeta)$-rank of U_S (seen as a matrix over $\mathbb{Q}_p(\zeta)$), which is at most the number of rows $|S|$. Since S is a Kakeya set, for all $u \in R^n$, there is a $b_u \in R^n$ such that $b_u + \lambda u \in S$ for all $\lambda \in R$. For each $u \in R^n$, let us fix a $b_u \in R^n$ with this property. Let V be the $q^n \times q^n$ matrix over $\mathbb{Q}_p(\zeta)[z]/(z^q - 1)$, with rows indexed by $u \in R^n$ and columns indexed by $v \in R^n$, with the entry in row u and column v equal to

$$V_{u,v} = \zeta^{\langle b_u,v\rangle} z^{\langle u,v\rangle} \in \mathbb{Q}_p(\zeta)[z]/(z^q - 1).$$

For all $u \in R^n$ and all $v \in R^n$,

$$\zeta^{\langle b_u,v\rangle} z^{\langle u,v\rangle} = \zeta^{\langle b_u,v\rangle} \sum_{\lambda\in R} \sum_{l=0}^{q-1} q^{-1} \zeta^{\lambda(\langle u,v\rangle - l)} z^l$$

$$= \sum_{\lambda \in R} \sum_{l=0}^{q-1} q^{-1} \zeta^{-\lambda l} z^l \zeta^{\langle b_u + \lambda u, v\rangle}. \tag{2.1}$$

Since $b_u + \lambda u \in S$ for all $u \in R^n$ and all $\lambda \in R$, equation (2.1) implies that every row of V is a $\mathbb{Q}_p(\zeta)[z]/(z^q - 1)$-linear combination of the rows of U_S. I.e., $V = CU_S$ for some matrix C over $\mathbb{Q}_p(\zeta)[z]/(z^q - 1)$. Therefore, any non-trivial $\mathbb{Z}_p[\zeta]$-linear dependency of the columns of U_S (which is a non-zero vector c with entries in $\mathbb{Z}_p[\zeta]$ such that $U_S c = 0$) gives a non-trivial $\mathbb{Z}_p[\zeta]$-linear dependency of the corresponding columns of V (since $Vc = CU_S c = 0$). In particular, the maximum number of $\mathbb{Z}_p[\zeta]$-linearly independent columns of V is at most $r_S \leqslant |S|$. All entries of V belong to the lattice $\mathbb{Z}_p[\zeta][z]/(z^q - 1)$, so we may reduce V modulo p. Reduction modulo p maps $\zeta \in \mathbb{Z}_p[\zeta]$ to 1, so the resulting matrix $\overline{V}$ is over $\mathbb{F}[z]/(z^q - 1) = \overline{T}$. To be more specific, $\overline{V}$ is the $q^n \times q^n$ matrix over $\overline{T}$, with rows indexed by $u \in R^n$ and columns indexed by $v \in R^n$, with the entry in row u and column v equal to

$$\overline{V}_{u,v} = z^{\langle u,v\rangle} \in \overline{T}.$$

So $\overline{V} = M_n$. Any non-trivial $\mathbb{Z}_p[\zeta]$-linear dependency of the columns of V gives a non-trivial $\mathbb{F}$-linear dependency of the corresponding columns of $\overline{V}$ (as, by suitably re-normalizing, we can ensure that some coefficient of the $\mathbb{Z}_p[\zeta]$-linear dependency is a p-adic unit). So the maximum number of $\mathbb{F}$-linearly independent columns of $\overline{V} = M_n$ is at most $r_S \leqslant |S|$, implying that $\operatorname{rank}_{\mathbb{F}} M_n \leqslant |S|$. $\square$

Before proceeding to the proof of theorem 2, let us prove a technical lemma concerning the decomposition of a certain Vandermonde matrix.

Lemma 1. *Let W be the $q \times q$ matrix over $T = \mathbb{Z}[z]$ defined by*

$$W = \left(z^{ij}\right)_{i,j \in \{0,\dots,q-1\}}.$$

There is a lower triangular matrix L over T with 1's on the diagonal, and an upper triangular matrix U over T with jth diagonal entry (for $j \in \{0, \dots, q-1\}$) equal to $\prod_{w=0}^{j-1}(z^j - z^w)$, such that $W = LU$.

Proof. Proof. For $l \in \{0, \dots, q-1\}$, let $f_l \in T[X]$ be the polynomial

$$f_l(X) = \prod_{w=0}^{l-1} (X - z^w)$$

(so that $f_0(X) = 1$). These polynomials are monic and

$$\deg f_l = l,$$

so there exist $a_{i,l} \in T$ for $i, l \in \{0, \dots, q-1\}$ such that $a_{i,l} = 0$ when $i < l$, $a_{i,i} = 1$ for all $i \in \{0, \dots, q-1\}$, and

$$X^i = \sum_{l=0}^{i} a_{i,l} f_l(X)$$

for all $i \in \{0, \ldots, q-1\}$. Let

$$L = (a_{i,l})_{i,l \in \{0,\ldots,q-1\}}, \text{ and } U = \left(f_l(z^j)\right)_{l,j \in \{0,\ldots,q-1\}}.$$

Then $W = LU$; L is lower triangular, over T, and with 1's on the diagonal; for $l, j \in \{0, \ldots, q-1\}$ such that $l > j$, $f_l(X)$ is divisible by $X - z^j$, implying that $f_l(z^j) = 0$, implying in turn that U is upper triangular, over T, with jth diagonal entry (for $j \in \{0, \ldots, q-1\}$) equal to $f_j(z^j) = \prod_{w=0}^{j-1}(z^j - z^w)$. □

Proof. Proof of theorem 2. Let $\overline{W}, \overline{U}, \overline{L}$ be the reductions modulo $(p, z^q - 1)$ of W, U, L from lemma 1. Then $M_1 = \overline{W} = \overline{LU}$; $\overline{L}$ is a lower triangular matrix over $\overline{T}$ with 1's on the diagonal; and $\overline{U}$ is an upper triangular matrix over $\overline{T}$ with jth diagonal entry (for $j \in \{0, \ldots, q-1\}$) equal to

$$\overline{U}_{j,j} = \prod_{w=0}^{j-1}(z^j - z^w) = (1+t)^{\binom{j}{2}} \prod_{l=1}^{j}((1+t)^l - 1).$$

Moreover, M_n is the nth tensor power (over $\overline{T}$) of M_1, so

$$M_n = M_1^{\otimes_{\overline{T}} n} = (\overline{LU})^{\otimes_{\overline{T}} n} = \overline{L}^{\otimes_{\overline{T}} n}\overline{U}^{\otimes_{\overline{T}} n}.$$

Then $L_n = \overline{L}^{\otimes_{\overline{T}} n}$ is a lower triangular matrix over $\overline{T}$ with 1's on the diagonal, and $U_n = \overline{U}^{\otimes_{\overline{T}} n}$ is an upper triangular matrix over $\overline{T}$. In particular, L_n is invertible, and $\operatorname{rank}_{\mathbb{F}} U_n$ is at least as large as the number of non-zero diagonal entries of U_n. The invertibility of L_n implies that a vector v is a non-trivial $\mathbb{F}$-linear dependency of the columns of U_n if and only if the entries of $v \neq 0$ are in $\mathbb{F}$ and $U_n v = 0$, if and only if the entries of $v \neq 0$ are in $\mathbb{F}$ and $M_n v = L_n U_n v = 0$, if and only if v is a non-trivial $\mathbb{F}$-linear dependency of the columns of M_n. Therefore,

$$\operatorname{rank}_{\mathbb{F}} M_n = \operatorname{rank}_{\mathbb{F}} U_n \geqslant \# \text{ of non-zero diagonal entries of } U_n.$$

The q^n diagonal entries of U_n are precisely the elements of the multiset

$$\left\{ \prod_{i=1}^{n} \overline{U}_{j_i,j_i} \mid (j_1, \ldots, j_n) \in \{0, \ldots, q-1\}^n \right\}.$$

Let $J = \{0, \ldots, \lceil \frac{q}{kn} \rceil - 1\}$. Suppose that $j \in J$. By using Kummer's theorem on the p-adic valuations of binomial coefficients, which implies that $\binom{l}{w}$ is a unit in $\mathbb{F}$ if and only if every p-adic digit of w is at most as large as the corresponding p-adic digit of l, we can deduce that the smallest integer α_l such that $(1+t)^l - 1 \in t^{\alpha_l}\overline{T}^{\times}$ is equal to $p^{v_p(l)}$ (whenever $l \in \{1, \ldots, q-1\}$). Therefore, the smallest integer β_j such that

$$\overline{U}_{j,j} \in t^{\beta_j}\overline{T}^{\times}$$

is equal to

$$\begin{aligned} \min\left\{q, \sum_{l=1}^{j} p^{v_p(l)}\right\} &\leqslant \sum_{y=0}^{\lfloor \log_p j \rfloor} \left(\left\lfloor \frac{j}{p^y} \right\rfloor - \left\lfloor \frac{j}{p^{y+1}} \right\rfloor \right) p^y \leqslant j(1 + \lfloor \log_p j \rfloor) \\ &< \frac{q(k + \lfloor 1 - \log_p(q/j) \rfloor)}{kn} \leqslant \frac{q}{n}. \end{aligned}$$

Suppose that $(j_1, \ldots, j_n) \in J^n$. Then the smallest integer $\beta_{(j_1,\ldots,j_n)}$ such that

$$\prod_{i=1}^{n} \overline{U}_{j_i,j_i} \in t^{\beta_{(j_1,\ldots,j_n)}} \overline{T}^{\times}$$

is equal to

$$\min\left\{q, \sum_{i=1}^{n} \beta_{j_i}\right\} < q \text{ (since } \beta_{j_i} < \frac{q}{n} \text{ for all } i \in \{1,\ldots,n\}).$$

In particular, $\prod_{i=1}^{n} \overline{U}_{j_i,j_i}$ is non-zero. So U_n has at least $|J^n| \geqslant (kn)^{-n} q^n$ non-zero diagonal entries, implying that

$$\operatorname{rank}_{\mathbb{F}} M_n = \operatorname{rank}_{\mathbb{F}} U_n \geqslant (kn)^{-n} q^n = (kn)^{-n} p^{kn}.$$

In light of propositon 2.1, this completes the proof. $\square$

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