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CONCERNING REAL FUNCTIONS WITH VALUES IN THE CANTOR SET

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Abstract. This article in particular indicates that there exist at least continuum-many "essentially nonconstant" (requiring the existence of at least two level sets having positive finite measure) almost everywhere continuous functions from the real field $\mathbb{R}$ to the Cantor ternary set $\mathcal{C}$, although it is a basic fact that there exists no nonconstant continuous function $\mathbb{R} \to \mathcal{C}$.

1. Introduction

A *Cantor space* (i.e. [following [3]], a topological space homeomorphic to the Cantor ternary set), denoted $\mathcal{C}$ hereafter, topologized as a topological subspace of the real field $\mathbb{R}$, seems to be asymmetrically studied as the codomain of a function on $\mathbb{R}$. One apparent reason would be the basic fact that every continuous function $f : \mathbb{R} \to \mathcal{C}$ is constant: If not, then the f-image $f^{\rightarrow}(\mathbb{R}) \subset \mathcal{C}$ of $\mathbb{R}$ has at least two elements; since $\mathcal{C}$ is Hausdorff and zero-dimensional (i.e. [following [3]], having a basis consisting of clopen sets), it then follows that $\mathbb{R}$ has a nonempty clopen proper subset, contradicting the fact that $\mathbb{R}$ is connected.

For our purposes, we introduce the following

Definition 1.1. *Let $q \geq 2$ be a positive integer; let $Y \subset \mathbb{R}$; let $f : \mathbb{R} \to Y$. The function f is said to be* essentially (q-)nonconstant *if and only if there exist some distinct $y_1, \ldots, y_q \in Y$ such that the f-preimage $f^{\leftarrow}(\{y_i\})$ of $\{y_i\}$ has positive finite (Lebesgue-)measure for all $1 \leq i \leq q$.*

Thus every essentially nonconstant function is automatically a nonconstant function.

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In what follows, we give another proof of an "almost everywhere continuous extension" result in a suitably generic setting, which is then applied to show that there are at least continuum-many essentially nonconstant almost everywhere continuous functions $\mathbb{R} \to \mathcal{C}$, in dire contrast to the nonexistence of a nonconstant continuous function $\mathbb{R} \to \mathcal{C}$.

2. Results

The continuous extendability of a continuous map from a dense subspace of a topological space to a compact Hausdorff space admits a well-known characterization (e.g., Theorem 3.2.1 in [2]). In view of the inherent restrictions in particular due to this necessary and sufficient condition, such a continuous map need not be continuously extendable in an automatic manner. However, one does have a generic result (Theorem 3.2 in [1]) indicating "how continuous" an extension of such a continuous map can be.

If $f : X \to Y$, we denote by $f^{\to}(A)$ the f-image of A for all $A \subset X$, and by $f^{\leftarrow}(B)$ the f-preimage of B for all $B \subset Y$.

We give another proof of Theorem 3.2 in [1] for compact codomains, which is directly relevant with respect to our situation:

Proposition 2.1. *Let Y be a compact Hausdorff space; let X be a topological space. If $A \subset X$ is a dense subspace, then every continuous map $A \to Y$ is extendable to some map $g : X \to Y$ that is continuous at every point of A.*

Proof. Let $f : A \to Y$ be continuous. If X is empty or $A = X$, then there is certainly nothing to prove; we consider the other cases. Since Y is compact, every net in Y has a cluster point, i.e. (following [2]), for every net $(y_\theta)_\theta$ in Y there exists some $y \in Y$ such that every neighborhood of y contains y_θ frequently in θ. On the other hand, by the denseness assumption, for every $x \in X \setminus A$ we can choose some net $(a_\theta^x)_{\theta \in \Theta_x}$ in A, where Θ_x is nonempty, converging in X to x. Then each net $(f(a_\theta^x))_{\theta \in \Theta_x}$ in Y has a cluster point in Y; for every $x \in X \setminus A$, let $\widehat{\lim}_{\theta \in \Theta_x} f(a_\theta^x)$ be a cluster point of the net $(f(a_\theta^x))_{\theta \in \Theta_x}$ by acknowledging the Axiom of Choice.

We claim that the map

$$g : X \to Y, \begin{cases} x \mapsto f(x), & \text{if } x \in A, \\ x \mapsto \widehat{\lim}_{\theta \in \Theta_x} f(a_\theta^x), & \text{if } x \in X \setminus A \end{cases}$$

is a desired extension of f over X. Let $x \in A$, and let G be a neighborhood of $g(x)$ in Y. Then, since Y is by using assumption in particular a regular space, we can choose some neighborhood V of $g(x) = f(x)$ such that $\mathrm{cl}(V) \subset G$. In turn, we can choose some neighborhood W_x of x in X, by the continuity of f, such that

$$f^{\to}(W_x \cap A) \subset V.$$

We show that $g^{\to}(W_x) \subset G$, so that the continuity of g at x is verified. To this end, it suffices to show that $g(x') \in \mathrm{cl}(V)$ for all $x' \in W_x$. Let $x' \in W_x$. If $x' \in W_x \cap A$, then $g(x') = f(x') \in V$, and there is nothing to prove. Suppose $x' \in W_x \cap (X \setminus A)$, and let O be a neighborhood of $g(x')$. Then the set $\Theta_{x'}(O) :=$

$\{\theta \in \Theta_{x'} \mid f(a_\theta^{x'}) \in O\}$ is nonempty and cofinal in $\Theta_{x'}$ by the construction of g. Since $(a_\theta^{x'})_{\theta\in\Theta_{x'}}$ converges in X to x' by definition, the neighborhood W_x of x' contains $a_\theta^{x'}$ eventually in θ. But each $a_\theta^{x'} \in A$ by definition; it follows that $f(a_\theta^{x'}) \in V$ eventually in θ, and so, in particular, there exists some $\theta \in \Theta_{x'}(O)$ such that $f(a_\theta^{x'}) \in V$ and hence $f(a_\theta^{x'}) \in V \cap O$. Thus $g(x') \in \mathrm{cl}(V)$. We have shown that g is continuous at every point of A; this completes the proof. □

We record as a relevant corollary of Proposition 2.1 the following evident measure-theoretic application:

Corollary 0.1. *Let Y be a compact Hausdorff space; let X be a topological space that is also a complete measure space with M_X denoting the given complete measure and with the given sigma-algebra including the Borel sigma-algebra of X.*

If $A \subset X$ is a dense co-null topological subspace (i.e., if A is a subset of X, being dense in the topological space X and being the complement of some M_X-null set in the measure space X, which is topologized as a subspace of the topological space X), then every continuous map $A \to Y$ can be extended to be some M_X-almost everywhere continuous map $X \to Y$. □

We are now in a position to prove

Theorem 1. *For every integer $q \geq 2$, there exist at least continuum-many essentially q-nonconstant almost everywhere continuous functions $\mathbb{R} \to \mathcal{C}$.*

Proof. Denote by A the set of all irrational numbers. Since the subspace A of $\mathbb{R}$ is second countable and zero-dimensional, by an elementary argument using the basic fact that being second countable implies being Lindelöf, we can choose some countable basis $\widehat{\mathscr{T}_A}$ of A consisting of clopen sets in A. Upon fixing any $a_0 \in A$ and choosing a neighborhood G of a_0 in A with positive finite measure, e.g., $G :=]a_0 - 1, a_0 + 1[\cap A$, we can in turn choose some element W_1 of $\widehat{\mathscr{T}_A}$ such that $W_1 \subset G$ and W_1 has positive finite measure in view of the countable subadditivity of a measure.

Let $q \geq 2$ be an integer; we can then choose some distinct $r_1, \ldots, r_{q-1} \in \mathbb{Q}$ such that if $t_{r_i} : x \mapsto r_i + x$ on $\mathbb{R}$ for all $1 \leq i \leq q-1$ then the sets

$$]a_0 - 1, a_0 + 1[, t_{r_1}^{\rightarrow}(]a_0 - 1, a_0 + 1[), \cdots, t_{r_{q-1}}^{\rightarrow}(]a_0 - 1, a_0 + 1[)$$

are pairwise disjoint. Thus each $W_{i+1} := t_{r_i}^{\rightarrow}(W_1)$ has the same positive finite measure as W_1 by the translation invariance of (Lebesgue) measure. Moreover, as each $t_{r_i}|_A$ is a homeomorphism of A onto A, the sets W_i are all clopen in A.

If $c_1 \in \mathcal{C}$, if $c_i \in \mathcal{C} \setminus \{c_1, \ldots, c_{i-1}\}$ for all $2 \leq i \leq q$, and if $c \in \mathcal{C} \setminus \{c_1, \ldots, c_q\}$, define

$$f : A \to \mathcal{C}, \begin{cases} a \mapsto c_1, & \text{if } a \in W_1; \\ \vdots & \vdots \\ a \mapsto c_q, & \text{if } a \in W_q; \\ a \mapsto c, & \text{if } a \in A \setminus \cup_{i=1}^{q} W_i. \end{cases}$$

Since the set $A \setminus \cup_{i=1}^{q} W_i$ is also (cl)open in A, the map f is continuous. Then we can choose by Proposition 2.1 some almost everywhere continuous extension $g : \mathbb{R} \to \mathcal{C}$ of f. But g is essentially q-nonconstant: We have $W_1, \ldots, W_q$ having positive finite measure by construction, and we have $c_1, \ldots, c_q \in \mathcal{C}$ being distinct; since $g^{\leftarrow}(\{c_i\}) = W_i \cup E$ for some $E \subset \mathbb{Q}$, the measure of $g^{\leftarrow}(\{c_i\})$ equals that of W_i for all $1 \leq i \leq q$.

Manifestly, given any finite subset Z of $\mathcal{C}$, the set $\mathcal{C} \setminus Z$ is in bijection with $\mathbb{R}$; for, we have $\mathbb{R}$ being equinumerous to $\mathbb{R} \setminus Z$ and $\mathbb{R} \setminus Z$ being equinumerous to $\mathcal{C} \setminus Z$. Thus the choices of f as c runs through $\mathcal{C} \setminus \{c_1, \ldots, c_q\}$, with each c_i and each W_i being fixed, are at least continuum-many; the proof is complete. □

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