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IMAGE OF QUASICOMPONENTS BY PROXIMATELY CHAIN REFINABLE FUNCTIONS

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Abstract. Proximate chain refinable functions were defined in [1] as a coarser class of functions from proximate refinable maps. In this paper we investigate some properties of these functions, we show that proximately chain refinable functions maps quasicomponents to quasiocomponents if codomain is with open connected components and in the end we provide some counterexamples.

1. Introduction

In [4, 5, 6] quasicomponents are characterized considering only the existence of chains between points for every open covering. Proximately chain refinable functions, as a generalization of proximately refinable maps, provide some similar properties for more general spaces, see [1]. Mapping quasicomponents to quasicomponents is an important property of functions, proximately chain refinable functions fulfill this condition assuming that the codomain is with open connected components. In the end we present classes of spaces where we always can contruct proximately chain refinable functions and investigate the images of graphs by proximately chain refinable functions.

2. Definitions and notations

Along this paper by a covering we mean an open covering of the space.

By $f : X \to Y$ we denote a function (not necessarily continuous) from X to Y and by $f : X \twoheadrightarrow Y$ we denote a surjective function from X to Y.

Let $x, y \in X$ and let $\mathcal{U}$ be an open covering for X. We say that x and y are $\mathcal{U}$-near if there exists $U \in \mathcal{U}$ such that $x, y \in U$.

Let $\mathcal{V}$ be an arbitrary covering of the space X. If $V', V'' \in \mathcal{V}$, then a *chain* from V' to V'' is a finite sequence $V_1, V_2, ..., V_n$ of members of $\mathcal{V}$ such that $V' \cap V_1 \neq \emptyset, V'' \cap V_n \neq \emptyset$ and $V_i \cap V_{i+1} \neq \emptyset$, for $1 \leq i \leq n-1$.

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For arbitrary covering $\mathcal{V}$ of the topological space X and for $V \in \mathcal{V}$ by $chainV$ we denote the set $\bigcup\{W \in \mathcal{V}|$ there exists e chain in $\mathcal{V}$ from V to W $\}$. By $chain\mathcal{V}$ we denote the covering $\{chainV|\ V \in \mathcal{V}\}$.

Suppose $\mathcal{F}$ is an open family of subsets of X and x and y are two points in X. A *chain* in $\mathcal{F}$ from x to y is a finite sequence $F_1, F_2, ..., F_n$ of members of $\mathcal{F}$ such that $x \in F_1, y \in F_n$ and $F_i \cap F_{i+1} \neq \emptyset$, for $1 \leq i \leq n-1$. We say that the points x and y are $chain\mathcal{F}-$near.

If $F', F'' \in \mathcal{F}$, then a *chain* from F' to F'' is a finite sequence $F_1, F_2, ..., F_n$ of members of $\mathcal{F}$ such that $F' \cap F_1 \neq \emptyset, F'' \cap F_n \neq \emptyset$ and $F_i \cap F_{i+1} \neq \emptyset$, for $1 \leq i \leq n-1$.

The collection $\mathcal{F}$ is said to be *connected* if for every pair F', F'' of elements of $\mathcal{F}$ there exists a chain from F' to F''. For every collection $\mathcal{F}$ the *components* of $\mathcal{F}$ are defined as maximal connected subcollections of $\mathcal{F}$.

Let $f : X \to Y$ be a function and let $\mathcal{V}$ be a covering of Y. We say that $g : X \to Y$ is $\mathcal{V}-$near to f if for every $x \in X$, $f(x)$ and $g(x)$ are in the same member of $\mathcal{V}$.

Definition 2.1. Let X, Y be topological spaces, and $\mathcal{V}$ a covering of Y . The function $f : X \to Y$ is $\mathcal{V}-$*continuous*, if for any $x \in X$, there exists a neighborhood U of x, such that $f(U) \subseteq V$ for some member $V \in \mathcal{V}$.
(The family of all such U form a covering $\mathcal{U}$ of X. Shortly, we say that $f : X \to Y$ is $\mathcal{V}-$continuous, if there exists $\mathcal{U}$ such that $f(\mathcal{U}) \prec V$.)

Let X, Y be arbitrary topological spaces.

Definition 2.2. Let $\mathcal{U}$ be a covering of the space X. The function $f : X \twoheadrightarrow Y$ is *strong* $\mathcal{U}-$*function* if for every $y \in Y$ there exists a neighborhood D of y in Y such that $f^{-1}(D)$ is contained in some member of $\mathcal{U}$.

Definition 2.3. The function $f : X \twoheadrightarrow Y$ is *proximately chain refinable* if for every covering $\mathcal{V}$ of Y and for every covering $\mathcal{U}$ for X, there exists a $\mathcal{V}-$continuous strong $\mathcal{U}-$function $g : X \twoheadrightarrow Y$ which is $chain\mathcal{V}-$near to f. We say that g is a $chain(\mathcal{U}, \mathcal{V})$ refinement of f.

3. Images of quasicomponents by proximately chain refinable functions

Quasicomponent of a point x of the space X is the intersection of all clopen subsets of X that contain x. It is known that quasicomponents are closed and every component is contained in a quasicomponent.

We will prove that quasicomponents are mapped to quasicomponents by proximately chain refinable functions, if Y is with open connected components.

Theorem 1. *Let X and Y be topological spaces and Y be with open connected components. If $f : X \twoheadrightarrow Y$ is a proximately chain refinable surjective function, then $f(Q(x)) = Q(f(x))$, for all $x \in X$, where $Q(f(x))$ is the quasicomponent of $f(x)$ in Y.*

Proof. First, we will show that $f(Q(x)) \subseteq Q(f(x))$. Suppose the contrary, that $x' \in Q(x)$ is mapped on $Y \setminus Q(f(x))$. Then, by Theorem 1.1 in [4] (also see [5] and [6]), there exists a covering $\mathcal{V}$ of Y such that no chain from $\mathcal{V}$ connects the points $f(x)$ and $f(x')$. Since f is a proximately chain refinable function, there exists a $chain(\{X\}, \mathcal{V})$ refinement g of f such that $g(\mathcal{U}) \prec \mathcal{V}$ for some covering $\mathcal{U}$ of X. Now, since x, x' lie in the same quasicomponent $Q(x)$ of X, from [4], there exists a chain $U_1, U_2, ..., U_n$ from $\mathcal{U}$ connecting them. But, for every fixed $i \in \{1, 2, ..., n\}$ we have $g(U_i) \subseteq V_i$ for some $V_i \in \mathcal{V}$, hence the sets $V_1, V_2, ..., V_n$ form a chain from $\mathcal{V}$ connecting the points $g(x), g(x')$. On the other hand, since g is $chain\mathcal{V}$-near to f, the pairs of points $g(x), f(x)$ and $g(x'), f(x')$ are already connected by chains from $\mathcal{V}$. This contradicts with the assumption that $f(x), f(x')$ are separated by the covering $\mathcal{V}$. Hence $f(Q(x)) \subseteq Q(f(x))$.

Since f is surjective, it is enough to show that different quasicomponents are mapped inside different ones. If we assume the contrary then there will exist points $x \in Q(x)$ and $x'' \notin Q(x)$ such that $f(x), f(x'') \in f(Q(x))$. In the space X fix a covering $\mathcal{U}'$ such that no chain from $\mathcal{U}'$ connects x, x''. Take a $chain(\mathcal{U}', \mathcal{C})$ refinement g' of f, where $\mathcal{C}$ is the disjoint clopen covering of Y consisting of its components (since Y is with open connected components, components and quasicomponents of Y coincide). The points $f(x), g(x)$ and $f(x''), g(x'')$ must lie in the same quasicomponent $Q(f(x))$. Also, considering the fact that g' is $\mathcal{U}'-$strong, there exists a covering $\mathcal{O} = \{O_y | y \in Y\}$ of Y such that for every $y \in Y$ we have $g^{-1}(O_y) \subseteq U''$ for some $U'' \in \mathcal{U}'$. Since, $g'(x), g'(x'') \in Q(f(x))$ there exists a chain $O_1, O_2, ..., O_m$ from $\mathcal{O}$ connecting $g(x), g(x'')$. Now, the sets $U_1', U_2', ..., U_m'$ where $g^{-1}(O_i) \subseteq U_i'$ for $i \in \{1, 2, ..., m\}$ form a chain from $\mathcal{U}'$ between points x, x'' which contradicts to our assumption. Consequently, we have that $Q(f(x)) \cap Q(f(x'')) = \emptyset$ whenever $Q(x) \cap Q(x'') = \emptyset$, so $f(Q(x)) = Q(f(x))$. □

Remark: In the first inclusion of the theorem the assumption of being strong is not used.

Corollary 1.1. *Let X and Y be topological spaces and Y locally connected. If $f : X \twoheadrightarrow Y$ is a proximately chain refinable surjective function, then $f(Q(x)) = Q(f(x))$, for all $x \in X$, where $Q(f(x))$ is the quasicomponent of $f(x)$ in Y.*

Corollary 1.2. *Let X and Y be topological spaces and Y compact Hausdorff. If $f : X \twoheadrightarrow Y$ is a proximately chain refinable surjective function, then $f(Q(x)) = Q(f(x))$, for all $x \in X$, where $Q(f(x))$ is the quasicomponent of $f(x)$ in Y.*

Example 3.1. In the previous theorem, if a bijection $f : X \to Y$ maps quasicomponents to quasicomponents, it doesnt need to be proximately chain refinable.

Take $X = \{(x, 1/n) | x \in [0, 1], n \in \{1, 2, 3, ...\}\} \cup \{(0, 0)\}$ and $Y = \{(x, n) | x \in [0, 1], n \in \{1, 2, 3, ...\}\} \cup \{(0, 0)\}$. Define a bijection $f : X \to Y$ by $f(x, 1/n) = (x, n)$ for $x \in [0, 1]$, $n \in N$ and $f(0, 0) = (0, 0)$. Then f maps quasicomponents to quasicomponents, but there is no $chain(\{X\}, \mathcal{V})-$ refinement g for the covering $\mathcal{V} = \{B_{(0,0)}(1/2)\} \cup \{V_n | V_n$ is an open neighborhood in Y that contains only the segment $\{(x, n) | x \in [0, 1]\}$. The last is not possible since g could not be $\mathcal{V}-$continuous in the point $(0, 0)$. (Every neighborhood O of the point $(0, 0)$

intersects with some quasicomponent $\{(x, 1/n)|x \in [0,1]\}$ of X so it could not be mapped inside $B_{(0,0)}(1/2) \in \mathcal{V}$). $\square$

In Theorem 1 if the space Y is not with open connected components (locally connected), then we could not claim the validity of the same property. The Example 3 from [2], given below, shows that even if there exists proximately refinable map between spaces X and Y (every proximately refinable map is proximately chain refinable function, [1]), quasicomponents are not mapped to quasicomponents since they differ in cardinality.

Example 3.2. Let K be a Cantor set and $p_1, p_2, ...$ be a sequence of points, in the complement of K, having K as its limiting set. Let $X = K \cup \{p_1, p_2, ...\}$, and let $Y = (X \setminus K) \cup \{K\}$ with the decomposition topology. We will denote the set K as "k" considering it as a point from Y.

Since the quasicomponents of the Cantor set are singletons, the space X has continuum number of quasicomponents. On the other hand, Y has at most countably many because its number of quasicomponents could not overcome the number of elements of the whole set Y. The space Y is not locally connected since the element k of Y doesn't have a connected neighborhood. Indeed, the set K is not open in X, hence $\{k\}$ is not open in Y so every open neighborhood of k should contain an element from the set $\{p_1, p_2, ...p_n\}$ and it will become disconnected in that way. $\square$

For a given function, being continuous doesn't imply the property of proximate chain near refinable. We will give an example to demonstrate this.

Example 3.3. Let $X = \mathbb{N} \cup \{0\}$ and $Y = \{1/n|n \in \mathbb{N}\} \cup \{0\}$. The function $f : X \to Y$ defined by $f(n) = 1/n$ for $n \in \mathbb{N}$ and $f(0) = 0$ is a continuous surjection. On the other hand, we can't find strong $\{\{x\}|x \in \mathbb{N} \cup \{0\}\}$ function from the discrete space X to the space $Y = \{1/n|n \in \mathbb{N}\} \cup \{0\}$ (relative topology of $\mathbb{R}$) since for any surjection from X to Y the preimage of 0 will contain more than one element and will not be inside the singleton open neighborhood of 0.

Definition 3.1. A surjective function $f : X \twoheadrightarrow Y$ between topological spaces X and Y is said to be *monotone* function if $f^{-1}(y)$ is connected for each $y \in Y$.

Lemma 1. *Let X be a topological space and Y is locally-connected. If the surjection $f : X \twoheadrightarrow Y$ is proximately chain refinable, then f is monotone.*

Proof. Let $y \in Y$, and suppose to the contrary that $f^{-1}(y)$ is not connected. Then there exist two disjoint open sets U, V in X such that $f^{-1}(y) \subseteq U \cup V$ and $f^{-1}(y) \cap U \neq \emptyset$ and $f^{-1}(y) \cap V \neq \emptyset$. Take $u \in f^{-1}(y) \cap U$ and $v \in f^{-1}(y) \cap V$, then the quasicomponents $Q(u), Q(v)$ are mapped in same quasicomponent $Q(y)$ which contradicts with Theorem 1. $\square$

Now we will give the definition of graph from [3].

Definition 3.2. A *graph* is a compact metric space that is the union of a finite, disjoint collection of sets each of which is either a singleton set or an open, connected subset of an arc in the space.

Lemma 2. *If f is continuous proximately chain refinable function from a graph G to a locally-connected space Y. Then Y is a graph.*

Proof. Since f is monotone continuous function, we have that Y is a graph, since any monotone image of a graph is a graph ([3]). □

Lemma 3. *Let X be a countable disjoint union of arcs and Y locally connected separable continuum (compact connected metric space). Then every continuous surjection $f : X \twoheadrightarrow Y$ is proximately chain refinable.*

Proof. Let $\mathcal{U}, \mathcal{V}$ be open coverings of the spaces X, Y respectively. The space Y is with countable number of components, so for every component C_x of X from [1] we could define $chain(\mathcal{U} \cap C_x, \mathcal{V} \cap C_f(x))$ refinement g_x of f in C_x which is $chain(\mathcal{V} \cap C_f(x))-$near to f. Now, by applying the pasting (gluing) -like procedure over the functions g_x for the partition $\{C_x | x \in X, C_x \text{ a component of } x\}$ of X, we obtain a $chain(\mathcal{U}, \mathcal{V})-$refinement g of f in the whole space. □

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