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A PARAMETRIZATION OF RATIONAL $D(q)$-TRIPLES

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Abstract. We parametrize rational $D(q)$-triples for $q \in \mathbb{Q}$ which is not a square. We use this parametrization to describe all rational $D(q)$-triples which are also $D(0)$-triples, and to describe all rational $D(q)$-quadruples which contain at least one regular $D(q)$-triple.

1. Introduction

Let $q \in \mathbb{Q}$ be a nonzero rational number. A set of n distinct nonzero rationals $\{a_1, a_2, \ldots, a_n\}$ is called a rational $D(q)$-n-tuple if $a_i a_j + q$ is a square for all $1 \leqslant i < j \leqslant n$. If $\{a_1, a_2, \ldots, a_n\}$ is a rational $D(q)$-n-tuple, then for all $r \in \mathbb{Q}$, the set $\{ra_1, ra_2, \ldots, ra_n\}$ is a $D(qr^2)$-n-tuple, since $(ra_1)(ra_2) + qr^2 = (a_1a_2 + q)r^2$. With this in mind, we restrict to square-free integers q. $D(1)$-n-tuples are called Diophantine n-tuples.

The first example of a rational Diophantine quadruple was the set

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$$

found by Diophantus, while the first example of an integer Diophantine quadruple, the set

$$\{1, 3, 8, 120\}$$

is due to Fermat.

In the case of integer Diophantine n-tuples, it is known that there are infinitely many Diophantine quadruples (e.g. $\{k-1, k+1, 4k, 16k^3-4k\}$, for $k \geq 2$). Dujella [7] showed there are no Diophantine sextuples and only finitely many Diophantine quintuples, while recently He, Togbé and Ziegler [16] proved there are no integer Diophantine quintuples, which was a long standing conjecture.

Gibbs [15] found the first example of a rational Diophantine sextuple using a computer, and Dujella, Kazalicki, Mikić and Szikszai [12] constructed infinite families of rational Diophantine sextuples. Dujella and Kazalicki parametrized Diophantine quadruples with a fixed product of elements using triples of points on

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a specific elliptic curve, and used that parametrization for counting Diophantine quadruples over finite fields [11] and for constructing rational sextuples [10]. There is no known rational Diophantine septuple.

Regarding rational $D(q)$-n-tuples, Dujella [6] has shown that there are infinitely many rational $D(q)$-quadruples for any $q \in \mathbb{Q}$. Dražić and Kazalicki [4] parametrized rational $D(q)$-quadruples (a, b, c, d) with a fixed product of elements $m = abcd$, using triples of points on the elliptic curve $E_m\colon y^2 = x^3 + (4q^2 - 2m)x^2 + m^2x$, and for each $q \in \mathbb{Q}$ they found all $m \in \mathbb{Q}$ such that there exists a rational $D(q)$-quadruple with product of elements equal to m. Dujella and Fuchs in [9] have shown that, assuming the Parity Conjecture, for infinitely squarefree integers $q \neq 1$ there exist infinitely many rational $D(q)$-quintuples, Dražić improved their results in [3] by proving the same statement for a larger class of numbers q. There is no known rational $D(q)$-sextuple for $q \neq a^2, a \in \mathbb{Q}$.

Adžaga, Dujella, Kreso and Tadić in [1] constructed infinite families of integer Diophantine triples which are $D(n)$-triples for two other different integers n. Dujella, Kazalicki and Petričević [13] constructed infinitely many rational Diophantine quintuples which are also $D(0)$-quintuples. There are several parametrizations of rational Diophantine triples. One was used by Dujella [8, §2] to construct rational Diophantine sextuples with mixed signs. Kazalicki and Naskręcki [17] discovered a new parametrization of rational Diophantine triples which is birationally equivalent to the parametrization of Lasić, mentioned in the appendix of their article.

The main result of this paper is the first parametrization of rational $D(q)$-triples, where q is not a square.

Theorem 1. *Let $u_1, u_2, u, q \in \mathbb{Q}$, where q is not a square. The triple $\{a, b, c\}$ defined by*

$$\begin{aligned} a &= \frac{(u^2 - q)(u_1^2 - q)}{(u^2 + q)(u_2 - u_1) + 2u(u_1u_2 - q)}, \\ b &= \frac{(u^2 - q)(u_2^2 - q)}{(u^2 + q)(u_2 - u_1) + 2u(u_1u_2 - q)}, \\ c &= \frac{(u^2 + q)(u_2 - u_1) + 2u(u_1u_2 - q)}{u^2 - q} \end{aligned} \tag{1.1}$$

is a rational $D(q)$-triple if a, b and c are nonzero distinct rational numbers. Every rational $D(q)$-triple is realized by this parametrization.

Remark 1.1: The non-degeneracy conditions, which are requirements that a, b and c are nonzero and distinct numbers, are discussed in Proposition 3.1

In Section 2 we adapt the construction used by Kazalicki and Naskręcki in [17] to parametrize all $\mathbb{Q}(\sqrt{q})$-rational $D(q)$-triples. In Section 3 we use the result from Section 2 to prove Theorem 1 and discuss degeneracy conditions (a triple is *degenerate* if any element is not a well defined number, if any element is equal to zero or if any two elements are not distinct). In Section 4 we give some direct applications of our parametrization, namely, in Theorem 2 we describe all rational

$D(q)$-triples which are also $D(0)$-triples (pairwise products of elements are squares) and in Theorem 4 we describe all rational $D(q)$-quadruples which contain at least one regular $D(q)$-triple (see Definition 4.2).

2. Initial construction

We start by following [17, Introduction]. Let K be a field with $\mathrm{char} K \neq 2$, and $\mathbb{A}$ the affine space over K. Let

$$\begin{aligned} \mathcal{D}&: \quad ab+q=r^2, ac+q=s^2, bc+q=t^2, \\ \widetilde{\mathcal{D}}&: \quad \mathcal{D} \setminus \{abc(a-b)(a-c)(b-c)=0\}. \end{aligned}$$

be affine varieties in $\mathbb{A}^6$, and define

$$\begin{aligned} X&: \quad (x^2-q)(y^2-q)(z^2-q)=k^2, \\ \widetilde{X}&: \quad X \setminus \{k(x^2-y^2)(x^2-z^2)(y^2-z^2)=0\}, \end{aligned}$$

to be affine varieties in $\mathbb{A}^4$. The birational map

$$p: \mathcal{D} \to X, p(a,b,c,r,s,t)=(r,s,t,abc)$$

and its inverse

$$p': X \to \mathcal{D}, p'(x,y,z,k)=\left(\frac{k}{z^2-q}, \frac{k}{y^2-q}, \frac{k}{x^2-q}, x, y, z\right) \tag{2.1}$$

define a bijection between the sets $\widetilde{\mathcal{D}}(K)$ and $\widetilde{X}(K)$.

Set $K=\mathbb{Q}(\sqrt{q})$ for some rational q which is not a square. Let $\overline{X} \subset \mathbb{P}^4(K)$ denote the projective closure of the variety X,

$$\overline{X}: (x^2-w^2q)(y^2-w^2q)(z^2-w^2q)=k^2w^4.$$

Proposition 2.1. *The projective variety $\overline{X}$ is birational to $\mathbb{P}^3(K)$, the projective 3-space over K.*

Proof. Let $\phi: \overline{X} \to \mathbb{P}^3(K)$ denote the following rational map

$$\begin{aligned} \phi([x:y:z:k:w]) = [&y \cdot (w\sqrt{q}-x)(w^2q-y^2) : \\ &z \cdot (w\sqrt{q}-x)(w^2q-y^2) : \\ &kw^3\sqrt{q} : \\ &w\sqrt{q} \cdot (w\sqrt{q}-x)(w^2q-y^2)] \end{aligned}$$

and let $\psi: \mathbb{P}^3(K) \to \overline{X}$ denote the map

$$\begin{aligned} \psi([t_1:t_2:t_3:t_0]) = [&t_0 \cdot (t_3^2(t_1^2-t_2^2)-(t_0^2-t_2^2)(t_0^2+t_3^2)) : \\ &t_1 \cdot ((t_0^2-t_2^2)(t_0^2-t_3^2)+t_3^2(t_1^2-t_2^2)) : \\ &t_2 \cdot ((t_0^2-t_2^2)(t_0^2-t_3^2)+t_3^2(t_1^2-t_2^2)) : \\ &2qt_3 \cdot (t_1^2-t_0^2)(t_2^2-t_0^2) : \\ &\frac{t_0}{\sqrt{q}} \cdot ((t_0^2-t_2^2)(t_0^2-t_3^2)+t_3^2(t_1^2-t_2^2))]. \end{aligned}$$

By direct computation we check that both $\phi \circ \psi$ and $\psi \circ \phi$ are identity maps, hence both are birational and we are done. $\square$

3. Parametrization of $D(q)$-triples

Using the maps p', defined at (2.1) and ψ, defined in Proposition 2.1 we can parametrize K-rational $D(q)$-triples, which are triples of distinct nonzero numbers $\{a, b, c\}$ in K such that $ab+q, bc+q, ac+q$ are squares in K.

We calculate

$$a = \frac{\frac{k}{w}}{\left(\frac{z}{w}\right)^2 - q}\Big|_{t_0=1} = \frac{2\sqrt{q}\cdot t_3(t_1^2-1)}{t_3^2(t_1^2-1)+1-t_2^2},$$
$$b = \frac{\frac{k}{w}}{\left(\frac{y}{w}\right)^2 - q}\Big|_{t_0=1} = \frac{2\sqrt{q}\cdot t_3(t_2^2-1)}{t_3^2(t_1^2-1)+1-t_2^2},$$
$$c = \frac{\frac{k}{w}}{\left(\frac{x}{w}\right)^2 - q}\Big|_{t_0=1} = \frac{\sqrt{q}\cdot (t_3^2(t_1^2-1)+1-t_2^2)}{2t_3}.$$

From this, it follows that

$$ac+q = qt_1^2, \quad bc+q = qt_2^2, \quad ab+q = q\cdot\left(\frac{t_3^2(t_1^2-1)+t_2^2-1}{t_3^2(t_1^2-1)-(t_2^2-1)}\right)^2.$$

We are interested in rational $D(q)$-triples. For a K-rational $D(q)$-triple to be a rational $D(q)$-triple, the numbers a, b and c have to be rational, and the numbers $ab+q, bc+q$ and $ac+q$ have to be rational squares. These statements simplify to $ab+q, bc+q$ and $ac+q$ being rational squares, and one of the numbers a, b, c being rational, the rationality of the other two follows easily.

The conditions $ac+q$ and $bc+q$ are rational squares directly imply that

$$t_1 = u_1\sqrt{q} \text{ and } t_2 = u_2\sqrt{q}, \quad u_1, u_2 \in \mathbb{Q}.$$

Let $t_3 = \alpha + \beta\sqrt{q}$, where $\alpha, \beta \in \mathbb{Q}$. Denote

$$r_1 = \alpha^2 + q\beta^2, \quad r_2 = t_2^2 - 1 = qu_2^2 - 1, \quad r_3 = 2\alpha\beta(t_1^2-1) = 2\alpha\beta(qu_1^2-1).$$

Then

$$ab+q = q\left(\frac{r_1+r_2+r_3\sqrt{q}}{r_1-r_2+r_3\sqrt{q}}\right)^2 = q\left(\frac{(r_1+r_2+r_3\sqrt{q})(r_1-r_2-r_3\sqrt{q})}{(r_1-r_2)^2-qr_3^2}\right)^2$$
$$= q\left(\frac{r_1^2-r_2^2-qr_3^2-2r_2r_3\sqrt{q}}{(r_1-r_2)^2-qr_3^2}\right)^2.$$

For the last expression to be a rational square, we must have

$$r_1^2 - r_2^2 - qr_3^2 = 0 \iff \alpha^2 - q\beta^2 = \pm\frac{qu_2^2-1}{qu_1^2-1} = \epsilon\frac{qu_2^2-1}{qu_1^2-1}, \tag{3.1}$$

where ϵ is either 1 or -1. Finally, we analyze the condition $c \in \mathbb{Q}$. Denote $\overline{t_3} = \alpha - \beta\sqrt{q}$.

$$\begin{aligned} c &= \frac{\sqrt{q}\cdot(t_3^2(t_1^2-1)+1-t_2^2)}{2t_3} = \frac{\sqrt{q}\cdot(t_3\cdot t_3\cdot \overline{t_3}(qu_1^2-1)+\overline{t_3}(1-qu_2^2))}{2t_3\cdot \overline{t_3}} \\ &= \frac{\sqrt{q}\cdot(t_3\cdot\epsilon(qu_2^2-1)+\overline{t_3}(1-qu_2^2))}{2(\alpha^2-q\beta^2)} = \frac{qu_2^2-1}{2(\alpha^2-q\beta^2)}\cdot\sqrt{q}\cdot(t_3\epsilon-\overline{t_3}). \end{aligned}$$

The number $\sqrt{q}(t_3\epsilon-\overline{t_3})$ must be rational from which we conclude $\epsilon=1$. The last thing to do is find all rational α,β which satisfy equation (3.1), depending on u_1,u_2. Notice that if we find one solution, we can find all of them since equation (3.1) is a conic in variables α,β. We use the norm on $\mathbb{Q}(\sqrt{q})$ to obtain one solution.

$$N(\alpha+\beta\sqrt{q}) = \alpha^2-q\beta^2 = \frac{qu_2^2-1}{qu_1^2-1} = \frac{N(1+u_2\sqrt{q})}{N(1+u_1\sqrt{q})} = N\left(\frac{1-u_1u_2q+(u_2-u_1)\sqrt{q}}{1-qu_1^2}\right)$$

which brings us to

$$\alpha_0 = \frac{1-u_1u_2q}{1-qu_1^2},\quad \beta_0 = \frac{u_2-u_1}{1-qu_1^2}.$$

The line $\alpha-\alpha_0 = u\cdot(\beta-\beta_0)$ for $u\in\mathbb{Q}$ intersects the conic defined by (3.1) at two rational points. One of them is (α_0,β_0), and the other is the general solution

$$\begin{aligned} \alpha &= \frac{(u^2+q)(1-u_1u_2q)+2qu(u_1-u_2)}{(u^2-q)(qu_1^2-1)} \\ \beta &= \frac{(u^2+q)(u_1-u_2)+2u(1-u_1u_2q)}{(u^2-q)(qu_1^2-1)} \end{aligned}$$

Plugging in t_1,t_2,t_3 depending on $u_1,u_2,u\in\mathbb{Q}$ into the equations for a,b,c we obtain

$$\begin{aligned} a &= \frac{(u^2-q)(1-u_1^2q)}{(u^2+q)(u_2-u_1)+2u(u_1u_2q-1)}, \\ b &= \frac{(u^2-q)(1-u_2^2q)}{(u^2+q)(u_2-u_1)+2u(u_1u_2q-1)}, \\ c &= -q\frac{(u^2+q)(u_2-u_1)+2u(u_1u_2q-1)}{u^2-q}. \end{aligned}$$

Finally, replacing the parameters u_1 and u_2 with u_1/q and u_2/q, as well as changing the sign of a,b and c, we obtain the parametrization from Theorem 1. For good measure, we note that

$$ac+q=u_1^2,\quad bc+q=u_2^2,\quad ab+q=\left(\frac{(q+u^2)(u_1u_2-q)+2qu(u_2-u_1)}{(q+u^2)(u_2-u_1)+2u(u_1u_2-q)}\right)^2. \tag{3.2}$$

This proves the fact that every rational $D(q)$-triple is attained by some $u,u_1,u_2\in\mathbb{Q}$ using the parametrization from Theorem 1. The other direction, the fact that every triple obtained by the parametrization is a rational $D(q)$-triple, ignoring degeneracies, is easily checked by direct calculation, for example using Magma [2].

Proposition 3.1. *The triple* (a, b, c) *obtained via the parametrization from Theorem 1 is degenerate if and only if any of the following conditions is true:*

(i) $u_1^2 = u_2^2$,

(ii) $u_2 = \dfrac{2uq + (u^2+q)u_1}{u^2+q+2uu_1}$,

(iii) $u_1 = \dfrac{w_1^2+q}{2w_1}$ *and* $\left(u_2 = \dfrac{q+uw_1}{u+w_1} \quad \text{or} \quad u_2 = \dfrac{q(u+w_1)}{q+uw_1}\right)$,
for some $w_1 \in \mathbb{Q}$,

(iv) $u_2 = \dfrac{w_2^2+q}{2w_2}$ *and* $\left(u_1 = \dfrac{q+uw_2}{u+w_2} \quad \text{or} \quad u_1 = \dfrac{q(u+w_2)}{q+uw_2}\right)$,
for some $w_2 \in \mathbb{Q}$.

Proof. The triple is degenerate if any element of the triple is equal to zero, if any element of the triple is not defined or if any two elements of the triple are equal.

Since q is not a square, a and b cannot be equal to zero, and c is a well defined (rational) number. The denominators of a and b, as well as the numerator of c are equal (at least in the non-reduced form of the parametrization from Theorem 1) and we need to calculate when this number is equal to zero. It is easy to see this is exactly case (ii) of our proposition. It is also easy to see that the condition $a = b$ is case (i).

The conditions $a = c$ and $b = c$ have symmetric computations (by switching variables u_1 and u_2) so it is enough to solve one of these conditions. Assume that $a = c$. Equating the parametric expressions for a and c leads to

$$u_1^2 - q = c^2 = \left(\frac{(u^2+q)(u_2-u_1) + 2u(u_1u_2-q)}{u^2-q}\right)^2. \tag{3.3}$$

We notice that $u_1^2 - q$ must be a square, which is true if and only if $u_1 = \dfrac{w_1^2+q}{2w_1}$, for some $w_1 \in \mathbb{Q}$. Taking the square root of equation (3.3) and after some manipulations we obtain

$$u_2 = \frac{(u^2+q)(w_1^2+q) + 2uq \pm (u^2-q)(w_1^2-q)}{2(u+w_1)(q+uw_1)},$$

which leads to solutions

$$u_2 = \frac{q+uw_1}{u+w_1} \quad \text{or} \quad u_2 = \frac{q(u+w_1)}{q+uw_1}.$$

The condition $a = c$ is case (iii) of our proposition. As previously mentioned, the condition $b = c$ is computationally symmetric and described by case (iv). This proves the proposition. □

4. Applications of the parametrization of $D(q)$-triples

Definition 4.1. *A* rational $D(0)$-triple (a, b, c) *is a triple of distinct nonzero rational numbers such that* ab, bc *and* ac *are squares.*

Motivated by the results of [1] and [13], we parametrize all $D(q)$-triples which are also $D(0)$-triples using the parametrization from Theorem 1. Notice that if any two numbers from the set $\{ab, bc, ac\}$ are squares, then the third one is as well, since $(ab)(bc)(ac) = (abc)^2$.

Theorem 2. *Every rational $D(q)$-triple which is also a $D(0)$-triple is parametrized by $w_1, w_2, u \in \mathbb{Q}$ as follows:*

$$\begin{aligned} a &= \frac{w_2(w_1^2-q)^2(u^2-q)}{2w_1(uw_1-uw_2-w_1w_2+q)(uw_1w_2-qu-qw_1+qw_2)}, \\ b &= \frac{w_1(w_2^2-q)^2(u^2-q)}{2w_2(uw_1-uw_2-w_1w_2+q)(uw_1w_2-qu-qw_1+qw_2)}, \\ c &= \frac{(uw_1-uw_2-w_1w_2+q)(uw_1w_2-qu-qw_1+qw_2)}{2w_1w_2(u^2-q)}. \end{aligned}$$

Proof. Since (a, b, c) is a $D(0)$-triple, we have $e^2 = ac, f^2 = bc$ for some $e, f \in \mathbb{Q}$. Now using the parametrization from Theorem 1 (specifically, (3.2)), we have $e^2 = ac = u_1^2 - q$ and $f^2 = bc = u_2^2 - q$. From this we easily obtain

$$u_1 = \frac{w_1^2+q}{2w_1}, \quad u_2 = \frac{w_2^2+q}{2w_2}, \qquad \text{for some } w_1, w_2 \in \mathbb{Q}.$$

Replacing u_1 and u_2 with the previous expressions depending on w_1 and w_2 proves one direction of the theorem. The other is checked by calculation. □

There is a more elegant, symmetric parametrization of such triples which makes it easier to characterize when the triple is non-degenerate.

Theorem 3. *All rational $D(q)$-triples which are $D(0)$-triples are parametrized by $t_1, t_2, t_3 \in \mathbb{Q}$ with*

$$a = \frac{1}{2}\frac{\left(\frac{q}{t_1}-t_1\right)\left(\frac{q}{t_2}-t_2\right)}{\frac{q}{t_3}-t_3}, \quad b = \frac{1}{2}\frac{\left(\frac{q}{t_2}-t_2\right)\left(\frac{q}{t_3}-t_3\right)}{\frac{q}{t_1}-t_1}, \quad c = \frac{1}{2}\frac{\left(\frac{q}{t_3}-t_3\right)\left(\frac{q}{t_1}-t_1\right)}{\frac{q}{t_2}-t_2}.$$

The triple is nondegenerate if the following conditions hold, with $i \neq j$:

$$t_i \neq 0, \quad t_i \neq \pm t_j, \quad t_it_j \neq \pm q.$$

Proof. Assume $ab + q = r^2$ and $ab = d^2$. As in the proof of Theorem 2, it is necessary and sufficient that $r = \frac{1}{2}\left(\frac{q}{t_1} + t_1\right)$ which makes $d = \frac{1}{2}\left(\frac{q}{t_1} - t_1\right)$, for some $t_1 \in \mathbb{Q}$. Similarly, if $ac = e^2$ and $bc = f^2$, we must have

$$e = \frac{1}{2}\left(\frac{q}{t_2} - t_2\right), \quad f = \frac{1}{2}\left(\frac{q}{t_3} - t_3\right), \qquad \text{for some } t_2, t_3 \in \mathbb{Q}.$$

Now we have

$$|a| = \sqrt{a^2} = \sqrt{(ab)(ac)/(bc)} = \sqrt{d^2e^2/f^2} = |de/f|,$$

from which we conclude $|b| = |fd/e|$ and $|c| = |ef/d|$. Notice that a, b and c are either all positive or all negative since ab, bc and ac are squares, hence positive. If the parameters (t_1, t_2, t_3) induce the triple (a, b, c), then the parameters $(-t_1, t_2, t_3)$ induce the triple $(-a, -b, -c)$, so every such triple is obtained via our parametrization.

A quick calculation checks that the triple (a, b, c) described by the parametrization from this theorem is indeed both a rational $D(q)$-triple and a $D(0)$-triple. □

Proposition 4.1. *The parametrizations from Theorems 2 and 3 are birationally equivalent.*

Proof. Define the maps $\phi_1 : \mathbb{A}^3(\mathbb{Q}) \to \mathbb{A}^3(\mathbb{Q})$ and $\psi_1 : \mathbb{A}^3(\mathbb{Q}) \to \mathbb{A}^3(\mathbb{Q})$ using the equations

$$\phi_1(w_1, w_2, u) = \left(w_1, \frac{w_2 w_1 u + w_2 q - w_1 q - qu}{w_2 w_1 + w_2 u - w_1 u - q}, w_2\right),$$

$$\psi_1(t_1, t_2, t_3) = \left(t_1, t_3, \frac{q(t_2 + t_3 - t_1) - t_1 t_2 t_3}{q + t_2 t_3 - t_1 t_2 - t_1 t_3}\right).$$

Using Magma, we check that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity map, as well as the following fact: If $\mathcal{P}_1(w_1, w_2, u)$ is the parametrization from Theorem 2, and $\mathcal{P}_2(t_1, t_2, t_3)$ the parametrization from Theorem 3, then the following holds:

$$\mathcal{P}_1 = \mathcal{P}_2 \circ \phi_1, \quad \mathcal{P}_2 = \mathcal{P}_1 \circ \psi_1.$$

□

Definition 4.2. *A rational $D(q)$-triple (a, b, c) is called* regular *if*

$$(c - b - a)^2 = 4(ab + q) \tag{4.1}$$

Remark 4.1: Equation (4.1) is symmetric under permutations of a, b, c. It is well known, and easily seen, that if (a, b, c) is a regular rational $D(q)$-triple then $c = c_+$ or $c = c_-$, where

$$c_\pm = a + b \pm 2r, \text{ with } ab + q = r^2.$$

We construct all rational $D(q)$-quadruples containing at least one regular triple. Let (a, b, c) be a rational $D(q)$-triple obtained from the parametrization of Theorem 1. Since $ac + q = u_1^2$, if we define $d = a + c + 2u_1$, then the triple (a, c, d) is regular. In order for (a, b, c, d) to be a rational $D(q)$-quadruple, the number $bd + q$ must be a square, and the quadruple (a, b, c, d) must be non-degenerate. The number $bd + q$ is equal to

$$bd + q = f_3(u, u_1, u_2, q)^{-2} \left(f_2(u, u_2, q) u_1^2 + f_1(u, u_2, q) u_1 + f_0(u, u_2, q)\right),$$

where f_i are polynomials given by the equations

$$\begin{aligned}
f_3(u, u_1, u_2, q) &= (u_2 u - u^2/2 - q/2) u_1 + u_2 u^2/2 + u_2 q/2 - uq, \\
f_2(u, u_2, q) &= u_2^4 u^2 - 2u_2^3 uq + u_2^2 q^2 - u_2 u^3 q + u_2 u q^2 + u^4 q/4 + u^2 q^2/2 - 3q^3/4, \\
f_1(u, u_2, q) &= u_2^4 u^3 + u_2^4 uq - 3u_2^3 u^2 q - u_2^3 q^2 + 2u_2^2 u q^2 - u_2 u^4 q/2 + u2q^3/2 \\
&\quad + u^3 q^2 - uq^3,
\end{aligned}$$

$$f_0(u,u_2,q) = \frac{1}{4}(u_2^4u^4 + 2u_2^4u^2q + u_2^4q^2 - 4u_2^3u^3q - 4u_2^3uq^2 - u_2^2u^4q \\ + 6u_2^2u^2q^2 - u_2^2q^3 + u^4q^2 - 2u^2q^3 + q^4).$$

We calculate u_1 for which

$$\mathcal{P}(u_1) := f_2(u,u_2,q)u_1^2 + f_1(u,u_2,q)u_1 + f_0(u,u_2,q),$$

is a square. By direct calculation we check that

$$\mathcal{P}(u_2) = f_2(u,u_2,q)u_2^2 + f_1(u,u_2,q)u_2 + f_0(u,u_2,q) = \left((u_2u + u^2/2 - q/2)(u_2^2 - q)\right)^2.$$

Similarly as when solving equation (3.1), we introduce a new variable z and solve the quadratic equation

$$\left(z(u_1 - u_2) + (u_2u + u^2/2 - q/2)(u_2^2 - q)\right)^2 = \mathcal{P}(u_1)$$

in variable u_1. We know one solution is $u_1 = u_2$, and the other solution is a rational function $u_1(u,u_2,z,q)$ such that $\mathcal{P}(u_1(u,u_2,z,q)) \in \left(\mathbb{Q}(u,u_2,z,q)^*\right)^2$. The general solution is

$$u_1 = \frac{\mathcal{B}(u,u_2,z,q)}{\mathcal{N}(u,u_2,z,q)} \tag{4.2}$$

where the polynomials $\mathcal{B}$ and $\mathcal{N}$ are given by

$$\begin{aligned}\mathcal{B}(u,u_2,z,q) =& - u_2^5u^2 - u_2^4u^3 + u_2^4uq + 3u_2^3u^2q + 2u_2^3uz + u_2^2u^3q + u_2^2u^2z \\ & - 3u_2^2uq^2 - u_2^2qz + u_2u^4q/4 - u_2u^2q^2/2 - 2u_2uqz + u_2q^3/4 \\ & - u_2z^2 - u^3q^2 - u^2qz + uq^3 + q^2z, \\ \mathcal{N}(u,u_2,z,q) =& u_2^4u^2 - 2u_2^3uq + u_2^2q^2 - u_2u^3q + u_2uq^2 \\ & + u^4q/4 + u^2q^2/2 - 3q^3/4 - z^2.\end{aligned}$$

Theorem 4. *Let $q,u,u_2,z \in \mathbb{Q}$, such that q is not a square. Define u_1 by (4.2), the numbers a,b and c by (1.1) and define $d = a + c + 2u_1$.*

The quadruple (a,b,c,d) parametrized by (u,u_2,z) is a rational $D(q)$-quadruple containing the regular triple (a,c,d), if the quadruple (a,b,c,d) is non-degenerate. Every rational $D(q)$-quadruple containing a regular triple is realized through this parametrization.

Remark 4.2: Many constructions of $D(q)$-m-tuples started from a $D(q)$-pair and expanded it to a $D(q)$-triple using regularity. Examples are the construction of rational $D(q)$-quintuples by Dujella [5] and Dražić [3], as well as the construction of *strong $D(q)$-triples* by Dujella, Paganin and Sadek [14]. The parametrizations we found may be used as a better starting point in similar constructions.

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