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# EQUIVALENT CHARACTERIZATION ON TRIEBEL-LIZORKIN SPACE

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**Abstract.** In this paper, we give an equivalent characterization of Triebel-Lizorkin spaces. This reveals the equivalent relation between the mixed derivative norm and single variable norm. Complex interpolation and Littlewood-Paley decomposition are applied in our proofs.

## 1. Introduction

In Sobolev spaces [1], it is known that $\|f\|_{H^2(\mathbb{R}^2)} \sim \|f\|_{L^2(\mathbb{R}^2)} + \sum_{i=1}^{2} \|\frac{\partial^2 f}{\partial x_i^2}\|_{L^2(\mathbb{R}^2)}$, where $\|f\|_{H^2(\mathbb{R}^2)} =: \|f\|_{L^2(\mathbb{R}^2)} + \|\partial_{x_1}\partial_{x_2} f\|_{L^2(\mathbb{R}^2)} + \|\partial_{x_1}^2 f\|_{L^2(\mathbb{R}^2)} + \|\partial_{x_2}^2 f\|_{L^2(\mathbb{R}^2)}$. Note that on the right hand side of the definition $\|f\|_{H^2(\mathbb{R}^2)}$, it contains the mixed derivative norm $\|\partial_{x_1}\partial_{x_2} f\|_{L^2(\mathbb{R}^2)}$. In application, like partial differential equation, this mixed derivative norm would make the calculation much more complicated or even infeasible to estimate partial differential equations with some anisotropy property, like Vlasov-Poisson equation [4]. So separating variables becomes necessary and meaningful.

In this paper, we aim to prove an equivalent characterization for Triebel-Lizorkin space, which extends this equivalent relation to fractional differential function spaces whose proof is far from obvious as in $H^2$.

## 2. Preliminaries

We first recall the definitions of Triebel-Lizorkin spaces [6, 7]. Given $f \in \mathscr{S}$ which is the Schwartz function, its Fourier transform $\mathscr{F} f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx,$$

and its inverse Fourier transform is defined by $\mathscr{F}^{-1} f(x) = \hat{f}(-x)$.

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We consider $\varphi \in \mathscr{S}$ satisfying

$$supp\ \varphi \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\},$$

and

$$\varphi(\xi) > 0, \qquad \text{if } \frac{1}{2} < |\xi| < 2.$$

Setting $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ with $j = \{1, 2, \cdots, \}$, we can adjust the normalization constant in front of $\varphi$ and choose $\varphi_0 \in \mathscr{S}$ satisfying

$$supp\ \varphi_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\},$$

such that

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1, \qquad \forall\ \xi \in \mathbb{R}^n.$$

We observe

$$supp\ \varphi_j \cap supp\ \varphi_{j'} = \emptyset \quad \text{if} \quad |j - j^{'}| \geq 2.$$

Given $f \in \mathscr{S}^{'}$, we denote $\Delta_j f = \mathscr{F}^{-1}\varphi_j \mathscr{F} f$. For $(s, p, r) \in \mathbb{R} \times [1, \infty) \times [1, \infty]$, we define the inhomogeneous Triebel-Lizorkin space by

$$F^s_{p,r} = \{f \in \mathscr{S}^{'} : \|f\|_{F^s_{p,r}} = \|(\sum_{j=0}^{\infty} 2^{jsr} |\Delta_j f|^r)^{1/r}\|_{L^p} \leq \infty\},$$

with the usual interpretation for $r = \infty$.

In what follows, we prepare some lemmas for later use. Firstly we recall a vector-valued Fourier multiplier theorem in $L^p(l^r)$, see [2] (Theorem 8.12 page 291).

**Lemma 1.** *([2]) Let $(p, r) \in (0, \infty) \times (0, \infty]$, $\Omega = \{\Omega_j\}_0^\infty$ which is compact set sequence in $\mathbb{R}^n$, and $d_l > 0, N > n/2 + n/\min\{p, r\}$. For any*

$$f = \{f_j\}_0^\infty \in L^p(l^r)^\Omega, m = \{m_j\}_0^\infty \subset H_2^N,$$

*we have*

$$\|\mathscr{F}^{-1} m_j \mathscr{F} f_j\|_{L^p(l^r)} \leq C \sup_l \|m_l(d_l \cdot)\|_{H_2^N} \|\{f_j\}\|_{L^p(l^r)},$$

*for some constant $C$, where $d_l$ is the diameter of $\Omega_l$ and*

$$L^p(l^r)^\Omega = \{f = \{f_j\} \subset \mathscr{S} : supp\ \mathscr{F} f_j \subset \Omega_j, \|\{f_j\}\|_{L^p(l^r)} < \infty\}.$$

The following unit decomposition is useful in the proof of the main theorem in Section 3. For its proof, see [1] (Lemma 6.2.6 on page 145).

**Lemma 2.** *([1]) Assume that $n \geq 2$, and take $\varphi$ as in the definition of Triebel-Lizorkin space. Then there exist functions $\chi_j \in \mathscr{S}(\mathbb{R}^n) (j = 1, \cdots, n)$, such that*

$$\sum_{j=1}^{n} \hat{\chi}_j = 1 \quad \text{on } supp\ \varphi = \{\xi : 1/2 \leq |\xi| \leq 2\},$$

*and*

$$supp\ \hat{\chi}_j \subset \{\xi \in \mathbb{R}^n : |\xi_j| \geq (3\sqrt{n})^{-1}\} \qquad (j = 1, \cdots, n).$$

We also need the following complex interpolation result [5] (see page 186 (e) in Remark 2).

**Lemma 3.** *([5]) Let* $-\infty < s_0, s_1 < \infty, 1 < p_0, p_1 < \infty, 0 < \theta < 1$, *and* $s = (1-\theta)s_0 + \theta s_1$. *We have*

$$[H^{s_0}_{p_0}, B^{s_1}_{p_1,p_1}]_\theta = F^s_{p,q},$$

*where*

$$\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{p_1}, \quad \|a\|_{[A_0,A_1]_\theta} = \inf_{h(\theta)=a} = \|h(z)\|_F$$

*with the infimum is taken over all* $h \in F(A_0, A_1)$ *with* $h(\theta) = a$, *and*

$$F(A_0, A_1, \gamma) = \{h(z) | h(z) \text{ is } (A_0 + A_1) \text{ continuous in } \bar{S} \text{ and}$$

$$(A_0 + A_1) - \text{analytic in } S;$$

$$\sup_{z\in\bar{S}} e^{-|\gamma|\cdot|Imz|} \|h(z)\|_{A_0+A_1} < \infty;$$

$$h(j + it) \in A_j, j = 0, 1, -\infty < t < \infty;$$

$$h(j + it) \text{ is } A_j - \text{continuous with respect to } t, j = 0, 1;$$

$$\|h\|_{F(\gamma)} = \max_{j=0,1} (\sup_t e^{-\gamma|t|} \|h(j + it)\|_{A_j}) < \infty\}.$$

The $S$ denotes the strip $S = \{z | 0 < Rez < 1\}$ in the complex plane and $\bar{S}$ is its closure, $\{A_0, A_1\}$ be an interpolation couple and $\gamma$ be a real number (we take $\gamma = 0$ i.e. $F(A_0, A_1) = F(A_0, A_1, 0)$).

## 3. Equivalent Characterization

In this section, we will state and prove the main theorem. To achieve this, we establish an identity decomposition which plays a fundamentally important role in one direction

$$\bigcap_{j=1}^{n} F^s_{p,q,x_j} \subset F^s_{p,q}.$$

Also, complex interpolation is exploited to prove the other direction

$$F^s_{p,q} \subset \bigcap_{j=1}^{n} F^s_{p,q,x_j}.$$

The main result in this paper is stated as below.

**Theorem 1.** *Suppose* $1 < p < \infty, 1 \le q \le \infty, 0 < \theta < 1, s = (1-\theta)s_0 + \theta s_1$. *We have*

$$F^s_{p,q} = \bigcap_{j=1}^{n} F^s_{p,q,x_j}, \tag{3.1}$$

*where*

$$\|f\|_{F^s_{p,q,x_j}} = \|(\sum_{k=0}^{\infty} 2^{ksq}|\mathscr{F}^{-1}\varphi_k^j \mathscr{F} f|^q)^{1/q}\|_{L^p},$$

*and* $\varphi_k^j$ *is the dyadic block of the unit decomposition for the* $j$ *th variable as in the definition of the Triebel-Lizorkin space.*

*Proof.* We split the proof of the main theorem into two steps. Now we start with step 1.

**Step I.** We prove

$$\bigcap_{j=1}^{n} F^s_{p,q,x_j} \subset F^s_{p,q}. \tag{3.2}$$

For $n = 1$, it is trivial.
For $n \geq 2$, we need the following key claim.

**Claim:** There exists a positive integer $m$ depending on $n$ only such that

$$\sum_{|l-k|\leq m} \varphi_k \hat{\chi}_{j,k} \varphi_l^j = \varphi_k \hat{\chi}_{j,k},$$

where

$$\varphi_k(\xi) = \varphi(2^{-k}\xi), \quad \hat{\chi}_{j,k}(\xi) = \hat{\chi}_j(2^{-k}\xi),$$

and

$$\varphi_l^j(\xi) = \varphi(2^{-l}\xi_j)$$

which is the dyadic block for $j$ *th* variable, and $\varphi_k$ is the usual $k$ *th* dyadic block as in the definition of the Triebel-Lizorkin spaces.

**Proof of the Claim.** By Lemma 2, we have

$$\varphi_k = \sum_{j=1}^{n} \varphi_k \hat{\chi}_{j,k}.$$

Note

$$\sum_{l\in\mathbb{Z}} \varphi_k \hat{\chi}_{j,k} \varphi_l^j = \varphi_k \hat{\chi}_{j,k}.$$

In order to get $\varphi_k \hat{\chi}_{j,k} \varphi_l^j \neq 0$, for any chosen $j$ and $k$, we must have

$$\begin{cases} 2^{k-1} \leq |\xi| \leq 2^{k+1}, \\ 2^{l-1} \leq |\xi_j| \leq 2^{l+1}, \\ |\xi_j| \geq 2^k (3\sqrt{n})^{-1}, \end{cases}$$

which implies that

$$|l-k| \leq m, \qquad \text{with } m = [\log_2 3\sqrt{n}] + 1.$$

Thus, we end the proof of the claim.

Now let us turn back to prove the main theorem. Young's inequality and vector valued multiplier Lemma 1 yields

$$\begin{aligned}
\|f\|_{F^s_{p,q}} &= \|2^{ks}\mathscr{F}^{-1}\varphi_k\mathscr{F}f\|_{L^p(l^q)} \\
&= \|2^{ks}\sum_{j=1}^{n}\mathscr{F}^{-1}\varphi_k\hat{\chi}_{j,k}\mathscr{F}f\|_{L^p(l^q)} \\
&\lesssim \sum_{j=1}^{n}\|2^{ks}\mathscr{F}^{-1}\varphi_k\hat{\chi}_{j,k}\mathscr{F}f\|_{L^p(l^q)} \\
&\lesssim \sum_{j=1}^{n}\|2^{ks}\sum_{|l-k|\le m}\mathscr{F}^{-1}\varphi_k\hat{\chi}_{j,k}\varphi_l^j\mathscr{F}f\|_{L^p(l^q)} \\
&\lesssim \sum_{j=1}^{n}\sup_k\|(\varphi_k\hat{\chi}_{j,k})(2^k\cdot)\|_{H^N}\|\sum_{k\in\mathbb{Z}}\sum_{|l-k|\le m}2^{ls}\mathscr{F}^{-1}\varphi_l^j\mathscr{F}f\|_{L^p(l^q)} \\
&\lesssim \sum_{j=1}^{n}\|f\|_{F^s_{p,q,x_j}},
\end{aligned}$$

where we applied the claim in the fourth line above. Thus, (3.2) holds.

**Step II.** We prove

$$F^s_{p,q} \subset \bigcap_{j=1}^{n} F^s_{p,q,x_j}. \tag{3.3}$$

Assume $f \in F^s_{p,q}$, by the complex interpolation Lemma 3, there exists

$$h(z) \in F(H^{s_0}_{p_0}, B^{s_1}_{p_1,p_1})$$

such that

$$h(\theta) = f_0 + f_1 \in H^{s_0}_{p_0} + B^{s_1}_{p_1,p_1} =: A_0 + A_1,$$

and

$$\|h\|_F = \max_{j=0,1}\big(\sup_t\|h(j+it)\|_{A_j}\big) < \infty.$$

Note that

$$H^s_p = \bigcap_{j=1}^{n} H^s_{p,x_j},$$

and

$$B^s_{p,q} = \bigcap_{j=1}^{n} B^s_{p,q,x_j},$$

see [3] (Theorem 4 and Theorem 5) for more details.
Thus, we have

$$\max_{j=0,1}\big(\sup_t\|h(j+it)\|_{\tilde{A}_j}\big) \le \max_{j=0,1}\big(\sup_t\|h(j+it)\|_{A_j}\big) < \infty, \tag{3.4}$$

where $\tilde{A}_0 := H^s_{p,x_j}, \tilde{A}_1 := B^s_{p,q,x_j}$. Taking the infimum on both sides of (3.4) with respect to $h(z)$ yields (3.3), which completes the proof of our main theorem. □

**Remark 3.1:** The methods could be adapted to the weighted Triebel-Lizorkin spaces, or even in anisotropic function space.

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