

SEVERAL INEQUALITIES FOR OPERATOR MONOTONE FUNCTIONS ON FINITE INTERVALS

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Abstract. In this paper we show that, if the function $f : (-1, 1) \rightarrow \mathbb{R}$ is operator monotone in $(-1, 1)$, then there exists a positive measure μ on $[-1, 1]$ such that

$$\begin{aligned} & [f(B) - f(A)](B - A) \\ &= f'(0) \int_{-1}^1 \left(\int_0^1 [1 - \lambda((1-t)A + tB)]^{-1} (B - A) \right)^2 dt \, d\mu(\lambda) \end{aligned}$$

for all A, B with $S_p(A), S_p(B) \subset (-1, 1)$. Some necessary and sufficient conditions for the operators A, B with $S_p(A), S_p(B) \subset (-1, 1)$ such that the inequality

$$f(B)B + f(A)A \geq f(A)B + f(B)A$$

holds for any operator monotone function f on $(-1, 1)$ are also given.

1. INTRODUCTION

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function $f(t)$ on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

In 1934, K. Löwner [10] had given a definitive characterization of operator monotone functions as follows, see for instance [2, p. 144-145]:

Theorem 1. *A function $f : [0, \infty) \rightarrow \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{ts}{t+s} dm(s) \tag{1.1}$$

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where $a \in \mathbb{R}$ and $b \geq 0$ and a positive measure m on $(0, \infty)$ such that

$$\int_0^\infty \frac{s}{1+s} dm(s) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0, 1]$, [8].

In [6], T. Furuta observed that for $\alpha_j \in [0, 1]$, $j = 1, \dots, n$ the functions

$$g(t) := \left(\sum_{j=1}^n t^{-\alpha_j} \right)^{-1} \quad \text{and} \quad h(t) = \sum_{j=1}^n (1+t^{-1})^{-\alpha_j}$$

are operator monotone in $(0, \infty)$.

Let $f(t)$ be a continuous function $(0, \infty) \rightarrow (0, \infty)$. It is known that $f(t)$ is operator monotone if and only if $g(t) = t/f(t) =: f^*(t)$ is also operator monotone, see for instance [6] or [11].

Consider the family of functions defined on $(0, \infty)$ and $p \in [-1, 2] \setminus \{0, 1\}$ by

$$f_p(t) := \frac{p-1}{p} \left(\frac{t^p - 1}{t^{p-1} - 1} \right)$$

and

$$f_0(t) := \frac{t}{1-t} \ln t, \quad f_1(t) := \frac{t-1}{\ln t} \quad (\text{logarithmic mean}).$$

We also have the functions of interest

$$f_{-1}(t) = \frac{2t}{1+t} \quad (\text{harmonic mean}), \quad f_{1/2}(t) = \sqrt{t} \quad (\text{geometric mean}).$$

In [5] the authors showed that f_p is operator monotone for $1 \leq p \leq 2$.

In the same category, we observe that the function

$$g_p(t) := \frac{t-1}{t^p-1}$$

is an operator monotone function for $p \in (0, 1]$, [7].

It is well known that the logarithmic function \ln is operator monotone and in [6] the author obtained that the functions

$$f(t) = t(1+t) \ln \left(1 + \frac{1}{t} \right), \quad g(t) = \frac{1}{(1+t) \ln \left(1 + \frac{1}{t} \right)}$$

are also operator monotone functions on $(0, \infty)$.

The case of operator monotone functions on finite intervals is as follows [2, p. 134]:

Theorem 2. *Let f be a nonconstant operator monotone function on $(-1, 1)$. Then there exists a unique probability measure μ on $[-1, 1]$ such that*

$$f(t) = f(0) + f'(0) \int_{-1}^1 \frac{t}{1-\lambda t} d\mu(\lambda) \quad (1.2)$$

for $t \in (-1, 1)$, where $f'(0) > 0$.

It is well known that the function $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$, $f(t) = \tan t$ is operator monotone on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the function g is operator monotone on $(0, \infty)$ if and only if the function $f : (-1, 1) \rightarrow \mathbb{R}$, $f(t) := g\left(\frac{1+t}{1-t}\right)$ is operator monotone on $(-1, 1)$, [13]. Therefore, the functions $f(t) = \left(\frac{1+t}{1-t}\right)^r$ and $f(t) = \ln\left(\frac{1+t}{1-t}\right)$ are operator monotone on $(-1, 1)$.

For recent operator inequalities related to operator monotone functions, see [1], [3], [9], [12] and [14].

In this paper we show that, if the function $f : (-1, 1) \rightarrow \mathbb{R}$ is operator monotone in $(-1, 1)$, then there exists a positive measure μ on $[-1, 1]$ such that

$$\begin{aligned} & [f(B) - f(A)](B - A) \\ &= f'(0) \int_{-1}^1 \left(\int_0^1 \left[(1 - \lambda((1-t)A + tB))^{-1} (B - A) \right]^2 dt \right) d\mu(\lambda) \end{aligned}$$

for all A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$. Some necessary and sufficient conditions for the operators A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$ such that the inequality

$$f(B)B + f(A)A \geq f(A)B + f(B)A$$

holds for any operator monotone function f on $(-1, 1)$ are also given.

2. MAIN RESULTS

We have the following representation results that is of interest in itself as well:

Theorem 3. *Let f be a nonconstant operator monotone function on $(-1, 1)$ that satisfies the representation (1.2). Then,*

$$\begin{aligned} f(B) - f(A) &= f'(0) \int_{-1}^1 \left(\int_0^1 (1 - \lambda((1-t)A + tB))^{-1} \right. \\ &\quad \left. \times (B - A)(1 - \lambda((1-t)A + tB))^{-1} dt \right) d\mu(\lambda) \end{aligned} \quad (2.1)$$

for all A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$.

Proof. For C a selfadjoint operator with $\text{Sp}(C) \subset (-1, 1)$, then by (1.2) we have

$$f(C) = f(0) + f'(0) \int_{-1}^1 C(1 - \lambda C)^{-1} d\mu(\lambda).$$

Therefore, for A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$,

$$f(B) - f(A) = f'(0) \int_{-1}^1 \left[B(1 - \lambda B)^{-1} - A(1 - \lambda A)^{-1} \right] d\mu(\lambda).$$

For $\lambda \in [-1, 1]$, $\lambda \neq 0$ we put

$$U_\lambda := B(1 - \lambda B)^{-1} - A(1 - \lambda A)^{-1}.$$

Then

$$\begin{aligned}
\lambda U_\lambda &= \lambda B (1 - \lambda B)^{-1} - \lambda A (1 - \lambda A)^{-1} \\
&= (1 - 1 + \lambda B) (1 - \lambda B)^{-1} - (1 - 1 + \lambda A) (1 - \lambda A)^{-1} \\
&= (1 - \lambda B)^{-1} - 1 - \left((1 - \lambda A)^{-1} - 1 \right) \\
&= (1 - \lambda B)^{-1} - (1 - \lambda A)^{-1},
\end{aligned}$$

namely

$$U_\lambda = \frac{(1 - \lambda B)^{-1} - (1 - \lambda A)^{-1}}{\lambda}$$

for $\lambda \in [-1, 1]$, $\lambda \neq 0$.

Using the identity

$$X^{-1} - Y^{-1} = X^{-1} (Y - X) Y^{-1},$$

we get

$$\begin{aligned}
&(1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} && (2.2) \\
&= (1 - \lambda B)^{-1} (1 - \lambda A - 1 + \lambda B) (1 - \lambda A)^{-1} \\
&= \lambda (1 - \lambda B)^{-1} (B - A) (1 - \lambda A)^{-1}
\end{aligned}$$

and then

$$\lambda U_\lambda = \lambda (1 - \lambda B)^{-1} (B - A) (1 - \lambda A)^{-1}$$

for $\lambda \in [-1, 1]$, $\lambda \neq 0$.

Therefore

$$U_\lambda = (1 - \lambda B)^{-1} (B - A) (1 - \lambda A)^{-1}, \quad \lambda \in [-1, 1]$$

and we get the identity of interest

$$f(B) - f(A) = f'(0) \int_{-1}^1 (1 - \lambda B)^{-1} (B - A) (1 - \lambda A)^{-1} d\mu(\lambda).$$

For small t around 0 we have

$$\begin{aligned}
&f(A + tC) - f(A) \\
&= f'(0) t \int_{-1}^1 \left[(1 - \lambda(A + tC))^{-1} C (1 - \lambda A)^{-1} \right] d\mu(\lambda),
\end{aligned}$$

for $C \in B(H)$, C selfadjoint and A with $\text{Sp}(A) \subset (-1, 1)$, which implies that the Gâteaux derivative in A of f exists and

$$\nabla_A f(C) = f'(0) \int_{-1}^1 (1 - \lambda A)^{-1} C (1 - \lambda A)^{-1} d\mu(\lambda). \quad (2.3)$$

Let A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$. Consider the auxiliary function $\varphi_{A,B} : [0, 1] \rightarrow B(H)$,

$$\varphi_{A,B}(t) := f((1-t)A + tB).$$

Then

$$\frac{d\varphi_{A,B}(t)}{dt} = \nabla_{(1-t)A + tB} f(B - A), \quad t \in (-1, 1)$$

and by (2.3) we get

$$\begin{aligned} \frac{d\varphi_{A,B}(t)}{dt} &= f'(0) \int_{-1}^1 (1 - \lambda((1-t)A + tB))^{-1} \\ &\quad \times (B - A)(1 - \lambda((1-t)A + tB))^{-1} d\mu(\lambda) \end{aligned} \quad (2.4)$$

for $t \in (-1, 1)$.

Since

$$f(B) - f(A) = \varphi_{A,B}(1) - \varphi_{A,B}(0) = \int_0^1 \frac{d\varphi_{A,B}(t)}{dt} dt,$$

hence by (2.4) we get

$$\begin{aligned} f(B) - f(A) &= f'(0) \int_0^1 \left(\int_{-1}^1 (1 - \lambda((1-t)A + tB))^{-1} \right. \\ &\quad \left. \times (B - A)(1 - \lambda((1-t)A + tB))^{-1} d\mu(\lambda) \right) dt \end{aligned}$$

and by Fubini's theorem we get (2.1). \square

Corollary 3.1. *With the assumptions of Theorem 3, we have the norm inequality*

$$\begin{aligned} \|f(B) - f(A)\| &\leq f'(0) \|B - A\| \\ &\quad \times \int_{-1}^1 \left(\int_0^1 \left\| (1 - \lambda((1-t)A + tB))^{-1} \right\|^2 dt \right) d\mu(\lambda). \end{aligned} \quad (2.5)$$

Theorem 4. *Let f be a nonconstant operator monotone function on $(-1, 1)$ that satisfies the representation (1.2), then*

$$\begin{aligned} [f(B) - f(A)](B - A) \\ = f'(0) \int_{-1}^1 \left(\int_0^1 \left[(1 - \lambda((1-t)A + tB))^{-1} (B - A) \right]^2 dt \right) d\mu(\lambda) \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} (B - A)[f(B) - f(A)] \\ = f'(0) \int_{-1}^1 \left(\int_0^1 \left[(B - A)(1 - \lambda((1-t)A + tB))^{-1} \right]^2 dt \right) d\mu(\lambda) \end{aligned} \quad (2.7)$$

for all A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$.

Proof. If we multiply the equality (2.1) at the right with $B - A$ we get

$$\begin{aligned} (f(B) - f(A))(B - A) \\ = f'(0) \int_{-1}^1 \left(\int_0^1 (1 - \lambda((1-t)A + tB))^{-1} \right. \\ \left. \times (B - A)(1 - \lambda((1-t)A + tB))^{-1} (B - A) dt \right) d\mu(\lambda) \\ = f'(0) \int_{-1}^1 \left(\int_0^1 \left[(1 - \lambda((1-t)A + tB))^{-1} (B - A) \right]^2 dt \right) d\mu(\lambda), \end{aligned} \quad (2.8)$$

which proves (2.6).

The proof of (2.7) follows in a similar way. \square

We also have:

Theorem 5. *Let f be a nonconstant operator monotone function on $(-1, 1)$ that satisfies the representation (1.2), then for A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$ and*

$$M \geq B - A \geq m,$$

for some real numbers $M > m$, we have

$$\begin{aligned} Mf'(0) \int_{-1}^1 \left[\int_0^1 (1 - \lambda((1-t)A + tB))^{-2} dt \right] d\mu(\lambda) & \quad (2.9) \\ & \geq f(B) - f(A) \\ & \geq mf'(0) \int_{-1}^1 \left[\int_0^1 (1 - \lambda((1-t)A + tB))^{-2} dt \right] d\mu(\lambda). \end{aligned}$$

Proof. If $B - A \geq m$, then

$$\begin{aligned} (1 - \lambda((1-t)A + tB))^{-1} (B - A) (1 - \lambda((1-t)A + tB))^{-1} dt \\ \geq m (1 - \lambda((1-t)A + tB))^{-2} \end{aligned}$$

for $t \in [0, 1]$ and $\lambda \in [-1, 1]$, which implies that

$$f(B) - f(A) \geq mf'(0) \int_{-1}^1 \left[\int_0^1 (1 - \lambda((1-t)A + tB))^{-2} dt \right] d\mu(\lambda).$$

If $M \geq B - A$, then in a similar way

$$Mf'(0) \int_{-1}^1 \left[\int_0^1 (1 - \lambda((1-t)A + tB))^{-2} dt \right] d\mu(\lambda) \geq f(B) - f(A)$$

and the theorem is proved. \square

Corollary 5.1. *With the assumptions of Theorem 5 and if $B - A \geq m > 0$, then*

$$\begin{aligned} f(B) - f(A) & \geq mf'(0) \left[1 + \left(\int_{-1}^1 \lambda d\mu(\lambda) \right) (A + B) \right. & (2.10) \\ & \quad \left. + \frac{1}{3} \left(\int_{-1}^1 \lambda^2 d\mu(\lambda) \right) \left(A^2 + B^2 + \frac{AB + BA}{2} \right) \right] \\ & \geq 0. \end{aligned}$$

Proof. Since for $x < 1$, then

$$\frac{1}{1-x} \geq 1+x,$$

which implies that

$$(1-x)^{-2} \geq (1+x)^2 = 1 + 2x + x^2 \geq 0.$$

We then get

$$(1 - \lambda((1-t)A + tB))^{-2}$$

$$\begin{aligned}
&\geq 1 + 2\lambda((1-t)A + tB) + \lambda^2((1-t)A + tB)^2 \\
&= 1 + 2\lambda((1-t)A + tB) \\
&\quad + \lambda^2 \left[(1-t)^2 A^2 + t(1-t)AB + t(1-t)BA + t^2 B^2 \right] \geq 0
\end{aligned}$$

for $t \in [0, 1]$, $\lambda \in [-1, 1]$.

If we take the integral over t in this inequality, then we get

$$\begin{aligned}
&\int_0^1 (1 - \lambda((1-t)A + tB))^{-2} dt \\
&\geq 1 + \lambda(A + B) + \frac{1}{3}\lambda^2 \left(A^2 + B^2 + \frac{AB + BA}{2} \right) \geq 0
\end{aligned}$$

for $\lambda \in [-1, 1]$.

Taking the integral over $\mu(\lambda)$ we get

$$\begin{aligned}
&\int_{-1}^1 \left[\int_0^1 (1 - \lambda((1-t)A + tB))^{-2} dt \right] d\mu(\lambda) \\
&\geq 1 + \left(\int_{-1}^1 \lambda d\mu(\lambda) \right) (A + B) \\
&\quad + \frac{1}{3} \left(\int_{-1}^1 \lambda^2 d\mu(\lambda) \right) \left(A^2 + B^2 + \frac{AB + BA}{2} \right) \geq 0,
\end{aligned}$$

which, by the second inequality in (2.9) gives the desired result (2.10). \square

In the following, in order to simplify terminology, when we write $T \geq 0$ we automatically assume that the operator T is selfadjoint.

We give now some necessary and sufficient conditions for the operators A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$ such that the inequality $(B - A)(f(B) - f(A)) \geq 0$ holds.

Theorem 6. *Let A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$. The following statements are equivalent:*

(i) For all for $\lambda \in [-1, 1]$,

$$(1 - \lambda B)^{-1} (1 - \lambda A) + (1 - \lambda A)^{-1} (1 - \lambda B) \geq 2. \quad (2.11)$$

(ii) For all for $\lambda \in [-1, 1]$,

$$\int_0^1 \left[(1 - \lambda[(1-t)B + \lambda tA])^{-1} (B - A) \right]^2 dt \geq 0.$$

(iii) For all $\lambda \in [-1, 1]$

$$(\kappa_\lambda(B) - \kappa_\lambda(A))(B - A) \geq 0,$$

where $\kappa_\lambda(t) = t(1 - \lambda t)^{-1}$, $t \in [-1, 1]$.

(iv) For all operator monotone function f on $(-1, 1)$,

$$(f(B) - f(A))(B - A) \geq 0. \quad (2.12)$$

(v) For all operator monotone function f on $(-1, 1)$,

$$(B - A)(f(B) - f(A)) \geq 0. \quad (2.13)$$

Proof. Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0, 1]$ by

$$f_{C,D}(t) = f((1-t)C + tD), \quad t \in [0, 1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C, D] := \{(1-t)C + tD, t \in [0, 1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t)) dt = \int_0^1 \nabla f_{(1-t)C+tD}(D - C) dt.$$

If we write this equality for the function $f(t) = -t^{-1}$ and $C, D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1}(D - C)((1-t)C + tD)^{-1} dt. \quad (2.14)$$

From (2.14) we have

$$\begin{aligned} & (1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \\ &= \int_0^1 ((1-t)(1 - \lambda B) + t(1 - \lambda A))^{-1} ((1 - \lambda A) - (1 - \lambda B)) \\ & \times ((1-t)(1 - \lambda B) + t(1 - \lambda A))^{-1} dt \\ &= \lambda \int_0^1 (1 - \lambda[(1-t)B + \lambda tA])^{-1} (B - A) \\ & \times (1 - \lambda[(1-t)B + \lambda tA])^{-1} dt \end{aligned}$$

for $\lambda \in [-1, 1]$.

If we multiply this equality at right by $B - A$, we get

$$\begin{aligned} & \left[(1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \right] (B - A) \\ &= \lambda \int_0^1 (1 - \lambda[(1-t)B + \lambda tA])^{-1} (B - A) \\ & \times (1 - \lambda[(1-t)B + \lambda tA])^{-1} (B - A) dt \\ &= \lambda \int_0^1 \left[(1 - \lambda[(1-t)B + \lambda tA])^{-1} (B - A) \right]^2 dt \end{aligned}$$

for $\lambda \in [-1, 1]$, $\lambda \neq 0$, namely

$$\begin{aligned} V_\lambda &:= \frac{1}{\lambda} \left[(1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \right] (B - A) \\ &= \int_0^1 \left[(1 - \lambda[(1-t)B + \lambda tA])^{-1} (B - A) \right]^2 dt. \end{aligned}$$

Also,

$$\begin{aligned}
& \left[(1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \right] (B - A) \\
&= \frac{1}{\lambda} \left[(1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \right] (\lambda B - \lambda A) \\
&= \frac{1}{\lambda} \left[(1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \right] (1 - \lambda A - (1 - \lambda B)) \\
&= \frac{1}{\lambda} \left[(1 - \lambda B)^{-1} (1 - \lambda A) + (1 - \lambda A)^{-1} (1 - \lambda B) - 2 \right],
\end{aligned}$$

which gives that

$$\begin{aligned}
V_\lambda &= \frac{1}{\lambda} \left[(1 - \lambda B)^{-1} - (1 - \lambda A)^{-1} \right] (B - A) \\
&= \frac{1}{\lambda^2} \left[(1 - \lambda B)^{-1} (1 - \lambda A) + (1 - \lambda A)^{-1} (1 - \lambda B) - 2 \right]
\end{aligned}$$

for $\lambda \in [-1, 1]$, $\lambda \neq 0$.

We conclude that

$$\begin{aligned}
& (\kappa_\lambda(B) - \kappa_\lambda(A))(B - A) \tag{2.15} \\
&= \frac{1}{\lambda^2} \left[(1 - \lambda B)^{-1} (1 - \lambda A) + (1 - \lambda A)^{-1} (1 - \lambda B) - 2 \right] \\
&= \int_0^1 \left[(1 - \lambda[(1-t)B + \lambda t A])^{-1} (B - A) \right]^2 dt
\end{aligned}$$

for $\lambda \in [-1, 1]$, $\lambda \neq 0$.

We observe that the equivalence between the statements (i), (ii) and (iii) follows directly from the identity (2.15).

Since the function $\kappa_\lambda(t) = t(1 - \lambda t)^{-1}$ is operator monotone on $(-1, 1)$, the statement (iv) implies (iii).

If (ii) holds, then by representation (2.6) we have

$$\begin{aligned}
& [f(B) - f(A)](B - A) \\
&= f'(0) \int_{-1}^1 \left(\int_0^1 \left[(1 - \lambda((1-t)A + tB))^{-1} (B - A) \right]^2 dt \right) d\mu(\lambda) \\
&\geq 0,
\end{aligned}$$

which shows that (iv) holds.

Define the operator $K := (f(B) - f(A))(B - A)$. Since

$$\begin{aligned}
K^* &= [(f(B) - f(A))(B - A)]^* = (B - A)^* (f(B) - f(A))^* \\
&= (B - A)(f(B) - f(A))
\end{aligned}$$

then the fact that K is selfadjoint is equivalent to

$$(f(B) - f(A))(B - A) = (B - A)(f(B) - f(A)),$$

which is also equivalent to the fact that

$$f(A)B + f(B)A = Bf(A) + Af(B).$$

These prove the equivalence between (iv) and (v). \square

Therefore, we can state:

Corollary 6.1. *Let A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$. The statement (i) is equivalent to the inequality*

$$f(B)B + f(A)A \geq f(A)B + f(B)A, \quad (2.16)$$

for all f an operator monotone function on $(-1, 1)$.

In the note [4] Fujii and Nakamoto showed that the inequality (2.13) does not hold in general for A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$. They also proved that if $C, D > 0$ and $CD^{-1} + DC^{-1}$ is selfadjoint, then

$$CD^{-1} + DC^{-1} \geq 2. \quad (2.17)$$

Indeed, as shown in [4], if we put $T = CD^{-1}$, then $V = T + T^{-1}$ is selfadjoint by the assumption. Note that the spectrum $\text{Sp}(T)$ of T is included in $(0, \infty)$, because $C, D > 0$ and $\text{Sp}(T) = \text{Sp}(C^{1/2}D^{-1}C^{1/2})$. Since $\text{Sp}(V) = \{t + \frac{1}{t}, t \in \text{Sp}(T)\}$ by the spectral mapping theorem for rational functions, hence we have $T + T^{-1} \geq 2$.

Observe that, in fact we have:

Proposition 2.1. *Let A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$, then the statements (i) and*

(i') The operator $A(1 - \lambda B)^{-1} + B(1 - \lambda A)^{-1}$ is selfadjoint for all $\lambda \in [-1, 1]$, are equivalent.

Proof. Notice that for all $\lambda \in [-1, 1]$,

$$\begin{aligned} & (1 - \lambda B)^{-1}(1 - \lambda A) + (1 - \lambda A)^{-1}(1 - \lambda B) \\ &= (1 - \lambda B)^{-1} + (1 - \lambda A)^{-1} - \lambda \left[(1 - \lambda B)^{-1}A + (1 - \lambda A)^{-1}B \right]. \end{aligned} \quad (2.18)$$

Also, the operator $(1 - \lambda B)^{-1} + (1 - \lambda A)^{-1}$ is selfadjoint for $\lambda \in [-1, 1]$.

If the statement (i) holds, then $(1 - \lambda B)^{-1}(1 - \lambda A) + (1 - \lambda A)^{-1}(1 - \lambda B)$ is selfadjoint and by (2.18) we must have that $(1 - \lambda B)^{-1}A + (1 - \lambda A)^{-1}B$ is selfadjoint, which shows that

$$\left((1 - \lambda B)^{-1}A + (1 - \lambda A)^{-1}B \right)^* = A(1 - \lambda B)^{-1} + B(1 - \lambda A)^{-1}$$

is selfadjoint, namely (i') is true.

If the statement (i') holds, then by (2.18) we get

$$(1 - \lambda B)^{-1}(1 - \lambda A) + (1 - \lambda A)^{-1}(1 - \lambda B)$$

is selfadjoint and by (2.17) for $C = (1 - \lambda B)^{-1}$, $D = (1 - \lambda A)^{-1}$ we obtain the inequality (2.11), namely (i) is true. \square

We define the class of operators

$$\mathfrak{CI}_{(-1,1)}(H) := \{(A, B) \mid \text{Sp}(A), \text{Sp}(B) \subset (-1, 1) \text{ and satisfy condition (i')}\}.$$

We observe that if $(A, B) \in \mathfrak{CI}_{(-1,1)}(H)$ then $(B, A) \in \mathfrak{CI}_{(-1,1)}(H)$.

Also if $AB = BA$ with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$ then $V_\lambda := (1 - \lambda B)^{-1} (1 - \lambda A)$ and $V_\lambda^{-1} = (1 - \lambda A)^{-1} (1 - \lambda B)$, $\lambda \in [-1, 1]$, are selfadjoint and since $V_\lambda + V_\lambda^{-1} \geq 2$, $\lambda \in [-1, 1]$ we derive that $(A, B) \in \mathfrak{C}\mathfrak{I}_{(-1,1)}(H)$. Therefore, if $\mathfrak{C}\mathfrak{O}_{(-1,1)}(H)$ is the class of all pairs of *commutative operators* A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1, 1)$, then we have

$$\emptyset \neq \mathfrak{C}\mathfrak{O}_{(-1,1)}(H) \subset \mathfrak{C}\mathfrak{I}_{(-1,1)}(H). \quad (2.19)$$

3. DISCRETE INEQUALITIES

We have the following Chebychev type operator inequality:

Proposition 3.1. *Assume that the function $f : (-1, 1) \rightarrow \mathbb{R}$ is operator monotone in $(-1, 1)$. If $g : I \rightarrow (-1, 1)$ is continuous, then for all selfadjoint operators A_k , $k = 1, \dots, n$ with spectra in I such that $(g(A_k), g(A_j)) \in \mathfrak{C}\mathfrak{I}_{(-1,1)}(H)$ for all $k, j = 1, \dots, n$ and $p_k \geq 0$, $k = 1, \dots, n$ with $\sum_{k=1}^n p_k = 1$, we have*

$$\sum_{k=1}^n p_k (f \circ g)(A_k) g(A_k) \geq \sum_{k=1}^n p_k g(A_k) \sum_{k=1}^n p_k (f \circ g)(A_k). \quad (3.1)$$

If A_k , $k = 1, \dots, n$, with $\text{Sp}(A_k) \subset (-1, 1)$, $(A_k, A_j) \in \mathfrak{C}\mathfrak{I}_{(-1,1)}(H)$ for all $k, j = 1, \dots, n$ and $p_k \geq 0$, $k = 1, \dots, n$ with $\sum_{k=1}^n p_k = 1$, then

$$\sum_{k=1}^n p_k f(A_k) A_k \geq \sum_{k=1}^n p_k A_k \sum_{k=1}^n p_k f(A_k). \quad (3.2)$$

Proof. From (2.16) we get

$$\begin{aligned} & (f \circ g)(A_k) g(A_k) + (f \circ g)(A_j) g(A_j) \\ & \geq (f \circ g)(A_j) g(A_k) + (f \circ g)(A_k) g(A_j) \end{aligned} \quad (3.3)$$

for all $k, j \in \{1, \dots, n\}$.

If we multiply (3.3) by $p_k p_j \geq 0$ and sum over k and j from 1 to n , we get

$$\begin{aligned} & \sum_{k=1}^n \sum_{j=1}^n p_k p_j (f \circ g)(A_k) g(A_k) + \sum_{k=1}^n \sum_{j=1}^n p_k p_j (f \circ g)(A_j) g(A_j) \\ & \geq \sum_{k=1}^n \sum_{j=1}^n p_k p_j (f \circ g)(A_j) g(A_k) + \sum_{k=1}^n \sum_{j=1}^n p_k p_j (f \circ g)(A_k) g(A_j), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \sum_{k=1}^n p_k (f \circ g)(A_k) g(A_k) + \sum_{j=1}^n p_j (f \circ g)(A_j) g(A_j) \\ & \geq \sum_{j=1}^n p_j (f \circ g)(A_j) \sum_{k=1}^n p_k g(A_k) + \sum_{k=1}^n p_k (f \circ g)(A_k) \sum_{j=1}^n p_j g(A_j) \end{aligned}$$

that is equivalent to the first part of (3.1). \square

Remark 3.1: If A_k , $k = 1, \dots, n$, with $\text{Sp}(A_k) \subset (-1, 1)$, $(A_k, A_j) \in \mathfrak{C}\mathfrak{I}_{(-1,1)}(H)$ for all $k, j = 1, \dots, n$ and $p_k \geq 0$, $k = 1, \dots, n$ with $\sum_{k=1}^n p_k = 1$, then by (3.2),

$$\sum_{k=1}^n p_k A_k \tan(A_k) \geq \sum_{k=1}^n p_k A_k \sum_{k=1}^n p_k \tan(A_k). \quad (3.4)$$

Also for $r \in (0, 1)$,

$$\begin{aligned} & \sum_{k=1}^n p_k A_k \left[(1_H + A_k)(1_H - A_k)^{-1} \right]^r \\ & \geq \sum_{k=1}^n p_k A_k \sum_{k=1}^n p_k \left[(1_H + A_k)(1_H - A_k)^{-1} \right]^r. \end{aligned} \quad (3.5)$$

We also have

$$\begin{aligned} & \sum_{k=1}^n p_k A_k \ln \left[(1_H + A_k)(1_H - A_k)^{-1} \right] \\ & \geq \sum_{k=1}^n p_k A_k \sum_{k=1}^n p_k \ln \left[(1_H + A_k)(1_H - A_k)^{-1} \right]. \end{aligned} \quad (3.6)$$

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