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SOME COVERING PROPERTIES USING MAXIMAL OPEN COVERS

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Abstract. The aim of this article is to study some covering properties using maximal open covers in topological spaces. We see that in a m-Lindelöf topological space each uncountable subset has a complete m-accumulation point as well as m-accumulation point. Furthermore we see that a Hausdorff m-paracompact topological space is minimal c-normal.

1. Introduction

We simply write X to denote a topological space $(X, \mathscr{T})$. By a proper open set (resp., closed set) of a topological space X, we mean an open set $G \neq \emptyset, X$ (resp., $E \neq \emptyset, X$). We also write $|A|$ to denote the cardinality of the subset A of X. By $A \subsetneq B$ we mean that $A \subseteq B$ but $A \neq B$.

In recent years, the notions of minimal, maximal and mean open and closed sets become a centre of attraction in the literature. There are significance number of research articles investigating several concepts using minimal, maximal and mean open and closed set. The investigative aspects of such kind of sets are still yielding new and interesting concepts. Nakaoka and Oda [9] introduced and studied the concept of maximal open sets in topological spaces: a proper open set U of a topological space X is said to be a maximal open set if U is contained in an open set G of X, then $G = U$ or $G = X$. Therefore if U is a maximal open set in a topological space X, there is no proper open set V in X such that $U \subsetneq V$. Considering this nature of maximal open sets, Mukharjee et al. [7] define s-refinement (Definition 2.3) which immediately leads to define maximal open covers of topological spaces (Definition 2.4).

2. Basic definitions and results

Firstly, we recall the followings.

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Definition 2.1 (Nakaoka and Oda, [10]). A proper closed set E of X is said to be a *minimal closed set* if F is a closed set of X contained in E, then $E = \emptyset$ or $E = F$.

Definition 2.2 (Nakaoka and Oda, [8, 9]). A proper open set U of X is said to be a *maximal open set* if U is contained in an open set G of X, then $G = U$ or $G = X$.

Definition 2.3 (Mukharjee et al., [7]). Let $\mathscr{A}$ and $\mathscr{B}$ be two covers of a topological space X. $\mathscr{A}$ is said to be an *s-refinement* of $\mathscr{B}$ if for each $A \in \mathscr{A}$ there is a $B \in \mathscr{B}$ such that $A \subsetneq B$. An s-refinement $\mathscr{A}$ of $\mathscr{B}$ is said to be an *open s-refinement* of $\mathscr{B}$ if all members of $\mathscr{A}$ are open.

Definition 2.4 (Mukharjee et al., [7]). An open cover $\mathscr{A}$ is said to be a *maximal open cover* of X if it is not an s-refinement of any other open cover of X.

Definition 2.5 (Mukharjee et al., [7]). A topological space X is said to be *m-compact* if each maximal open cover of X has an open finite s-refinement.

Definition 2.6 (Mukharjee et al., [7]). Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *m-continuous* if $f^{-1}(U)$ is a maximal open set in X for each proper open set U in Y.

Definition 2.7 (Mukharjee et al., [7]). A point x of a topological space X is said to be *m-complete accumulation point* of a subset K of X if $|G \cap K| = |K|$ for each maximal open set G containing x.

Definition 2.8 (Benchalli et al., [2]). A topological space X is said to be *minimal c-regular* if for each $x \in X$ and each minimal closed set F with $x \notin F$, there exist disjoint open sets U, V such that $x \in U$ and $F \subseteq V$.

Definition 2.9 (Benchalli et al., [3]). A topological space X is said to be *minimal c-normal* if for each pair of distinct minimal closed sets E, F, there exist disjoint open sets U, V such that $E \subseteq U$ and $F \subseteq V$.

Definition 2.10 (Mukharjee and Bagchi, [6]). An open set M of X is said to be a *mean open set* X if there exist two proper open sets U and V of X satisfying $U \subsetneq M \subsetneq V$.

Lemma 1 (Mukharjee et al., [7]). *An open cover containing a maximal open set is maximal.*

Theorem 1 (Willard, [11]). *If X is a T_3 topological space, then following are equivalent:*

(i) X is paracompact,
(ii) each open cover of X has an open σ-locally finite refinement,
(iii) each open cover of X has a locally finite refinement (not necessarily open),
(iv) each open cover of X has a closed locally finite refinement,

Theorem 2 (Nakaoka and Oda, [9]). *If U is a maximal open set and W is an open set in X, then either $U \cup W = X$ or $W \subseteq U$. If W is also a maximal open set distinct from U, then $U \cup W = X$.*

Theorem 3 (Nakaoka and Oda, [10]). *If E is a minimal closed set and F is a closed set in X, then either $E \cap F = \emptyset$ or $E \subseteq F$. If F is also a minimal closed set distinct from E, then $E \cap F = \emptyset$.*

Theorem 4 (Mukharjee et al., [7]). *Every infinite T_1 connected topological space is m-compact.*

3. Main results

Definition 3.1. A topological space X is said to be *weakly m-compact* if each maximal open cover of X has an open finite refinement.

Let $Y \subseteq X$. Y is said to be a *weakly m-compact subset of* X if $(Y, \mathscr{T}_Y)$ is weakly m-compact.

Theorem 5. *Let X be a m-compact topological space and F be minimal closed in X. Then F is weakly m-compact.*

Proof. Let $\mathscr{A}$ be a maximal open cover of the minimal closed set F. Then for each $A \in \mathscr{A}$, there is an open set B_A in X such that $A = F \cap B_A$. Now $X - F$ is a maximal open set in X, and so, by the Lemma 1, $\mathscr{B} = \{B_A : A \in \mathscr{A}\} \cup \{X - F\}$ is a maximal open cover of X. By m-compactness of X, $\mathscr{B}$ has an open finite s-refinement $\{U_1, U_2, ..., U_n\}$, say. Clearly $\{U_1 \cap F, U_2 \cap F, ..., U_n \cap F\}$ is an open finite refinement of $\mathscr{A}$. □

Definition 3.2. Let X be a topological space, $A \subseteq X$ and $x \in X$. x is said to be an *m-accumulation point* of A if for each maximal open set containing x contains at least one point of A other than x.

Theorem 6. *Let X be an m-compact topological space. Then every infinite subset of X has an m-accumulation point.*

Proof. Let A be an infinite subset of X. If possible, let A has no m-accumulation point. Then for each $x \in X$, there is a maximal open set U_x in X such that $x \in U_x$ and $U_x \cap A = \emptyset$ or $U_x \cap A = \{x\}$. Now $\mathscr{A} = \{U_x : x \in X\}$ is a maximal open cover of X (by the Lemma 1). By the m-compactness of X, there is a finite s-refinement $\mathscr{B}$ of $\mathscr{A}$. Let us write $\mathscr{B} = \{B_{x_1}, B_{x_2}, ..., B_{x_n}\}$. Then $A \subseteq X = \bigcup_{i=1}^{n} B_{x_i}$. But for each $i \in \{1, 2, ..., n\}$, $A \cap B_{x_i} = \emptyset$ or $A \cap B_{x_i} = \{x_i\}$. This means that cardinality of A is at most n. Which contradicts the fact that A is infinite. □

Definition 3.3. A topological space X is said to be *m-Lindelöf* if every maximal open cover of X has an open countable s-refinement.

Theorem 7. *Let X and Y be topological spaces, where X is m-Lindelöf and $f : X \to Y$ be a bijective m-continuous function. Then Y is also m-Lindelöf.*

Proof. Let $\mathscr{A}^{(Y)}$ be a maximal open cover of Y. Since f is a bijective m-continuous function, $\mathscr{A}^{(X)} = \{f^{-1}(A) : A \in \mathscr{A}^{(Y)}\}$ is a maximal open cover of X (by Definition 2.6 and Lemma 1). By m-Lindelöfness of X, $\mathscr{A}^{(X)}$ has an open countable s-refinement $\mathscr{A}_1^{(X)} = \{B_\lambda : \lambda \in \Lambda\}$, say, where the index set Λ is countable. Since f is bijective, it follows that $\mathscr{A}_1^{(Y)} = \{f(B_\lambda) : \lambda \in \Lambda\}$ covers Y. Now, let $f(B_\lambda)$

be a member of $\mathscr{A}_1^{(Y)}$. Then $B_\lambda \in \mathscr{A}_1^{(X)}$. Since $\mathscr{A}_1^{(X)}$ is an s-refinement of $\mathscr{A}^{(X)}$, we have $B_\lambda \subsetneq f^{-1}(A)$, for some $A \in \mathscr{A}^{(Y)}$. Again since f is bijective, it follows that $f(B_\lambda) \subsetneq A$. Therefore $\mathscr{A}_1^{(Y)}$ is an open countable s-refinement of $\mathscr{A}^{(Y)}$. □

Theorem 8. *Let X be an m-Lindelöf topological space and K be a subset of X with $|K| \geq \mathrm{c}$. Then K has a complete m-accumulation point.*

Proof. Suppose for each $x \in X$, there is a maximal open set V_x containing x and satisfying $|V_x \cap K| < |K|$. Then $|V_x \cap K| \leq \aleph_0$, for each $x \in X$. Since $\{V_x : x \in X\}$ is an open cover of X consists of maximal open sets, by Lemma 1, $\{V_x : x \in X\}$ is a maximal open cover of X. So there is an open countable s-refinement $\{V_{x_i} : x_i \in X, i \in \Lambda\}$, where the index set Λ is a countable subset, of $\{V_x : x \in X\}$. Now $|K| = \bigcup_{i\in\Lambda}(V_{x_i} \cap K) \leq \aleph_0$. But this implies that $|K| \leq \aleph_0 < \mathrm{c} \leq |K|$, which is a contradiction. □

Theorem 9. *Let X be an m-Lindelöf topological space and K be an uncountable subset of X. Then K has a m-accumulation point.*

Proof. If possible, let K has no m-accumulation point. Then for each $x \in X$, there is a maximal open set U_x in X such that $x \in U_x$ and $U_x \cap K = \emptyset$ or $U_x \cap K = \{x\}$. Now $\mathscr{A} = \{U_x : x \in X\}$ is a maximal open cover of X (by the Lemma 1). By the m-Lindelöfness of X, there is an open countable s-refinement $\mathscr{B}$ of $\mathscr{A}$. Let us write $\mathscr{B} = \{B_{x_1}, B_{x_2}, ..., B_{x_n}, ...\}$. Then $K \subseteq X = \bigcup_{i=1}^{\infty} B_{x_i}$. But for each $i \in \{1, 2, ..., n, ...\}$, $K \cap B_{x_i} = \emptyset$ or $K \cap B_{x_i} = \{x_i\}$. This means that cardinality of K is at most $\aleph_0$. Which contradicts the fact that K is uncountable. □

Definition 3.4. A topological space X is said to be *weakly m-Lindelöf* if each maximal open cover of X has a countable open refinement.

Let $Y \subseteq X$. Y is said to be a *weakly m-compact subset* of X if $(Y, \mathscr{T}_Y)$ is weakly m-compact.

Theorem 10. *Let X be a m-Lindelöf topological space and F be minimal closed in X. Then F is weakly m-Lindelöf.*

Proof. Proof is similar to the proof of Theorem 5. □

Definition 3.5. A topological space X is said to be *countably m-compact* if every countable maximal open cover has a finite open s-refinement.

Obviously an m-compact topological space is countably m-compact.

Theorem 11. *Let X be a Lindelöf topological space containing a minimal closed set F. Then following are equivalent:*

(i) X is m-compact.
(ii) X is countably m-compact.

Proof. $(i) \Rightarrow (ii)$
This part is obvious.
$(ii) \Rightarrow (i)$
Let $\mathscr{A}$ be a maximal open cover of X. By Lindelöfness of X, $\mathscr{A}$ has a countable subcollection $\mathscr{B}$, say, that covers X. Then by the Lemma 1, $\mathscr{B} \cup \{X - F\}$ is

a countable maximal open cover of X. By the countably m-compactness of X, $\mathscr{B} \cup \{X - F\}$ has a finite open s-refinement of X, i.e., X is m-compact. □

Theorem 12. *An infinite T_1 connected topological space is countably m-compact.*

Proof. Proof follows from Theorem 4. □

Definition 3.6. A topological space X is said to be an *m-paracompact* topological space if each maximal open cover of X has an open locally finite s-refinement.

Theorem 13. *If X is an m-paracompact topological space, then each maximal open cover of X has an open σ-locally finite s-refinement.*

Proof. Proof is trivial. □

Lemma 2. *Let $\mathscr{A}$ be an s-refinement (resp., refinement) of $\mathscr{B}$ and $\mathscr{B}$ be a refinement (resp., s-refinement) of $\mathscr{C}$. Then $\mathscr{A}$ is an s-refinement of $\mathscr{C}$.*

Proof. Proof is very easy and hence omitted. □

Theorem 14. *If X is a m-paracompact topological space, then each maximal open cover of X has a locally finite s-refinement (not necessarily open).*

Proof. Proof follows from the Lemma 2 and the Theorem 1. □

Theorem 15. *A Hausdorff m-paracompact topological space is minimal c-regular.*

Proof. Let X be a Hausdorff m-paracompact topological space. Also let $x \in X$ and F be a minimal closed set such that $x \notin F$. Then for each $y \in F$, there exist disjoint open sets U_y, V_y such that $x \in U_y$ and $y \in V_y$. Clearly $x \notin cl(V_y)$. Then $\mathscr{V} = \{V_y : y \in F\} \cup \{X - F\}$ is a maximal open cover of X, by the Lemma 1. Since X is m-paracompact, there is an open locally finite s-refinement $\mathscr{W}$, say, of $\mathscr{V} = \{V_y : y \in F\} \cup \{X - F\}$.

Let $V = \bigcup\{W \in \mathscr{W} | W \cap F \neq \emptyset\}$. Then V is an open set which contains F. Since $\{W \in \mathscr{W} | W \cap F \neq \emptyset\}$ is a subcollection of a locally finite family, it is locally finite and therefore $cl(V) = \bigcup\{cl(W) : W \in \mathscr{W}, W \cap F \neq \emptyset\}$. Now for each $W \in \mathscr{W}$, there is a $V_y \in \mathscr{V}$ such that $W \subsetneq V_y$, i.e., $cl(W) \subseteq cl(V_y)$. Thus $x \notin cl(V)$, i.e., $x \in X - cl(V)$. Thus X is minimal c-regular. □

Corollary 15.1. *A Hausdorff m-paracompact topological space is minimal c-normal.*

Proof. Let E and F distinct minimal closed sets in a Hausdorff m-paracompact topological space X. For each $y \in E$, by the Theorem 15, there exist disjoint open sets U_y, V_y such that $y \in U_y$ and $B \subseteq V_y$. Again by the Lemma 1,

$$\mathscr{U} = \{U_y : y \in E\} \cup \{X - E\}$$

is a maximal open cover of X. Now if we proceed in a similar way as the proof of the Theorem, we can get disjoint open sets U, V such that $E \subseteq U$ and $F \subseteq V$. □

Definition 3.7. A topological space X is said to be *weakly m-paracompact* if each maximal open cover of X has an open locally finite refinement.

Let $Y \subseteq X$. Y is said to be a *weakly m-paracompact subset of X* if $(Y, \mathscr{T}_Y)$ is weakly m-paracompact.

Theorem 16. *Let X be an m-paracompact topological space and F be minimal closed in X. Then F is weakly m-paracompact.*

Proof. Proof is similar to the proof of Theorem 5. □

Theorem 17. *Let X be a topological space in which all proper open sets are mean open. Then X is m-compact.*

Proof. If possible, let X be not m-compact. Then there exists a maximal open cover $\mathscr{A}$ of X which has no open finite s-refinement. Now by the Definition 2.10, for each $A \in \mathscr{A}$ we have can a proper open set B_A such that $A \subsetneq B_A$. Then $\mathscr{A}$ is an s-refinement of $\{B_A : A \in \mathscr{A}\}$, which contradicts the maximality of $\mathscr{A}$. Thus X is m-compact. □

Corollary 17.1. *Let X be a topological space in which all proper open sets are mean open. Then X is m-Lindelöf, countably m-compact, m-paracompact, weakly m-compact, weakly m-Lindelöf and weakly m-paracompact.*

Proof. Proof is similar to the proof of the previous theorem. □

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