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NEW HERMITE–HADAMARD TYPE INEQUALITIES VIA CONFORMABLE FRACTIONAL INTEGRALS CONCERNING DIFFERENTIABLE RELATIVE SEMI–$(r; m, h_1, h_2)$–CONVEX MAPPINGS AND THEIR APPLICATIONS

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Abstract. In this article, we first presented a new identity via conformable fractional integrals. By applied the concept of relative semi–$(r; m, h_1, h_2)$–convexity and the obtained identity as an auxiliary result, some new estimates with respect to the left hand side of the Hermite–Hadamard type inequalities via conformable fractional integrals are given. Also, some applications to special means and error estimates for the midpoint formula are provided as well.

1. Introduction and Preliminaries

The fractional calculus attracted many researches in the last and present centuries. The impact of this fractional calculus in both pure and applied branches of science and engineering started to increase substantially during the last two decades apparently. Many researches started to deal with the discrete versions of this fractional calculus benefitting from the theory of time scales and the references therein. The main idea behind setting this fractional calculus is summarized into two approaches. The first approach is Riemann–Liouville which based on iterating the integral operator n times and then replaced it by one integral via the famous cauchy formula where then $n!$ is changed to the Gamma function and hence the fractional integral of non integer is defined. Then integrals were used to define Riemann and Caputo fractional derivatives. The second approach is the Grünwald-letnikov approach which based on iterating the derivative n times and then fractionalizing by using the Gamma function in the binomial coefficients. The obtained fractional derivatives in this calculus seemed complicated and lost some of the basic properties that usual derivatives have such as the product rule and the chain rule. However, the semigroup properties of these fractional operators behave well in some cases. Recently the author in define a new well-behaved simple fractional derivative called the conformable fractional derivative depending just

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on the basic limit definition of the derivative. For other recent results interested readers are referred to [1]-[6],[15]-[17] and references therein.

Definition 1.1. [18] Given a function $\phi : [0, +\infty) \to \mathbb{R}$. Then the *conformable fractional derivative* of ϕ of order α is defined by

$$\mathtt{D}_\alpha(\phi)(\gamma) = \lim_{\epsilon \to 0^+} \frac{\phi(\gamma + \epsilon\gamma^{1-\alpha}) - \phi(\gamma)}{\epsilon}, \tag{1.1}$$

for all $\gamma > 0$, $\alpha \in (0, 1)$. If the conformable fractional derivative of ϕ of order α exists, then we say that ϕ is α-differentiable. We will, sometimes, write $\phi^\alpha(\gamma)$ and $\frac{d_\alpha}{d_\alpha\gamma}(\phi)$ for $\mathtt{D}_\alpha(\phi)(\gamma)$, to denote the conformable fractional derivatives of ϕ of order α. Let ϕ be α-differentiable in $(0, a)$, and $\lim_{\gamma\to 0^+} \phi^\alpha(\gamma)$ exists, then define

$$\phi^\alpha(0) = \lim_{\gamma \to 0^+} \phi^\alpha(\gamma). \tag{1.2}$$

Theorem 1. [18] *Let $\alpha \in (0, 1]$ and ϕ, ψ be α–differentiable at a point $\gamma > 0$. Then*

i. $\frac{d_\alpha}{d_\alpha\gamma}(\gamma^n) = n\gamma^{n-\alpha}$, *for all $n \in \mathbb{R}$.*
ii. $\frac{d_\alpha}{d_\alpha\gamma}(a_2) = 0$, *for all constant functions $\phi(\gamma) = a_2$.*
iii. $\frac{d_\alpha}{d_\alpha\gamma}(a_1\phi(\gamma) + a_2\psi(\gamma)) = a_1\frac{d_\alpha}{d_\alpha\gamma}(\phi(\gamma)) + a_2\frac{d_\alpha}{d_\alpha\gamma}(\psi(\gamma))$, *for all $a_1, a_2 \in \mathbb{R}$.*
iv. $\frac{d_\alpha}{d_\alpha\gamma}(\phi(\gamma)\psi(\gamma)) = \phi(\gamma)\frac{d_\alpha}{d_\alpha\gamma}(\psi(\gamma)) + \psi(\gamma)\frac{d_\alpha}{d_\alpha\gamma}(\phi(\gamma))$.
v. $\frac{d_\alpha}{d_\alpha\gamma}\left(\frac{\phi(\gamma)}{\psi(\gamma)}\right) = \frac{\psi(\gamma)\frac{d_\alpha}{d_\alpha\gamma}(\phi(\gamma)) - \phi(\gamma)\frac{d_\alpha}{d_\alpha\gamma}(\psi(\gamma))}{(\psi(\gamma))^2}$.
vi. $\frac{d_\alpha}{d_\alpha\gamma}((\phi\circ\psi)(\gamma)) = \phi'(\psi(\gamma))\frac{d_\alpha}{d_\alpha\gamma}(\psi(\gamma))$, *for ϕ differentiable at $\psi(\gamma)$.*

If, in addition, the function ϕ is differentiable, then

$$\frac{d_\alpha}{d_\alpha\gamma}(\phi(\gamma)) = \gamma^{1-\alpha}\frac{d}{d\gamma}(\phi(\gamma)). \tag{1.3}$$

Also, it is important to note the following:

(1) $\frac{d_\alpha}{d_\alpha\gamma}(1) = 0$.
(2) $\frac{d_\alpha}{d_\alpha\gamma}(e^{n\gamma}) = n\gamma^{1-\alpha}e^{n\gamma}, n \in \mathbb{R}$.
(3) $\frac{d_\alpha}{d_\alpha\gamma}(\sin(n\gamma)) = n\gamma^{1-\alpha}\cos(n\gamma), n \in \mathbb{R}$.
(4) $\frac{d_\alpha}{d_\alpha\gamma}(\cos(n\gamma)) = -n\gamma^{1-\alpha}\sin(n\gamma), n \in \mathbb{R}$.
(5) $\frac{d_\alpha}{d_\alpha\gamma}(\frac{1}{\alpha}\gamma^\alpha) = 1$.
(6) $\frac{d_\alpha}{d_\alpha\gamma}\left(\sin\left(\frac{\gamma^\alpha}{\alpha}\right)\right) = \cos\left(\frac{\gamma^\alpha}{\alpha}\right)$.
(7) $\frac{d_\alpha}{d_\alpha\gamma}\left(\cos\left(\frac{\gamma^\alpha}{\alpha}\right)\right) = -\sin\left(\frac{\gamma^\alpha}{\alpha}\right)$.
(8) $\frac{d_\alpha}{d_\alpha\gamma}\left(e^{\left(\frac{\gamma^\alpha}{\alpha}\right)}\right) = e^{\left(\frac{\gamma^\alpha}{\alpha}\right)}$.

Definition 1.2. [4] (**Conformable fractional integral**). Let $\alpha \in (0, 1)$ and $0 \le a_1 < a_2$. A function $\phi : [a_1, a_2] \to \mathbb{R}$ is *α–fractional integrable on* $[a_1, a_2]$, if the integral

$$\int_{a_1}^{a_2} \phi(\gamma)d_\alpha\gamma := \int_{a_1}^{a_2} \phi(\gamma)\gamma^{\alpha-1}d\gamma \tag{1.4}$$

exists and is finite. All α–fractional integrable functions on $[a_1, a_2]$ is indicated by $\mathtt{L}^1_\alpha([a_1, a_2])$.

Definition 1.3. A function $\phi : \mathtt{I} \to \mathbb{R}$, $\mathtt{I} \subseteq \mathbb{R}$, is said to be *convex on* $\mathtt{I}$, if the inequality

$$\phi(\gamma a_1 + (1-\gamma)a_2) \leq \gamma\phi(a_1) + (1-\gamma)\phi(a_2) \tag{1.5}$$

holds for all $a_1, a_2 \in \mathtt{I}$ and $\gamma \in [0, 1]$. Also, we say that ϕ is *concave*, if the inequality (1.5) is reversed. It is well–known that one of the most fundamental and interesting inequalities for classical convex functions is that associated with the name of Hermite-Hadamard inequality: If $\phi : \mathtt{I} \to \mathbb{R}$ is a convex function on the interval $\mathtt{I}$, then for any $a_1, a_2 \in \mathtt{I}$ with $a_1 \neq a_2$, we have the following double inequality:

$$\phi\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \phi(\gamma)d\gamma \leq \frac{\phi(a_1) + \phi(a_2)}{2}. \tag{1.6}$$

The above inequality (1.6) was firstly discovered by Hermite in 1881, (see Mitrinović and Lacković [25]) But, this beautiful results (see Pečarić et al. [31]). For more recent results which generalize, improve, and extend this classical Hermite–Hadamard inequality, one can see [3, 9, 30, 29] and references therein. Meanwhile, fractional integrals and derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. It draws a great application in nonlinear oscillation of earth quakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model, for more recent development on fractional calculus one can see monographs [7, 10, 19, 21]. The convexity of function and its generalized form play an important role in many fields such as Economic science, Biology, optimization. In recent years, several extensions, refinements, and generalizations have been considered for classical convexity [8, 11, 12, 22, 24].

In [13] Dragomir et al. proved the following results connected with the right hand part of Hermite–Hadamard inequality.

Lemma 1. [13] *Let $\phi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° (the interior of I) and $a_1, a_2 \in I^\circ$ with $a_1 < a_2$. If $\phi' \in L([a_1, a_2])$, then the following identity holds:*

$$\frac{\phi(a_1) + \phi(a_2)}{2} - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \phi(s)ds = \frac{a_2 - a_1}{2}\int_0^1 (1 - 2\gamma)\phi'(\gamma a_1 + (1-\gamma)a_2)d\gamma. \tag{1.7}$$

Theorem 2. [13] *Let $\phi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a_1, a_2 \in I^\circ$ with $a_1 < a_2$. If $\phi' \in L([a_1, a_2])$ and $|\phi'|$ is convex on $[a_1, a_2]$, then we have the following inequality:*

$$\left|\frac{\phi(a_1) + \phi(a_2)}{2} - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \phi(s)ds\right| \leq \frac{(a_2 - a_1)(|\phi'(a_1)| + |\phi'(a_2)|)}{8}. \tag{1.8}$$

In [20], Kirmaci proved the following results:

Lemma 2. [20] *Let $\phi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a_1, a_2 \in I^\circ$ with $a_1 < a_2$. If $\phi' \in L([a_1, a_2])$, then the following equality holds:*

$$\frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \phi(s)ds - \phi\left(\frac{a_1 + a_2}{2}\right) \tag{1.9}$$
$$= (a_2 - a_1)\left[\int_0^{\frac{1}{2}} \gamma\phi'(\gamma a_1 + (1-\gamma)a_2)d\gamma + \int_{\frac{1}{2}}^1 (\gamma - 1)\phi'(\gamma a_1 + (1-\gamma)a_2)d\gamma\right].$$

Theorem 3. [20] *Let $\phi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a_1, a_2 \in I^\circ$ with $a_1 < a_2$. If $\phi' \in L([a_1, a_2])$ and $|\phi'|$ is convex on $[a_1, a_2]$, then we have the following inequality:*

$$\left|\frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \phi(s)ds - \phi\left(\frac{a_1 + a_2}{2}\right)\right| \le \frac{(a_2 - a_1)(|\phi'(a_1)| + |\phi'(a_2)|)}{8}. \tag{1.10}$$

Very recently, Anderson [4] investigated the following conformable integral version of Hermite–Hadamard inequality:

Theorem 4. [4] *Let $\alpha \in (0, 1]$ and $\phi : [a_1, a_2] \to \mathbb{R}$ be an α–differentiable function with $0 < a_1 < a_2$, such that $D_\alpha(\phi)$ is increasing, then we have the following inequality*

$$\frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} \phi(s)d_\alpha s \le \frac{\phi(a_1) + \phi(a_2)}{2}. \tag{1.11}$$

Moreover, if the function ϕ is decreasing on $[a_1, a_2]$, then we have

$$\phi\left(\frac{a_1 + a_2}{2}\right) \le \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} \phi(s)d_\alpha s. \tag{1.12}$$

Remark 1.1: It is obvious that, if we choose $\alpha = 1$, then the inequalities (1.11) and (1.12) reduce to the inequality (1.6).

Let us recall some special functions and evoke some basic definitions as follows:

Definition 1.4. [36] Let $S \subseteq \mathbb{R}^n$ be an open set. A function $\phi : S \longrightarrow [0, +\infty)$ is said to be *s–convex* (or *s–Breckner convex*) with $s \in (0, 1]$, if for every $x, y \in S$ and $t \in [0, 1]$,

$$\phi((1-t)x + ty) \le (1-t)^s\phi(x) + t^s\phi(y). \tag{1.13}$$

Definition 1.5. [27] A function $\phi : K \longrightarrow \mathbb{R}$ is said to be *s–Godunova–Levin–Dragomir convex of second kind*, if

$$\phi((1-t)x + ty) \le (1-t)^{-s}\phi(x) + t^{-s}\phi(y), \tag{1.14}$$

for each $x, y \in K, t \in (0, 1)$ and $s \in (0, 1]$.

Definition 1.6. [35] A non-negative function $\phi : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a *tgs–convex on K*, if

$$\phi((1-t)x + ty) \le t(1-t)[\phi(x) + \phi(y)] \tag{1.15}$$

holds for all $x, y \in K$ and $t \in (0, 1)$.

Definition 1.7. [23] A function $\phi : \mathtt{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to *MT-convex*, if it is non-negative and $\forall\, x, y \in \mathtt{I}$ and $t \in (0,1)$ satisfies the following inequality

$$\phi(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}\phi(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}\phi(y). \tag{1.16}$$

Definition 1.8. [28] A function: $\phi : \mathtt{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be *m–MT–convex*, if ϕ is positive and for $\forall\, x, y \in \mathtt{I}$, and $t \in (0,1)$, among $m \in [0,1]$, satisfies the following inequality

$$\phi(tx + m(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}\phi(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}}\phi(y). \tag{1.17}$$

Motivated by above results an literatures, the main purpose of the paper is to present an identity for conformable fractional integrals and by applying the concept of relative semi–$(r; m, h_1, h_2)$–convexity and the obtained identity as an auxiliary result, some new estimates with respect to the left hand side of the Hermite–Hadamard type inequalities via conformable fractional integrals will given. Also, some inequalities for certain special means of two positive real numbers are deduced and at the end of the paper we give the error estimations for the midpoint formula.

2. Main Results

Firstly, we introduce a new class called relative semi–$(r; m, h_1, h_2)$–convex mappings as follows:

Definition 2.1. *Let $I \subseteq \mathbb{R}$ and $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ are continuous functions. A mapping $\phi : I \longrightarrow (0,+\infty)$ is said to be relative semi–$(r; m, h_1, h_2)$–convex, if*

$$\phi(\gamma a_1 + (1-\gamma)a_2) \leq M_r(h_1(\gamma), h_2(\gamma); m\phi(a_1), \phi(a_2)) \tag{2.1}$$

holds for all $a_1, a_2 \in I$ and $\gamma \in [0,1]$ and for some fixed $m \in (0,1]$, where

$$M_r(h_1(\gamma), h_2(\gamma); m\phi(a_1), \phi(a_2)) := \begin{cases} [mh_1(\gamma)\phi^r(a_1) + h_2(\gamma)\phi^r(a_2)]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ [m\phi(a_1)]^{h_1(\gamma)}[\phi(a_2)]^{h_2(\gamma)}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $\phi(a_1)$ and $\phi(a_2)$.

Remark 2.1: In Definition 2.1, if we choose $m = r = 1$, this definition reduces to the definition considered by Noor in [26] and Fulga et. al. in [14].

Remark 2.2: For $r = 1$, let us discuss some special cases in Definition 2.1.

(I) Taking $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0,1]$, then we have relative semi–(m, s)–Breckner convex mappings.
(II) Choosing $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get relative semi–(m, s)–Godunova–Levin–Dragomir convex mappings.
(III) Taking $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we obtain relative semi–(m, h)–convex mappings.
(IV) Choosing $h_1(t) = h_2(t) = t(1-t)$, then we have relative semi–(m, tgs)–convex

mappings.

(V) Taking $h_1(t)=\dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get relative semi–m–MT–convex mappings. It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

In order to prove our main results we need the following Lemma.

Lemma 3. *Let $0\le a_1<a_2$, and let $\phi:[a_1,a_2]\longrightarrow\mathbb{R}$ be a differentiable function on (a_1,a_2) for $\alpha\in(0,1]$. If $D_\alpha(\phi)\in L^1_\alpha([a_1,a_2])$, then the following identity holds:*

$$\begin{aligned}
&\phi\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}\phi(s)d_\alpha s\\
=&\ \frac{(a_2-a_1)}{2\left(a_2^\alpha-a_1^\alpha\right)}\Big[\int_0^1\left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^{2\alpha-1}-a_1^\alpha\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^{\alpha-1}\right)\\
\times&\ D_\alpha(\phi)\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)\gamma^{1-\alpha}d_\alpha\gamma\\
+&\ \int_0^1\left(\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^{2\alpha-1}-a_2^\alpha\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^{\alpha-1}\right)\\
\times&\ D_\alpha(\phi)\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)\gamma^{1-\alpha}d_\alpha\gamma\Big].
\end{aligned}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
I_1 =&\ \int_0^1\left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^{2\alpha-1}-a_1^\alpha\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^{\alpha-1}\right)\\
\times&\ D_\alpha(\phi)\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)\gamma^{1-\alpha}d_\alpha\gamma\\
=&\ \int_0^1\left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^{\alpha}-a_1^\alpha\right)\phi'\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)d\gamma\\
=&\ \left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^{\alpha}-a_1^\alpha\right)\frac{\phi\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)}{\frac{a_1-a_2}{2}}\Bigg|_0^1\\
-&\ \int_0^1\alpha\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^{\alpha-1}\left(\frac{a_1-a_2}{2}\right)\frac{\phi\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)}{\frac{a_1-a_2}{2}}d\gamma\\
=&\ \frac{2}{a_2-a_1}\Big[\left(\left(\frac{a_1+a_2}{2}\right)^{\alpha}-a_1^\alpha\right)\phi\left(\frac{a_1+a_2}{2}\right)-\alpha\int_{a_1}^{\frac{a_1+a_2}{2}}\phi(s)d_\alpha s\Big];\\
I_2 =&\ \int_0^1\left(\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^{2\alpha-1}-a_2^\alpha\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^{\alpha-1}\right)\\
\times&\ D_\alpha(\phi)\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)\gamma^{1-\alpha}d_\alpha\gamma\\
=&\ \int_0^1\left(\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^{\alpha}-a_2^\alpha\right)\phi'\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)d\gamma
\end{aligned}$$

$$= \left(\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^{\alpha}-a_2^{\alpha}\right)\frac{\phi\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)}{\frac{a_2-a_1}{2}}\Bigg|_0^1$$

$$- \int_0^1 \alpha\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^{\alpha-1}\left(\frac{a_2-a_1}{2}\right)\frac{\phi\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)}{\frac{a_2-a_1}{2}}d\gamma$$

$$= \frac{2}{a_2-a_1}\Big[\left(a_2^{\alpha}-\left(\frac{a_1+a_2}{2}\right)^{\alpha}\right)\phi\left(\frac{a_1+a_2}{2}\right)-\alpha\int_{\frac{a_1+a_2}{2}}^{a_2}\phi(s)d_{\alpha}s\Big],$$

where we have used the change of variable $s=\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}$ in I_1 and $s=\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}$ in I_2. Then multiplying both sides by $\frac{a_2-a_1}{2\left(a_2^{\alpha}-a_1^{\alpha}\right)}$, we get the desired result. □

Theorem 5. *Let $0\le a_1<a_2$, $0<r\le 1$ and $\phi:[a_1,a_2]\longrightarrow(0,+\infty)$ be a differentiable mapping on (a_1,a_2) for $\alpha\in(0,1]$. Suppose $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ are continuous functions. If $D_{\alpha}(\phi)\in L_{\alpha}^1([a_1,a_2])$ and ϕ' is relative semi–$(r;m,h_1,h_2)$–convex on $[a_1,a_2]$, then we have the following inequality:*

$$\begin{aligned}
&\left|\phi(\frac{a_1+a_2}{2})-\frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2}\phi(s)d_{\alpha}s\right| \\
&\le \frac{a_2-a_1}{2\left(a_2^{\alpha}-a_1^{\alpha}\right)}\Big\{\left[m(\phi'(a_1))^r I^r(h_1(\gamma);\alpha,r)+(\phi'(a_2))^r I^r(h_2(\gamma);\alpha,r)\right]^{\frac{1}{r}} \qquad (2.2)\\
&\quad +\left[m(\phi'(a_1))^r\overline{I}^r(h_1(\gamma);\alpha,r)+(\phi'(a_2))^r\overline{I}^r(h_2(\gamma);\alpha,r)\right]^{\frac{1}{r}}\Big\},
\end{aligned}$$

where

$$\begin{aligned}
A_{\alpha}(\gamma) &:= \left(\frac{(1-\gamma)a_2^{\alpha-1}}{2}+\frac{(1+\gamma)a_1^{\alpha-1}}{2}\right)\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)-a_1^{\alpha},\\
B_{\alpha}(\gamma) &:= a_2^{\alpha}-\left(\frac{(1-\gamma)a_1^{\alpha}}{2}+\frac{(1+\gamma)a_2^{\alpha}}{2}\right),\\
I(h_i(\gamma);\alpha,r) &:= \int_0^1 A_{\alpha}(\gamma)h_i^{\frac{1}{r}}\left(\frac{1+\gamma}{2}\right)d\gamma,\quad \forall\, i=1,2,\\
\overline{I}(h_i(\gamma);\alpha,r) &:= \int_0^1 B_{\alpha}(\gamma)h_i^{\frac{1}{r}}\left(\frac{1-\gamma}{2}\right)d\gamma,\quad \forall\, i=1,2.
\end{aligned}$$

Proof. First of all, we consider Lemma 3 and then using the convexity of $y^{\alpha-1}$ and $-y^{\alpha}$ $(y>0)$ for $\alpha\in(0,1]$. Also, by Minkowski inequality, properties of the modulus and since the mapping ϕ' is relative semi–$(r;m,h_1,h_2)$–convex, we have

$$\begin{aligned}
&\left|\phi\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2}\phi(s)d_{\alpha}s\right|\\
\le\ & \frac{a_2-a_1}{2(a_2^{\alpha}-a_1^{\alpha})}\Big[\int_0^1\left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^{\alpha}-a_1^{\alpha}\right)\left|\phi'\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)\right|d\gamma\\
+\ & \int_0^1\left(a_2^{\alpha}-\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^{\alpha}\right)\left|\phi'\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)\right|d\gamma\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Big[\int_0^1\left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^{\alpha-1}\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)-a_1^\alpha\right)\\
&\times\ \phi'\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)d\gamma\\
&+\ \int_0^1\left(a_2^\alpha-\left(\frac{(1-\gamma)a_1^\alpha}{2}+\frac{(1+\gamma)a_2^\alpha}{2}\right)\right)\phi'\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)d\gamma\Big]\\
&\le \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Big\{\int_0^1\left(\left(\frac{(1-\gamma)a_2^{\alpha-1}}{2}+\frac{(1+\gamma)a_1^{\alpha-1}}{2}\right)\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)-a_1^\alpha\right)\\
&\times\ \left[mh_1\left(\frac{1+\gamma}{2}\right)(\phi'(a_1))^r+h_2\left(\frac{1+\gamma}{2}\right)(\phi'(a_2))^r\right]^{\frac{1}{r}}d\gamma\\
&+\ \int_0^1\left(a_2^\alpha-\left(\frac{(1-\gamma)a_1^\alpha}{2}+\frac{(1+\gamma)a_2^\alpha}{2}\right)\right)\\
&\times\ \left[mh_1\left(\frac{1-\gamma}{2}\right)(\phi'(a_1))^r+h_2\left(\frac{1-\gamma}{2}\right)(\phi'(a_2))^r\right]^{\frac{1}{r}}d\gamma\Big\}\\
&\le \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Big\{\Big[\left(\int_0^1 m^{\frac{1}{r}}\phi'(a_1)A_\alpha(\gamma)h_1^{\frac{1}{r}}\left(\frac{1+\gamma}{2}\right)d\gamma\right)^r\\
&+\ \left(\int_0^1\phi'(a_2)A_\alpha(\gamma)h_2^{\frac{1}{r}}\left(\frac{1+\gamma}{2}\right)d\gamma\right)^r\Big]^{\frac{1}{r}}\\
&+\ \Big[\left(\int_0^1 m^{\frac{1}{r}}\phi'(a_1)B_\alpha(\gamma)h_1^{\frac{1}{r}}\left(\frac{1-\gamma}{2}\right)d\gamma\right)^r+\left(\int_0^1\phi'(a_2)B_\alpha(\gamma)h_2^{\frac{1}{r}}\left(\frac{1-\gamma}{2}\right)d\gamma\right)^r\Big]^{\frac{1}{r}}\Big\}\\
&=\ \frac{a_2-a_1}{2\left(a_2^\alpha-a_1^\alpha\right)}\Big\{\left[m(\phi'(a_1))^rI^r(h_1(\gamma);\alpha,r)+(\phi'(a_2))^rI^r(h_2(\gamma);\alpha,r)\right]^{\frac{1}{r}}\\
&+\ \left[m(\phi'(a_1))^r\overline{I}^r(h_1(\gamma);\alpha,r)+(\phi'(a_2))^r\overline{I}^r(h_2(\gamma);\alpha,r)\right]^{\frac{1}{r}}\Big\}.
\end{aligned}
$$

Hence, we have the result in (2.2). □

Remark 2.3: By putting $\alpha=m=r=1$, $h_1(\gamma)=\gamma$ and $h_2(\gamma)=1-\gamma$, in (2.2), we get inequality (1.10).

Theorem 6. *Let $0\le a_1<a_2$, $0<r\le 1$ and $\phi:[a_1,a_2]\longrightarrow(0,+\infty)$ be a differentiable mapping on (a_1,a_2) for $\alpha\in(0,1]$. Suppose $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ are continuous functions. If $D_\alpha(\phi)\in L_\alpha^1([a_1,a_2])$ and ϕ'^q is relative semi–$(r;m,h_1,h_2)$–convex on $[a_1,a_2]$ for $q>1$, $p^{-1}+q^{-1}=1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\phi(\frac{a_1+a_2}{2})-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}\phi(s)d_\alpha s\right|\\
&\le\frac{a_2-a_1}{2\left(a_2^\alpha-a_1^\alpha\right)}\Big\{\left(A_1(\alpha,p)\right)^{\frac{1}{p}}\left[m(\phi'(a_1))^{rq}I^r(h_1(\gamma);r)+(\phi'(a_2))^{rq}I^r(h_2(\gamma);r)\right]^{\frac{1}{rq}}\\
&\qquad+\left(A_2(\alpha,p)\right)^{\frac{1}{p}}\left[m(\phi'(a_1))^{rq}I^r(h_1(-\gamma);r)+(\phi'(a_2))^{rq}I^r(h_2(-\gamma);r)\right]^{\frac{1}{rq}}\Big\},
\end{aligned}\tag{2.3}
$$

where

$$
\begin{aligned}
A_1(\alpha,p)\ &:=\ \int_0^1\left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^\alpha-a_1^\alpha\right)^p d\gamma\\
&=\ \frac{2}{(a_2-a_1)}\int_{a_1}^{\frac{a_1+a_2}{2}}\left(\gamma^\alpha-a_1^\alpha\right)^p d\gamma,
\end{aligned}
$$

$$\begin{aligned}
\mathtt{A}_2(\alpha,p) &:= \int_0^1\left(a_2^\alpha-\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^\alpha\right)^p d\gamma\\
&= \frac{2}{(a_2-a_1)}\int_{\frac{a_1+a_2}{2}}^{a_2}(a_2^\alpha-\gamma^\alpha)^p\,d\gamma,\\
I(h_i(\gamma);r) &:= \int_0^1 h_i^{\frac{1}{r}}\left(\frac{1+\gamma}{2}\right)d\gamma,\quad \forall\, i=1,2.
\end{aligned}$$

Proof. Using Lemma 3, relative semi–$(r;m,h_1,h_2)$–convexity of ϕ'^q on $[a_1,a_2]$, Hölder's inequality, Minkowski inequality and property of the modulus, it follows that

$$\begin{aligned}
&\left|\phi\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}\phi(s)d_\alpha s\right|\\
\le\ & \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Big[\int_0^1\left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^\alpha-a_1^\alpha\right)\left|\phi'\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)\right|d\gamma\\
+\ & \int_0^1\left(a_2^\alpha-\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^\alpha\right)\left|\phi'\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)\right|d\gamma\Big]\\
\le\ & \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Big\{\left(\int_0^1\left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^\alpha-a_1^\alpha\right)^p d\gamma\right)^{\frac{1}{p}}\\
\times\ & \left(\int_0^1\left(\phi'\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)\right)^q d\gamma\right)^{\frac{1}{q}}\\
+\ & \left(\int_0^1\left(a_2^\alpha-\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^\alpha\right)^p d\gamma\right)^{\frac{1}{p}}\\
\times\ & \left(\int_0^1\left(\phi'\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)\right)^q d\gamma\right)^{\frac{1}{q}}\Big\}\\
\le\ & \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Big\{\left(\int_0^1\left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^\alpha-a_1^\alpha\right)^p d\gamma\right)^{\frac{1}{p}}\\
\times\ & \left(\int_0^1\left[mh_1\left(\frac{1+\gamma}{2}\right)(\phi'(a_1))^{rq}+h_2\left(\frac{1+\gamma}{2}\right)(\phi'(a_2))^{rq}\right]^{\frac{1}{r}}d\gamma\right)^{\frac{1}{q}}\\
+\ & \left(\int_0^1\left(a_2^\alpha-\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^\alpha\right)^p d\gamma\right)^{\frac{1}{p}}\\
\times\ & \left(\int_0^1\left[mh_1\left(\frac{1-\gamma}{2}\right)(\phi'(a_1))^{rq}+h_2\left(\frac{1-\gamma}{2}\right)(\phi'(a_2))^{rq}\right]^{\frac{1}{r}}d\gamma\right)^{\frac{1}{q}}\Big\}\\
\le\ & \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Big\{(\mathtt{A}_1(\alpha,p))^{\frac{1}{p}}\Big[\left(\int_0^1 m^{\frac{1}{r}}(\phi'(a_1))^q h_1^{\frac{1}{r}}\left(\frac{1+\gamma}{2}\right)d\gamma\right)^r\\
+\ & \left(\int_0^1(\phi'(a_2))^q h_2^{\frac{1}{r}}\left(\frac{1+\gamma}{2}\right)d\gamma\right)^r\Big]^{\frac{1}{rq}}\\
+\ & (\mathtt{A}_2(\alpha,p))^{\frac{1}{p}}\Big[\left(\int_0^1 m^{\frac{1}{r}}(\phi'(a_1))^q h_1^{\frac{1}{r}}\left(\frac{1-\gamma}{2}\right)d\gamma\right)^r+\left(\int_0^1(\phi'(a_2))^q h_2^{\frac{1}{r}}\left(\frac{1-\gamma}{2}\right)d\gamma\right)^r\Big]^{\frac{1}{rq}}\Big\}\\
=\ & \frac{a_2-a_1}{2\left(a_2^\alpha-a_1^\alpha\right)}\Big\{(\mathtt{A}_1(\alpha,p))^{\frac{1}{p}}\left[m(\phi'(a_1))^{rq}I^r(h_1(\gamma);r)+(\phi'(a_2))^{rq}I^r(h_2(\gamma);r)\right]^{\frac{1}{rq}}\\
+\ & (\mathtt{A}_2(\alpha,p))^{\frac{1}{p}}\left[m(\phi'(a_1))^{rq}I^r(h_1(-\gamma);r)+(\phi'(a_2))^{rq}I^r(h_2(-\gamma);r)\right]^{\frac{1}{rq}}\Big\}.
\end{aligned}$$

Hence, we have the result in (2.3). □

Corollary 6.1. *By setting $p = q = 2$ in Theorem (6), we obtain the following inequality*

$$\begin{aligned}&\left|\phi(\frac{a_1+a_2}{2}) - \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2}\phi(s)d_\alpha s\right|\\&\leq \frac{a_2-a_1}{2\left(a_2^\alpha - a_1^\alpha\right)}\Big\{\sqrt{A_1(\alpha,p)}[m(\phi'(a_1))^{2r}I^r(h_1(\gamma);r) + (\phi'(a_2))^{2r}I^r(h_2(\gamma);r)]^{\frac{1}{2r}} \quad (2.4)\\&\quad + \sqrt{A_2(\alpha,p)}[m(\phi'(a_1))^{2r}I^r(h_1(-\gamma);r) + (\phi'(a_2))^{2r}I^r(h_2(-\gamma);r)]^{\frac{1}{2r}}\Big\}.\end{aligned}$$

Remark 2.4: By setting $\alpha = m = r = 1$, $h_1(\gamma) = \gamma$ and $h_2(\gamma) = 1 - \gamma$, in (2.3), we get the following inequality

$$\begin{aligned}&\left|\phi\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\phi(s)ds\right|\\&\leq \frac{1}{2}\left(\frac{2}{(a_2-a_1)(p+1)}\right)^{\frac{1}{p}}\left(\frac{a_2-a_1}{2}\right)^{\frac{p+1}{p}}\left[C_1^{\frac{1}{q}} + C_2^{\frac{1}{q}}\right],\end{aligned}$$

where $C_1 := \dfrac{3(\phi'(a_1))^q + (\phi'(a_2))^q}{4}, C_2 := \dfrac{(\phi'(a_1))^q + 3(\phi'(a_2))^q}{4}$.

Theorem 7. *Let $0 \leq a_1 < a_2$, $0 < r \leq 1$ and $\phi : [a_1, a_2] \longrightarrow (0, +\infty)$ be a differentiable mapping on (a_1, a_2) for $\alpha \in (0, 1]$. Suppose $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$ are continuous functions. If $D_\alpha(\phi) \in L_\alpha^1([a_1, a_2])$ and ϕ'^q is relative semi–$(r; m, h_1, h_2)$–convex on $[a_1, a_2]$ for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\phi(\frac{a_1+a_2}{2}) - \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2}\phi(s)d_\alpha s\right|\\&\leq \frac{a_2-a_1}{2\left(a_2^\alpha - a_1^\alpha\right)}\Big\{(A_1(\alpha))^{1-\frac{1}{q}}\left[m(\phi'(a_1))^{rq}I_1^r(h_1(\gamma);\alpha,r) + (\phi'(a_2))^{rq}I_1^r(h_2(\gamma);\alpha,r)\right]^{\frac{1}{rq}} \quad (2.5)\\&\quad + (B_1(\alpha))^{1-\frac{1}{q}}\left[m(\phi'(a_1))^{rq}\overline{I}_1^r(h_1(\gamma);\alpha,r) + (\phi'(a_2))^{rq}\overline{I}_1^r(h_2(\gamma);\alpha,r)\right]^{\frac{1}{rq}}\Big\},\end{aligned}$$

where

$$C_\alpha(\gamma) := \left(\tfrac{(1-\gamma)a_2}{2} + \tfrac{(1+\gamma)a_1}{2}\right)^\alpha - a_1^\alpha,$$

$$D_\alpha(\gamma) := a_2^\alpha - \left(\tfrac{(1-\gamma)a_1}{2} + \tfrac{(1+\gamma)a_2}{2}\right)^\alpha,$$

$$A_1(\alpha) := \int_0^1 C_\alpha(\gamma)d\gamma = \frac{2}{(a_1-a_2)}\left[\frac{a_1^{\alpha+1} - \left(\frac{a_1+a_2}{2}\right)^{\alpha+1}}{\alpha+1} - a_1^\alpha\left(\frac{a_1-a_2}{2}\right)\right],$$

$$B_1(\alpha) := \int_0^1 D_\alpha(\gamma)d\gamma = \frac{2}{(a_2-a_1)}\left[a_2^\alpha\left(\frac{a_2-a_1}{2}\right) - \frac{a_2^{\alpha+1} - \left(\frac{a_1+a_2}{2}\right)^{\alpha+1}}{\alpha+1}\right],$$

$$I_1(h_i(\gamma);\alpha,r) := \int_0^1 C_\alpha(\gamma)h_i^{\frac{1}{r}}\left(\tfrac{1+\gamma}{2}\right)d\gamma, \quad \forall\, i = 1, 2,$$

$$\overline{I}_1(h_i(\gamma);\alpha,r) := \int_0^1 D_\alpha(\gamma)h_i^{\frac{1}{r}}\left(\tfrac{1-\gamma}{2}\right)d\gamma, \quad \forall\, i = 1, 2.$$

Proof. Using Lemma 3, relative semi–$(r;m,h_1,h_2)$–convexity of ϕ'^q on $[a_1,a_2]$, Hölder's inequality, Minkowski inequality and property of the modulus, it follows that

$$\begin{aligned}
&\left|\phi\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}\phi(s)d_\alpha s\right|\\
\le\;& \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Big[\int_0^1\left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^\alpha-a_1^\alpha\right)\left|\phi'\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)\right|d\gamma\\
+\;& \int_0^1\left(a_2^\alpha-\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^\alpha\right)\left|\phi'\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)\right|d\gamma\Big]\\
\le\;& \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Big\{\left(\int_0^1\left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^\alpha-a_1^\alpha\right)d\gamma\right)^{1-\frac{1}{q}}\\
\times\;& \left(\int_0^1\left(\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)^\alpha-a_1^\alpha\right)\left(\phi'\left(\frac{(1-\gamma)a_2}{2}+\frac{(1+\gamma)a_1}{2}\right)\right)^q d\gamma\right)^{\frac{1}{q}}\\
+\;& \left(\int_0^1\left(a_2^\alpha-\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^\alpha\right)d\gamma\right)^{1-\frac{1}{q}}\\
\times\;& \left(\int_0^1\left(a_2^\alpha-\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)^\alpha\right)\left(\phi'\left(\frac{(1-\gamma)a_1}{2}+\frac{(1+\gamma)a_2}{2}\right)\right)^q d\gamma\right)^{\frac{1}{q}}\Big\}\\
\le\;& \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Big\{(A_1(\alpha))^{1-\frac{1}{q}}\\
\times\;& \left(\int_0^1 C_\alpha(\gamma)\Big[mh_1\left(\frac{1+\gamma}{2}\right)(\phi'(a_1))^{rq}+h_2\left(\frac{1+\gamma}{2}\right)(\phi'(a_2))^{rq}\Big]^{\frac{1}{r}}d\gamma\right)^{\frac{1}{q}}\\
+\;& (B_1(\alpha))^{1-\frac{1}{q}}\left(\int_0^1 D_\alpha(\gamma)\Big[mh_1\left(\frac{1-\gamma}{2}\right)(\phi'(a_1))^{rq}+h_2\left(\frac{1-\gamma}{2}\right)(\phi'(a_2))^{rq}\Big]^{\frac{1}{r}}d\gamma\right)^{\frac{1}{q}}\Big\}\\
\le\;& \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Big\{(A_1(\alpha))^{1-\frac{1}{q}}\Big[\left(\int_0^1 m^{\frac{1}{r}}(\phi'(a_1))^q C_\alpha(\gamma)h_1^{\frac{1}{r}}\left(\frac{1+\gamma}{2}\right)d\gamma\right)^r\\
+\;& \left(\int_0^1(\phi'(a_2))^q C_\alpha(\gamma)h_2^{\frac{1}{r}}\left(\frac{1+\gamma}{2}\right)d\gamma\right)^r\Big]^{\frac{1}{rq}}\\
+\;& (B_1(\alpha))^{1-\frac{1}{q}}\Big[\left(\int_0^1 m^{\frac{1}{r}}(\phi'(a_1))^q D_\alpha(\gamma)h_1^{\frac{1}{r}}\left(\frac{1-\gamma}{2}\right)d\gamma\right)^r\\
+\;& \left(\int_0^1(\phi'(a_2))^q D_\alpha(\gamma)h_2^{\frac{1}{r}}\left(\frac{1-\gamma}{2}\right)d\gamma\right)^r\Big]^{\frac{1}{rq}}\Big\}\\
=\;& \frac{a_2-a_1}{2\,(a_2^\alpha-a_1^\alpha)}\Big\{(\mathtt{A}_1(\alpha))^{1-\frac{1}{q}}\left[m(\phi'(a_1))^{rq}I_1^r(h_1(\gamma);\alpha,r)+(\phi'(a_2))^{rq}I_1^r(h_2(\gamma);\alpha,r)\right]^{\frac{1}{rq}}\\
+\;& (\mathtt{B}_1(\alpha))^{1-\frac{1}{q}}\left[m(\phi'(a_1))^{rq}\overline{I}_1^r(h_1(\gamma);\alpha,r)+(\phi'(a_2))^{rq}\overline{I}_1^r(h_2(\gamma);\alpha,r)\right]^{\frac{1}{rq}}\Big\}.
\end{aligned}$$

Hence, we have the result in (2.5). □

Corollary 7.1. *By setting $q=1$ in Theorem (7), we get the following inequality*

$$\left|\phi(\frac{a_1+a_2}{2})-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}\phi(s)d_\alpha s\right|$$

$$\leq \frac{a_2-a_1}{2\left(a_2^\alpha-a_1^\alpha\right)}\Big\{\left[m(\phi'(a_1))^r I_1^r(h_1(\gamma);\alpha,r)+(\phi'(a_2))^r I_1^r(h_2(\gamma);\alpha,r)\right]^{\frac{1}{r}} \quad (2.6)$$
$$+\left[m(\phi'(a_1))^r \overline{I}_1^r(h_1(\gamma);\alpha,r)+(\phi'(a_2))^r \overline{I}_1^r(h_2(\gamma);\alpha,r)\right]^{\frac{1}{r}}\Big\}.$$

Remark 2.5: By setting $\alpha=m=r=1$, $h_1(\gamma)=\gamma$ and $h_2(\gamma)=1-\gamma$, in (2.5), we get the following inequality

$$\left|\phi\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\phi(s)ds\right|$$
$$\leq \frac{1}{2}\left(\frac{a_2-a_1}{4}\right)^{1-\frac{1}{q}}\left\{\left[\mathtt{A}_2(1)(\phi'(a_1))^q+\mathtt{A}_3(1)(\phi'(a_2))^q\right]^{\frac{1}{q}}+\left[\mathtt{B}_2(1)(\phi'(a_1))^q+\mathtt{B}_3(1)(\phi'(a_2))^q\right]^{\frac{1}{q}}\right\},$$

where $\mathtt{A}_2(1):=\frac{1}{2}\left(\frac{4}{(a_2-a_1)^2}\left[\frac{a_1^3-\left(\frac{a_1+a_2}{2}\right)^3}{3}-a_2\left(\frac{a_1^2-\left(\frac{a_1+a_2}{2}\right)^2}{2}\right)\right]-\frac{3a_1}{2}\right)$,

$$\mathtt{B}_2(1):=\frac{1}{2}\left(\frac{a_2}{2}-\frac{4}{(a_2-a_1)^2}\left[a_2\left(\frac{a_2^2-\left(\frac{a_1+a_2}{2}\right)^2}{2}\right)-\frac{a_2^3-\left(\frac{a_1+a_2}{2}\right)^3}{3}\right]\right),$$
$$\mathtt{A}_3(1):=\frac{1}{2}\left(\frac{4}{(a_2-a_1)^2}\left[a_1\left(\frac{a_1^2-\left(\frac{a_1+a_2}{2}\right)^2}{2}\right)-\frac{a_1^3-\left(\frac{a_1+a_2}{2}\right)^3}{3}\right]-\frac{a_1}{2}\right),$$
$$\mathtt{B}_3(1):=\frac{1}{2}\left(\frac{3a_2}{2}-\frac{4}{(a_2-a_1)^2}\left[\frac{a_2^3-\left(\frac{a_1+a_2}{2}\right)^3}{3}-a_1\left(\frac{a_2^2-\left(\frac{a_1+a_2}{2}\right)^2}{2}\right)\right]\right).$$

Remark 2.6: Several important variants of Hermite–Hadamard inequality have been provided in the literature, such as the versions established by Anderson [4], Sarikaya et al. [32] and Set et al. [33] etc. Recently, Set et al. [34] presented some Hermite–Hadamard type inequalities for conformable fractional integrals. They have obtained results for the version of Hermite–Hadamard type inequalities given in [33], while our results are devoted to the version obtained by Anderson [4].

3. Applications

Several special cases of Hermite–Hadamard inequality for conformable fractional integrals can be deduced from our Theorems 5, 6 and 7, if we choose appropriate h_1, h_2 continuous functions such that ϕ'^q should be relative semi–$(r;m,h_1,h_2)$–convex for $q\geq 1$.

We begin, this section by considering some particular means for arbitrary positive real numbers a_1, a_2 such that $a_1\neq a_2$. So, for this purpose we recall the following well–known definitions in the literature:

(1) `The arithmetic mean`:

$$\mathtt{A}=\mathtt{A}(a_1,a_2):=\frac{a_1+a_2}{2}.$$

(2) `The generalized logarithmic` (α, r)`-th mean`:

$$\mathtt{L}_{(\alpha,r)} = \mathtt{L}_{(\alpha,r)}(a_1, a_2) = \left[\frac{\alpha(a_2^{r+\alpha} - a_1^{r+\alpha})}{(a_2^\alpha - a_1^\alpha)(r+\alpha)}\right]^{\frac{1}{r}}, \quad r \neq 0, -\alpha;\ \alpha \in (0,1],\ r \in \mathbb{R}.$$

Now, by making use of the results obtained in Section 2, we give some applications to special means as follows:

Proposition 3.1. *Let $0 \le a_1 < a_2$ and $\alpha \in (0,1]$. Then for $r_1 > 1$, the following inequality holds:*

$$|A^{r_1}(a_1, a_2) - L^{r_1}_{(\alpha,r_1)}(a_1, a_2)| \le \frac{r_1(a_2 - a_1)}{2\,(a_2^\alpha - a_1^\alpha)}[A(\alpha)a_1^{r_1-1} + B(\alpha)a_2^{r_1-1}],$$

where

$$A(\alpha) := \frac{13a_2^\alpha - 35a_1^\alpha + 11\left(a_1 a_2^{\alpha-1} + a_2 a_1^{\alpha-1}\right)}{96},$$
$$B(\alpha) := \frac{19a_2^\alpha - 29a_1^\alpha + 5\left(a_1 a_2^{\alpha-1} + a_2 a_1^{\alpha-1}\right)}{96}.$$

Proof. The result follows from Theorem 5, if we take $m = r = 1$, $r_1 > 1, h_1(\gamma) = \gamma$ and $h_2(\gamma) = 1 - \gamma$ for $\phi(x) = x^{r_1}$, $x > 0$. □

Proposition 3.2. *Let $0 \le a_1 < a_2$ and $\alpha \in (0,1]$. Then for $r_1 > 1$, $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds:*

$$\begin{aligned}
&|A^{r_1}(a_1, a_2) - L^{r_1}_{(\alpha,r_1)}(a_1, a_2)| \\
&\le \frac{r_1(a_2-a_1)}{2^{\frac{q+2}{q}}\left(a_2^\alpha - a_1^\alpha\right)}\Big[\left(A_1(\alpha,p)\right)^{\frac{1}{p}}\left(3a_1^{(r_1-1)q} + a_2^{(r_1-1)q}\right)^{\frac{1}{q}} + \\
&+ \left(A_2(\alpha,p)\right)^{\frac{1}{p}}\left(a_1^{(r_1-1)q} + 3a_2^{(r_1-1)q}\right)^{\frac{1}{q}}\Big],
\end{aligned}$$

where $A_1(\alpha, p)$ and $A_2(\alpha, p)$ are defined as in Theorem 6.

Proof. One can obtain the result from Theorem 6, if we choose $m = r = 1$, $r_1 > 1, h_1(\gamma) = \gamma$ and $h_2(\gamma) = 1 - \gamma$ for $\phi(x) = x^{r_1}, x > 0$. □

Proposition 3.3. *Let $0 \le a_1 < a_2$ and $\alpha \in (0,1]$. Then for $r_1 > 1$ and $q \ge 1$, the following inequality holds:*

$$\begin{aligned}
&|A^{r_1}(a_1, a_2) - L^{r_1}_{(\alpha,r_1)}(a_1, a_2)| \\
\le\ & \frac{r_1(a_2 - a_1)}{2\,(a_2^\alpha - a_1^\alpha)}\Big\{\left(A_1(\alpha)\right)^{1-\frac{1}{q}}\left[A_2(\alpha)a_1^{(r_1-1)q} + A_3(\alpha)a_2^{(r_1-1)q}\right]^{\frac{1}{q}} \\
+\ & \left(B_1(\alpha)\right)^{1-\frac{1}{q}}\left[B_2(\alpha)a_1^{(r_1-1)q} + B_3(\alpha)a_2^{(r_1-1)q}\right]^{\frac{1}{q}}\Big\},
\end{aligned}$$

where

$$A_1(\alpha) \ := \ \frac{2}{(a_1 - a_2)}\Big[\frac{a_1^{\alpha+1} - \left(\frac{a_1+a_2}{2}\right)^{\alpha+1}}{\alpha+1} - a_1^\alpha\left(\frac{a_1 - a_2}{2}\right)\Big],$$

$$\begin{aligned}
B_1(\alpha) &:= \frac{2}{(a_2-a_1)}\Big[a_2^{\alpha}\left(\frac{a_2-a_1}{2}\right)-\frac{a_2^{\alpha+1}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+1}}{\alpha+1}\Big],\\
A_2(\alpha) &:= \frac{1}{2}\left(\frac{4}{(a_2-a_1)^2}\Big[\frac{a_1^{\alpha+2}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+2}}{\alpha+2}-a_2\left(\frac{a_1^{\alpha+1}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+1}}{\alpha+1}\right)\Big]-\frac{3a_1^{\alpha}}{2}\right),
\end{aligned}$$

$$\begin{aligned}
B_2(\alpha) &:= \frac{1}{2}\left(\frac{a_2^{\alpha}}{2}-\frac{4}{(a_2-a_1)^2}\Big[a_2\left(\frac{a_2^{\alpha+1}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+1}}{\alpha+1}\right)-\frac{a_2^{\alpha+2}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+2}}{\alpha+2}\Big]\right),\\
A_3(\alpha) &:= \frac{1}{2}\left(\frac{4}{(a_2-a_1)^2}\Big[a_1\left(\frac{a_1^{\alpha+1}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+1}}{\alpha+1}\right)-\frac{a_1^{\alpha+2}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+2}}{\alpha+2}\Big]-\frac{a_1^{\alpha}}{2}\right),\\
B_3(\alpha) &:= \frac{1}{2}\left(\frac{3a_2^{\alpha}}{2}-\frac{4}{(a_2-a_1)^2}\Big[\frac{a_2^{\alpha+2}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+2}}{\alpha+2}-a_1\left(\frac{a_2^{\alpha+1}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+1}}{\alpha+1}\right)\Big]\right).
\end{aligned}$$

Proof. One can obtain the result from Theorem 7, if we take $m = r = 1$, $r_1 > 1, h_1(\gamma) = \gamma$ and $h_2(\gamma) = 1-\gamma$ for $\phi(x) = x^{r_1}$, $x > 0$. □

Proposition 3.4. *Let $0 \le a_1 < a_2$ and $\alpha \in (0,1)$. Then for $q \ge 1$, the following inequality holds:*

$$\begin{aligned}
|A^{-1}(a_1,a_2)-L^{-1}_{(\alpha,-1)}(a_1,a_2)| &\le \frac{a_2-a_1}{2\left(a_2^{\alpha}-a_1^{\alpha}\right)}\Big\{\left(A_1(\alpha)\right)^{1-\frac{1}{q}}\left[\frac{A_2(\alpha)}{a_1^{2q}}+\frac{A_3(\alpha)}{a_2^{2q}}\right]^{\frac{1}{q}}\\
&+ \left(B_1(\alpha)\right)^{1-\frac{1}{q}}\left[\frac{B_2(\alpha)}{a_1^{2q}}+\frac{B_3(\alpha)}{a_2^{2q}}\right]^{\frac{1}{q}}\Big\},
\end{aligned}$$

where $A_1(\alpha), B_1(\alpha), A_2(\alpha), B_2(\alpha), A_3(\alpha), B_3(\alpha)$ are defined as in Proposition 3.3.

Proof. The statement of results follows from Theorem 7, if we choose $m = r = 1$, $h_1(\gamma) = \gamma$ and $h_2(\gamma) = 1-\gamma$ for $\phi(x) = \dfrac{1}{x}$, $x > 0$. □

Proposition 3.5. *Let $0 \le a_1 < a_2$ and $\alpha \in (0,1)$. Then for $q > 1$ and $p^{-1}+q^{-1} = 1$, the following inequality holds:*

$$\begin{aligned}
&|A^{-1}(a_1,a_2)-L^{-1}_{(\alpha,-1)}(a_1,a_2)|\\
\le\ &\frac{a_2-a_1}{2^{\frac{q+2}{q}}\left(a_2^{\alpha}-a_1^{\alpha}\right)}\Big[\left(A_1(\alpha,p)\right)^{\frac{1}{p}}\left(\frac{3}{a_1^{2q}}+\frac{1}{a_2^{2q}}\right)^{\frac{1}{q}}+\left(A_2(\alpha,p)\right)^{\frac{1}{p}}\left(\frac{1}{a_1^{2q}}+\frac{3}{a_2^{2q}}\right)^{\frac{1}{q}}\Big],
\end{aligned}$$

where $A_1(\alpha,p)$ and $A_2(\alpha,p)$ are defined as in Theorem 6.

Proof. We can get the inequality from Theorem 6, if we take $m = r = 1$, $h_1(\gamma) = \gamma$ and $h_2(\gamma) = 1-\gamma$ for $\phi(x) = \dfrac{1}{x}$, $x > 0$. □

Let P be the partition of the points $a_1 = x_0 < x_1 < \ldots < x_{n-1} < x_n = a_2$ of the interval $[a_1, a_2]$ and consider the quadrature formula:

$$\int_{a_1}^{a_2} \phi(x) d_\alpha x = \mathtt{T}_\alpha(\phi, \mathtt{P}) + \mathtt{E}_\alpha(\phi, \mathtt{P}), \tag{3.1}$$

where

$$\mathtt{T}_\alpha(\phi, \mathtt{P}) = \sum_{i=0}^{n-1} \phi\left(\frac{x_i + x_{i+1}}{2}\right) \frac{\left(x_{i+1}^\alpha - x_i^\alpha\right)}{\alpha} \tag{3.2}$$

is the midpoint version and $\mathtt{E}_\alpha(\phi, \mathtt{P})$ denotes the associated approximation error. Here, we are going to derive some new error estimates for the midpoint formula.

Proposition 3.6. *Let $0 \leq a_1 < a_2$ and $\phi : [a_1, a_2] \longrightarrow (0, +\infty)$ be a differentiable mapping on (a_1, a_2) for $\alpha \in (0, 1]$. If $D_\alpha(\phi) \in L_\alpha^1([a_1, a_2])$ and ϕ' is relative semi–$(1; 1, \gamma, 1-\gamma)$–convex on $[a_1, a_2]$, then we have the following inequality:*

$$|E_\alpha(\phi, P)| \leq \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)}{2\alpha} [A(\alpha)(\phi'(x_i)) + B(\alpha)(\phi'(x_{i+1}))],$$

where

$$A(\alpha) := \frac{13x_{i+1}^\alpha - 35x_i^\alpha + 11\left(x_i x_{i+1}^{\alpha-1} + x_{i+1} x_i^{\alpha-1}\right)}{96},$$

$$B(\alpha) := \frac{19x_{i+1}^\alpha - 29x_i^\alpha + 5\left(x_i x_{i+1}^{\alpha-1} + x_{i+1} x_i^{\alpha-1}\right)}{96}.$$

Proof. Applying Theorem 5 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, \dots, n-1)$ of the partition $\mathtt{P}$, for $m = r = 1$, $h_1(\gamma) = \gamma$ and $h_2(\gamma) = 1 - \gamma$, we have

$$\begin{aligned} &\left|\phi\left(\frac{x_i + x_{i+1}}{2}\right) \frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha} - \int_{x_i}^{x_{i+1}} \phi(x) d_\alpha x\right| \\ \leq\ & \frac{(x_{i+1} - x_i)}{2\alpha} [A(\alpha)(\phi'(x_i)) + B(\alpha)(\phi'(x_{i+1}))]. \end{aligned}$$

Hence, from above, we obtain

$$\begin{aligned} &\left|\int_{a_1}^{a_2} \phi(x) d_\alpha x - \mathtt{T}_\alpha(\phi, \mathtt{P})\right| \\ =\ & \left|\sum_{i=0}^{n-1} \left\{\int_{x_i}^{x_{i+1}} \phi(x) d_\alpha x - \phi\left(\frac{x_i + x_{i+1}}{2}\right) \frac{\left(x_{i+1}^\alpha - x_i^\alpha\right)}{\alpha}\right\}\right| \\ \leq\ & \sum_{i=0}^{n-1} \left|\left\{\int_{x_i}^{x_{i+1}} \phi(x) d_\alpha x - \phi\left(\frac{x_i + x_{i+1}}{2}\right) \frac{\left(x_{i+1}^\alpha - x_i^\alpha\right)}{\alpha}\right\}\right| \\ \leq\ & \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)}{2\alpha} [A(\alpha)(\phi'(x_i)) + B(\alpha)(\phi'(x_{i+1}))]. \end{aligned}$$

□

Proposition 3.7. *Let $0 \leq a_1 < a_2$ and $\phi : [a_1, a_2] \longrightarrow (0, +\infty)$ be a differentiable mapping on (a_1, a_2) for $\alpha \in (0, 1]$. If $D_\alpha(\phi) \in L^1_\alpha([a_1, a_2])$ and ϕ'^q is relative semi–$(1; 1, \gamma, 1-\gamma)$–convex on $[a_1, a_2]$ with $q > 1$, $p^{-1} + q^{-1} = 1$, then the following inequality holds:*

$$\begin{aligned} |E_\alpha(\phi, P)| &\leq \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)}{2^{\frac{q+2}{q}}\alpha} \Big\{ (A_1(\alpha, p))^{\frac{1}{p}} \left[3(\phi'(x_i))^q + (\phi'(x_{i+1}))^q\right]^{\frac{1}{q}} \\ &+ (A_2(\alpha, p))^{\frac{1}{p}} \left[(\phi'(x_i))^q + 3(\phi'(x_{i+1}))^q\right]^{\frac{1}{q}} \Big\}, \end{aligned}$$

where $A_1(\alpha, p)$ and $A_2(\alpha, p)$ are defined as in Theorem 6.

Proof. The proof is analogous to that of Proposition 3.6 only by using Theorem 6 and taking $a_1 = x_i$ and $a_2 = x_{i+1}$. □

Proposition 3.8. *Let $0 \leq a_1 < a_2$ and $\phi : [a_1, a_2] \longrightarrow (0, +\infty)$ be a differentiable mapping on (a_1, a_2) for $\alpha \in (0, 1]$. If $D_\alpha(\phi) \in L^1_\alpha([a_1, a_2])$ and ϕ'^q is relative semi–$(1; 1, \gamma, 1-\gamma)$–convex on $[a_1, a_2]$ for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} |E_\alpha(\phi, P)| &\leq \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)}{2\alpha} \Big\{ (A_1(\alpha))^{1-\frac{1}{q}} \left[A_2(\alpha)(\phi'(x_i))^q + A_3(\alpha)(\phi'(x_{i+1}))^q\right]^{\frac{1}{q}} \\ &+ (B_1(\alpha))^{1-\frac{1}{q}} \left[B_2(\alpha)(\phi'(x_i))^q + B_3(\alpha)(\phi'(x_{i+1}))^q\right]^{\frac{1}{q}} \Big\}. \end{aligned}$$

where $A_1(\alpha), B_1(\alpha), A_2(\alpha), B_2(\alpha), A_3(\alpha), B_3(\alpha)$ are defined as in Proposition 3.3.

Proof. The proof is analogous to that of Proposition 3.6 only by using Theorem 7 and setting $a_1 = x_i$ and $a_2 = x_{i+1}$. □

4. Conclusion

In the present study, we obtained an integral identity associated with inequality (1.12), and by making use of it, we found some Hermite–Hadamard type inequalities for conformable fractional integrals. As a consequence of our main results, we established some new inequalities for certain bivariate means of positive real numbers, such as arithmetic mean and generalized logarithmic (α, r)-th mean and provided some new error estimations for the midpoint formula. Since the new class of convex functions have large applications in many mathematical areas, they can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences.

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