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ON THE BOREL THEOREM

FATHI HAGGUI [1] AND ABDESSAMI JALLED [2]

Abstract. In this note we give a new characterization of holomorphic curves which lies in the complement of $2n$ hyperplanes in general position. This is a generalisation of The Borel Theorem. By M.Green, any holomorphic curves which lies in the complement of $2n+1$ hyperplanes in general position is constant. Using our generalisation of The Borel Theorem, we can easily show Green's Theorem.

1. Introduction

The classical Picard Theorem [8] (see also [6] and [11]) states that every holomorphic map from the complex Euclidean space $\mathbb{C}$ to $\mathbb{C}P^1$ that omits three points, is constant. This Theorem has been extended to higher dimension by M.Green [9] (see also [2] and [7]), who provided examples of complex Kobayashi hyperbolic manifolds. We recall that given a complex manifold X, then X is said to be Kobayashi hyperbolic if the following kobayashi-Royden pseudo-metric is non degenerate:

$$K_X(p,v) := inf\{\alpha > 0; \exists f : \mathbb{D} \to X \ holomorphic,\ f(0) = p,\ f'(0) = \frac{v}{\alpha}\}.$$

With $p \in X$ and $v \in T_p(X)$.
We call a complex manifold X Brody hyperbolic (B-hyperbolic) if any holomorphic curve $f : \mathbb{C} \to X$ is constant. Obviously, Kobayashi hyperbolicity implies B-hyperbolicity. The reverse is true for compact complex manifolds [3]. There are many examples of compact complex manifolds that are hyperbolic according to Kobayashi, and so by Brody Theorem (see [3]), have the prorerty that each holomorphic curve $f : \mathbb{C} \to X$ is constant. We recall that if $H_1, ..., H_m$ are complex hyperplanes in $\mathbb{C}P^n$, then they are said to be in general position if $m \geqslant n+1$ and any $(n+1)$ of these hyperplanes are linearly independent. Since Bloch and Cartan, the hyperbolicity of the complement of arrangements of projective lines in general position in the complex projective plane has been the subject of numerous

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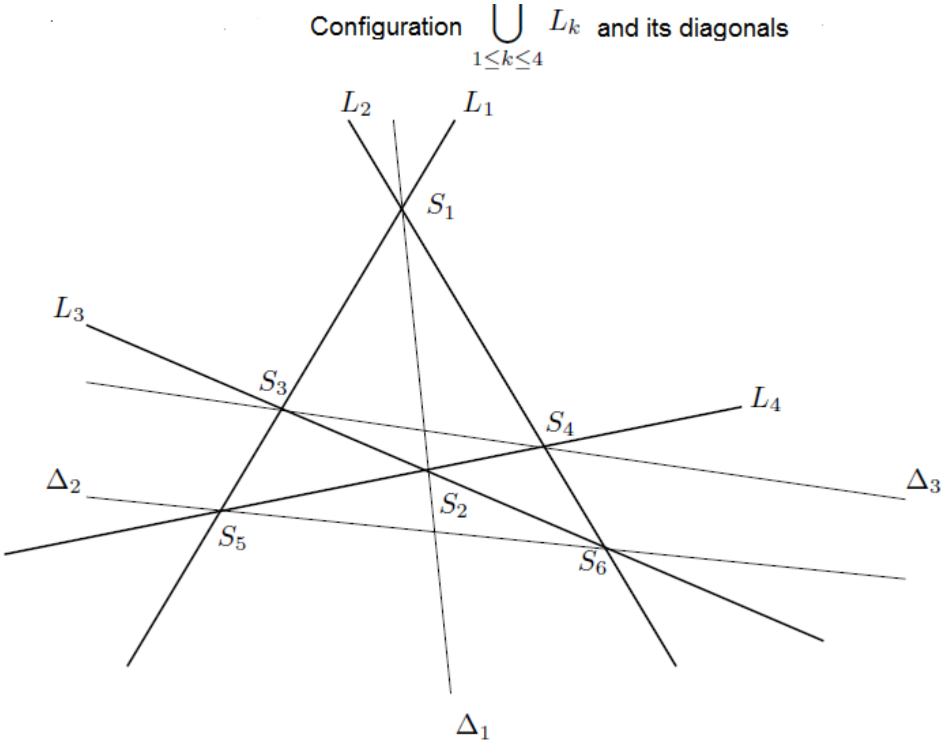
studies for many years and by several researchers (as Mikhail Zaidenberg, Adel Khalfallah and Hervé Gaussier). Different results were obtained for some special cases, especially the next Theorem due to Borel, stated by Cartan in the following form (see [4], [2] and also [1]):

Theorem 1 (Borel Theorem)**.** *Let L_1, L_2, L_3 and L_4 be four projective lines of $\mathbb{C}P^2$ in the general position, that is, such that the configuration $C = L_1 \cup L_2 \cup L_3 \cup L_4$ does not have a triple point. Let us note Δ be the union of the three diagonals Δ_1, Δ_2 and Δ_3 of the configuration C, that is the projective lines passing through double points of C. Then, any non-constant entire curve with values in $\mathbb{C}P^2 \setminus C$ degenerates in Δ. $(i.e$ there exists $i \in \{1, 2, 3\}$ such that $f(\mathbb{C}) \subset \Delta_i)$.*



Thus, via R. Brody's reparameterization Theorem (see [3], [5] and also [1]), M. Green was able to deduce from Borel's theorem the hyperbolically embedded character of the complement of five lines in the projective plane. He showed the following:

Theorem 2. [9], [11]
Let H_1, H_2,..., H_{2n+1} be $2n+1$ hyperplanes in general position in $\mathbb{C}P^n$. Then any holomorphic curves $f : \mathbb{C} \to \mathbb{C}P^n \setminus \cup_{i=1}^{2n+1} H_i$ is constant.

The main goal in this work is to generalize the Borel Theorem. We show the following:

Theorem 3. *Let $H_1, ..., H_{2n}$ be $(2n)$ projective hyperplanes in general position in $\mathbb{C}P^n$. Then there are $\frac{1}{2}C_{2n}^n$ diagonals $\Delta_1, ..., \Delta_{\frac{1}{2}C_{2n}^n}$ such that for any non constant holomorphic curve $f : \mathbb{C} \longrightarrow \mathbb{C}P^n \setminus \cup_{i=1}^{2n} H_i$, there exists $k_f \in \{1, ..., \frac{1}{2}C_{2n}^n\}$ such that $f(\mathbb{C}) \subset \Delta_{k_f}$.*

1.1. **The Fujimoto Theorem.** In 1972, Fujimoto [10] (see also M.Green[9]) showed a statement that characterizes the image of a holomorphic map $f : \mathbb{C} \to \mathbb{C}P^n$ omitting $(n+p)$ hyperplanes in general position. He proved the following:
Theorem (Serge Lang [10] **p.196**). Let $f : \mathbb{C} \to \mathbb{C}P^n$ be holomorphic. Assume that the image of f lies in the complement of $n+p$ hyperplanes in general position, then this image is contained in a complex projective subspace of complex dimension $\leqslant [n/p]$, where $[\,.\,]$ denotes the greatest integer.

The version of the Green Theorem stated in the introduction is a particular case of the previous Theorem, with $p = n+1$.

2. Proof of Theorems 3

Definition 2.1. *Let $H_1, ..., H_m$, $m \geqslant 2n$, be hyperplanes in general position of $\mathbb{C}P^n$. We call diagonal, the line passing through the two points $\cap_{i\in I} H_i$ and $\cap_{j \in J} H_j$, where $|I| = |J| = n$ and $I \cap J = \emptyset$. Here $|I|$ denotes the cardinal of I.*

Proof. (of Theorem 3). The proof is inspired by the Fujimoto Theorem, (see [10]).

Let $f : \mathbb{C} \longrightarrow \mathbb{C}P^n$ be holomorphic, such that $f(\mathbb{C}) \cap (\cup_{i=1}^{2n} H_i) = \emptyset$.
Let L_1,..., L_{2n} be linear forms defining the hyperplanes $H_1, ..., H_{2n}$, namely $H_k = L_k^{-1}(\{0\})$ for $k = 1, ..., 2n$. If $f = [f_1 : ... : f_{n+1}]$, we denote

$$h_k := L_k(f),\ k = 1, ..., 2n.$$

Let $I = \{1, \cdots, 2n\}$ be the set of indices and $\sim$ be the equivalence relation defined by $i \sim j$ if h_i/h_j is constant. We take a partition of the set of indices according to $\sim$. First, we know that the complement of a given class S has at most n elements. Hence S has at least n elements and there are at most two classes.
The case of one class is not possible. Indeed, suppose that there exists $\alpha_2, ..., \alpha_{2n} \in \mathbb{C}$ such that

$$\textbf{(S)} \left\{ \begin{array}{rcl} h_2 & = & \alpha_2 h_1 \\ h_3 & = & \alpha_3 h_1 \\ & \vdots & \\ h_{2n} & = & \alpha_{2n} h_1 \end{array} \right.$$

The first equation $h_2 = \alpha_2 h_1$ implies that $h_2 - \alpha_2 h_1 = 0$. Since, by construction, $h_k := L_k(f)$ then we obtain:

$$\begin{aligned} & L_2(f) - \alpha_2 L_1(f) = 0 \\ \Rightarrow \quad & (L_2 - \alpha_2 L_1)(f) = 0 \\ \Rightarrow \quad & f(\mathbb{C}) \subset \ker(L_2 - \alpha_2 L_1) \end{aligned}$$

Now, since $L_2 - \alpha_2 L_1$ is a combination of two linear forms defining the hyperplanes H_2 and H_1, then

$$f(\mathbb{C}) \subset \ker(L_2 - \alpha_2 L_1) \Longrightarrow f(\mathbb{C}) \subset H_2 \cap H_1$$

This is for the first equation. By doing the same for the other equations, we obtain

$$f(\mathbb{C}) \subset \cap_{k=2}^{n+1}(H_k \cap H_1) = \cap_{k=1}^{n+1} H_k = \emptyset,$$

which is impossible. Hence there are exactly two classes S_1 and S_2.
We know that each of the two classes S_1 and S_2 contains n elements. Then there exists a permutation $\sigma : \{1, ..., 2n\} \to \{1, ..., 2n\}$ such that

$$S_1 = \{\sigma(1), ..., \sigma(n)\},\ S_2 = \{\sigma(n+1), ..., \sigma(2n)\}.$$

Hence, there exist $\alpha_2, ..., \alpha_n, \beta_{n+1}, ..., \beta_{2n-1} \in \mathbb{C}$ such that $h_1, ..., h_{2n}$ satisfy the systems:

$$(S_1) \left\{ \begin{array}{rcl} h_{\sigma(2)} & = & \alpha_2 h_{\sigma(1)} \\ h_{\sigma(3)} & = & \alpha_3 h_{\sigma(1)} \\ & \vdots & \\ h_{\sigma(n)} & = & \alpha_n h_{\sigma(1)} \end{array} \right. \qquad (S_2) \left\{ \begin{array}{rcl} h_{\sigma(n+1)} & = & \beta_{n+1} h_{\sigma(2n)} \\ h_{\sigma(n+2)} & = & \beta_{n+2} h_{\sigma(2n)} \\ & \vdots & \\ h_{\sigma(2n-1)} & = & \beta_{2n-1} h_{\sigma(2n)} \end{array} \right.$$

Hence, by following the same procedure done in the previous system, we get

$$\left\{ \begin{array}{lclcl} f(\mathbb{C}) & \subset & \cap_{k=2}^{n}(H_{\sigma(k)} \cap H_{\sigma(1)}) & = & \cap_{k=1}^{n} H_{\sigma(k)} \\ f(\mathbb{C}) & \subset & \cap_{k=n+1}^{2n-1}(H_{\sigma(k)} \cap H_{\sigma(2n)}) & = & \cap_{k=n+1}^{2n} H_{\sigma(k)} \end{array} \right.$$

Then $f(\mathbb{C}) \subset \Delta_\sigma$, where Δ_σ is the unique diagonal (line) passing through the two points $\cap_{k=1}^{n} H_{\sigma(k)}$ and $\cap_{k=n+1}^{2n} H_{\sigma(k)}$.
Now the two points, and consequently Δ_σ, are completely determined by $S_1 = \{\sigma(1), ..., \sigma(n)\}$ since S_2 is automatically fixed once S_1 is chosen. Hence Δ_σ is completely determined by a choice of a partition of $\{1, ..., 2n\}$ into two subsets, each of them containing n elements. There are exactly $\frac{1}{2} C_{2n}^{n}$ such partitions. This proves the Theorem. □

Corollary 3.1. *Any holomorphic curve that lies in the complement of* $2n+1$ *hyperplanes in general position in* $\mathbb{C}P^n$*, is constant.*

Proof. Since $f(\mathbb{C})$ omits $\cup_{i=1}^{2n+1} H_i$, then in particular $f(\mathbb{C})$ omits $\cup_{i=1}^{2n} H_i$. Hence by Theorem 3 there exists a diagonal Δ_1 such that $f(\mathbb{C}) \subset \Delta_1$. On the other hand, since $f(\mathbb{C})$ omits $\cup_{i=2}^{2n+1} H_i$, then there exists Δ_2 such that $f(\mathbb{C}) \subset \Delta_2$. Hence

$$f(\mathbb{C}) \subset \Delta \cap \Delta' = \{pt\}$$

Thus f is constant. □

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[1] UNIVERSITY OF MONASTIR,
INSTITUT PRÉPARATOIRE AUX ETUDES D'INGÉNIEUR DE MONASTIR,
MONASTIR, TUNISIA
Email address: fathi.haggui@gmail.com

[2] UNIVERSITY OF MONASTIR,
FACULTY OF SCIENCES OF MONASTIR,
MONASTIR, TUNISIA
Email address: jalled.abdessami90@gmail.com

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