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NEW INTEGRAL INEQUALITIES PERTAINING CONVEX FUNCTIONS AND THEIR APPLICATIONS

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Abstract. In this paper, first we prove a new generalized midpoint identity. By applying this identity some interesting midpoint type integral inequalities via s–convex functions are given. Some special cases obtained from our main results are discussed in details. Finally, some applications on the Bessel functions, special means of distinct positive real numbers and error estimation about midpoint quadrature formula are presented to support our theoretical results.

1. Introduction and Preliminaries

The theory of convexity present an amazing and fascinating field of research and also played significant role in the development of the theory of inequalities. Many researchers endeavor, attempt and to define and introduced new ideas and concepts about convex functions and extend and generalize its variant forms in different ways using innovative ideas and fruitful techniques. Using the theory of convexity, mathematicians provides an amazing tool, numerical techniques to tackle and to solve a problems in mathematics. In diverse and opponent research, inequalities have a lot of applications in statistical problems, probability and numerical quadrature formulas. Many researchers always try to do and use new ideas for the enjoyment and beautification of convex analysis. Hudzik and Maligranda [6] introduced the class of s–convex functions in second sense. Further in this direction Dragomir and Fitzpatrick [4] put an efforts, established new integral inequalities via s–convex functions. İşcan [11], asserted that some integral inequalities via s–convex functions with the help of well known and remarkable inequalities, improved power–mean integral inequality and Hölder–İscan integral inequality. Muddassar [10] adds some contributions via s–convex functions in this dynamic field. Noor [13] keeping his work on generalizations, introduced and proved new

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versions of Hermite–Hadamard inequality for exponentially s–convex function via the Katugampola fractional integral.

Definition 1.1. A function $\phi : \mathcal{I} \subseteq \Re \to \Re$ is called *convex*, if

$$\phi(\varrho\zeta_1 + (1-\varrho)\zeta_2) \leq \varrho\phi(\zeta_1) + (1-\varrho)\phi(\zeta_2),$$

holds for all $\zeta_1, \zeta_2 \in \mathcal{I}$ ($\mathcal{I}$ is a (real) interval) and $\varrho \in [0,1]$. Likewise, ϕ is *concave* if $(-\phi)$ is convex.

Definition 1.2. [6] Let $s \in (0,1]$ be fixed. A function $\phi : \mathcal{I} \subseteq \Re \to \Re$ is said to be *s–convex* (in the second sense), if

$$\phi(\varrho\zeta_1 + (1-\varrho)\zeta_2) \leq \varrho^s\phi(\zeta_1) + (1-\varrho)^s\phi(\zeta_2),$$

holds for all $\zeta_1, \zeta_2 \in \mathcal{I}$ and $\varrho \in [0,1]$.

The following Hermite–Hadamard type inequality has remained an area of great interest due to its widespread view and applications in the field of mathematical analysis.

Theorem 1. *Let $\phi : \mathcal{I} \subseteq \Re \to \Re$ be a convex function on $\mathcal{I}$ for $\zeta_1, \zeta_2 \in \mathcal{I}$ and $\zeta_1 < \zeta_2$. Then the following double inequality holds:*

$$\phi\left(\frac{\zeta_1+\zeta_2}{2}\right) \leq \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(\varrho)d\varrho \leq \frac{\phi(\zeta_1)+\phi(\zeta_2)}{2}. \tag{1.1}$$

Various generalizations, refinements and improvements have appeared. Interested readers can refer to [1]–[18].

Motivated by the above result and literature, our paper is organized as follows: In Section 2, we will derive an interesting generalized midpoint identity. Using this identity many inequalities using s–convex functions will be establish. Various special cases will be identified from our general results. In Section 3, we give some applications of the Bessel functions, special means and error estimation about midpoint quadrature formula to support the main results.

2. Main Results

We need the following lemma in order to establish our main results.

Lemma 1. *Let $\phi : \mathcal{I} \subseteq \Re \to \Re$ be a differentiable function on $\mathcal{I}^\circ$ (the interior of $\mathcal{I}$) and $\zeta_1, \zeta_2 \in \mathcal{I}^\circ$ with $\zeta_1 < \zeta_2$. If $\phi' \in \mathcal{L}[\zeta_1, \zeta_2]$, then for $n \in \mathbb{N}$ the following equality holds:*

$$\mathcal{S}_n(\phi;\zeta_1,\zeta_2) := \frac{n}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(x)dx - \sum_{k=0}^{n-1}\phi\left(\frac{(2(n-k)-1)\zeta_1+(2k+1)\zeta_2}{2n}\right)$$

$$= \left(\frac{\zeta_2-\zeta_1}{4n}\right) \tag{2.1}$$

$$\times \sum_{k=0}^{n-1}\Big\{\int_0^1 \varrho\phi'\left(\frac{\varrho}{2}\frac{(n-k)\zeta_1+k\zeta_2}{n}+\frac{(2-\varrho)}{2}\frac{(n-k-1)\zeta_1+(k+1)\zeta_2}{n}\right)d\varrho$$
$$-\int_0^1 \varrho\phi'\left(\frac{\varrho}{2}\frac{(n-k-1)\zeta_1+(k+1)\zeta_2}{n}+\frac{(2-\varrho)}{2}\frac{(n-k)\zeta_1+k\zeta_2}{n}\right)d\varrho\Big\}.$$

Proof. Let denote, respectively

$$\mathcal{I}_1 := \int_0^1 \varrho\phi'\left(\frac{\varrho}{2}\frac{(n-k)\zeta_1+k\zeta_2}{n}+\frac{(2-\varrho)}{2}\frac{(n-k-1)\zeta_1+(k+1)\zeta_2}{n}\right)d\varrho \quad (2.2)$$

and

$$\mathcal{I}_2 := \int_0^1 \varrho\phi'\left(\frac{\varrho}{2}\frac{(n-k-1)\zeta_1+(k+1)\zeta_2}{n}+\frac{(2-\varrho)}{2}\frac{(n-k)\zeta_1+k\zeta_2}{n}\right)d\varrho. \quad (2.3)$$

Applying integration by parts on equality (2.2), we have

$$\mathcal{I}_1 = \left(\frac{2n}{\zeta_1-\zeta_2}\right)\Big[\varrho\phi\left(\frac{\varrho}{2}\frac{(n-k)\zeta_1+k\zeta_2}{n}+\frac{(2-\varrho)}{2}\frac{(n-k-1)\zeta_1+(k+1)\zeta_2}{n}\right)\Big|_0^1$$
$$-\int_0^1 \phi\left(\frac{\varrho}{2}\frac{(n-k)\zeta_1+k\zeta_2}{n}+\frac{(2-\varrho)}{2}\frac{(n-k-1)\zeta_1+(k+1)\zeta_2}{n}\right)d\varrho\Big]$$
$$= \left(\frac{2n}{\zeta_1-\zeta_2}\right)\Big[\phi\left(\frac{(2(n-k)-1)\zeta_1+(2k+1)\zeta_2}{2n}\right)-\int_{\frac{(n-k-1)\zeta_1+(k+1)\zeta_2}{n}}^{\frac{(2(n-k)-1)\zeta_1+(2k+1)\zeta_2}{2n}}\phi(x)dx\Big] \quad (2.4)$$

and similarly, from equality (2.3), we obtain

$$\mathcal{I}_2 = \left(\frac{2n}{\zeta_2-\zeta_1}\right)\Big[\phi\left(\frac{(2(n-k)-1)\zeta_1+(2k+1)\zeta_2}{2n}\right)-\int_{\frac{(n-k)\zeta_1+k\zeta_2}{n}}^{\frac{(2(n-k)-1)\zeta_1+(2k+1)\zeta_2}{2n}}\phi(x)dx\Big], \quad (2.5)$$

for all $k = 0, 1, 2, \ldots, n-1$.

Subtracting equality (2.5) from (2.4), multiplying by the factor $\left(\frac{\zeta_2-\zeta_1}{4n}\right)$ and summing over k from 0 to $n-1$, we can easily attain the desired identity (2.1). □

Remark 2.1: Taking $n = 1$ in Lemma 1, we have the following midpoint identity (see [14], Corollary 1):

$$\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(x)dx-\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)=\frac{(\zeta_2-\zeta_1)}{4} \quad (2.6)$$
$$\times\Big\{\int_0^1 \varrho\phi'\left(\frac{\varrho}{2}\zeta_1+\frac{(2-\varrho)}{2}\zeta_2\right)d\varrho-\int_0^1 \varrho\phi'\left(\frac{\varrho}{2}\zeta_2+\frac{(2-\varrho)}{2}\zeta_1\right)d\varrho\Big\}.$$

Throughout the rest of the paper, lets denote:

$$\mathbf{u}_{n,k} := \frac{(n-k)\zeta_1+k\zeta_2}{n} \quad \text{and} \quad \mathbf{u}_{n,k+1} := \frac{(n-k-1)\zeta_1+(k+1)\zeta_2}{n}.$$

Theorem 2. *Let $\phi : \mathcal{I} \subseteq [0, +\infty) \to \Re$ be a differentiable function on $\mathcal{I}^\circ$ and $\zeta_1, \zeta_2 \in \mathcal{I}^\circ$ with $\zeta_1 < \zeta_2$ such that $\phi' \in \mathcal{L}[\zeta_1, \zeta_2]$. Also, let $n \in \mathbb{N}$ and $s \in (0, 1]$ be fixed. If $|\phi'|$ is s–convex function on $[\zeta_1, \zeta_2]$, then the following inequality holds:*

$$|\mathcal{S}_n(\phi; \zeta_1, \zeta_2)| \leq \frac{(\zeta_2 - \zeta_1)}{n} \frac{\left(2^{s+1} - 1\right)}{2^{s+1}(s+1)(s+2)} \sum_{k=0}^{n-1} \left[|\phi'(\boldsymbol{u}_{n,k})| + |\phi'(\boldsymbol{u}_{n,k+1})|\right]. \quad (2.7)$$

Proof. By using Lemma 1, s–convexity of $|\phi'|$ and properties of modulus, we have

$$|\mathcal{S}_n(\phi; \zeta_1, \zeta_2)| \leq \left(\frac{\zeta_2 - \zeta_1}{4n}\right) \times$$

$$\times \sum_{k=0}^{n-1} \Big\{ \int_0^1 \varrho \left|\phi'\left(\frac{\varrho}{2}\mathbf{u}_{n,k} + \frac{(2-\varrho)}{2}\mathbf{u}_{n,k+1}\right)\right| d\varrho +$$

$$+ \int_0^1 \varrho \left|\phi'\left(\frac{(2-\varrho)}{2}\mathbf{u}_{n,k} + \frac{\varrho}{2}\mathbf{u}_{n,k+1}\right)\right| d\varrho \Big\}$$

$$\leq \left(\frac{\zeta_2 - \zeta_1}{4n}\right) \times \sum_{k=0}^{n-1} \Big\{ \int_0^1 \varrho \left[\left(\frac{\varrho}{2}\right)^s |\phi'(\mathbf{u}_{n,k})| + \left(1 - \frac{\varrho}{2}\right)^s |\phi'(\mathbf{u}_{n,k+1})|\right] d\varrho +$$

$$+ \int_0^1 \varrho \left[\left(1 - \frac{\varrho}{2}\right)^s |\phi'(\mathbf{u}_{n,k})| + \left(\frac{\varrho}{2}\right)^s |\phi'(\mathbf{u}_{n,k+1})|\right] d\varrho \Big\}$$

$$= \frac{(\zeta_2 - \zeta_1)}{n} \frac{\left(2^{s+1} - 1\right)}{2^{s+1}(s+1)(s+2)} \sum_{k=0}^{n-1} \left[|\phi'(\mathbf{u}_{n,k})| + |\phi'(\mathbf{u}_{n,k+1})|\right],$$

which completes the proof. □

Corollary 2.1. *Choosing $s = 1$ in Theorem 2, we get the following inequality for convex function:*

$$|\mathcal{S}_n(\phi; \zeta_1, \zeta_2)| \leq \frac{(\zeta_2 - \zeta_1)}{8n} \sum_{k=0}^{n-1} \left[|\phi'(\boldsymbol{u}_{n,k})| + |\phi'(\boldsymbol{u}_{n,k+1})|\right]. \quad (2.8)$$

Corollary 2.2. *Taking $n = 1$ in Theorem 2, we obtain*

$$\left|\frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \phi(x)dx - \phi\left(\frac{\zeta_1 + \zeta_2}{2}\right)\right| \leq$$

$$\leq (\zeta_2 - \zeta_1) \frac{\left(2^{s+1} - 1\right)}{2^{s+1}(s+1)(s+2)} \left[|\phi'(\zeta_1)| + |\phi'(\zeta_2)|\right]. \quad (2.9)$$

Theorem 3. *Let $f : \mathcal{I} \subseteq [0, +\infty) \to \Re$ be a differentiable function on $\mathcal{I}^\circ$ and $\zeta_1, \zeta_2 \in \mathcal{I}^\circ$ with $\zeta_1 < \zeta_2$ such that $\phi' \in \mathcal{L}[\zeta_1, \zeta_2]$. Also, let $n \in \mathbb{N}$ and $s \in (0, 1]$ be fixed. If $|\phi'|^q$ is s–convex function on $[\zeta_1, \zeta_2]$, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality holds:*

$$|\mathcal{S}_n(\phi; \zeta_1, \zeta_2)| \leq \frac{(\zeta_2 - \zeta_1)}{n} \left(\frac{1}{2}\right)^{2+\frac{s}{q}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{s+1}\right)^{\frac{1}{q}} \times \quad (2.10)$$

$$\sum_{k=0}^{n-1}\left\{\left[|\phi'(\boldsymbol{u}_{n,k})|^q+\left(2^{s+1}-1\right)|\phi'(\boldsymbol{u}_{n,k+1})|^q\right]^{\frac{1}{q}}+\right.$$
$$\left.+\left[\left(2^{s+1}-1\right)|\phi'(\boldsymbol{u}_{n,k})|^q+|\phi'(\boldsymbol{u}_{n,k+1})|^q\right]^{\frac{1}{q}}\right\}.$$

Proof. By using Lemma 1, Hölder's inequality, s–convexity of $|\phi'|^q$ and properties of modulus, we have

$$|\mathcal{S}_n(\phi;\zeta_1,\zeta_2)|\leq\left(\frac{\zeta_2-\zeta_1}{4n}\right)\times$$
$$\sum_{k=0}^{n-1}\left\{\int_0^1\varrho\left|\phi'\left(\frac{\varrho}{2}\mathbf{u}_{n,k}+\frac{(2-\varrho)}{2}\mathbf{u}_{n,k+1}\right)\right|d\varrho+\right.$$
$$\left.+\int_0^1\varrho\left|\phi'\left(\frac{(2-\varrho)}{2}\mathbf{u}_{n,k}+\frac{\varrho}{2}\mathbf{u}_{n,k+1}\right)\right|d\varrho\right\}$$
$$\leq\left(\frac{\zeta_2-\zeta_1}{4n}\right)\left(\int_0^1\varrho^p d\varrho\right)^{\frac{1}{p}}\times\sum_{k=0}^{n-1}\left\{\left(\int_0^1\left|\phi'\left(\frac{\varrho}{2}\mathbf{u}_{n,k}+\frac{(2-\varrho)}{2}\mathbf{u}_{n,k+1}\right)\right|^q d\varrho\right)^{\frac{1}{q}}+\right.$$
$$\left.+\left(\int_0^1\left|\phi'\left(\frac{(2-\varrho)}{2}\mathbf{u}_{n,k}+\frac{\varrho}{2}\mathbf{u}_{n,k+1}\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right\}\leq$$
$$\left(\frac{\zeta_2-\zeta_1}{4n}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\times\sum_{k=0}^{n-1}\left\{\left(\int_0^1\left[\left(\frac{\varrho}{2}\right)^s|\phi'(\mathbf{u}_{n,k})|^q+\left(1-\frac{\varrho}{2}\right)^s|\phi'(\mathbf{u}_{n,k+1})|\right]^q d\varrho\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left[\left(1-\frac{\varrho}{2}\right)^s|\phi'(\mathbf{u}_{n,k})|^q+\left(\frac{\varrho}{2}\right)^s|\phi'(\mathbf{u}_{n,k+1})|\right]^q d\varrho\right)^{\frac{1}{q}}\right\}$$
$$=\frac{(\zeta_2-\zeta_1)}{n}\left(\frac{1}{2}\right)^{2+\frac{s}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\times$$
$$\sum_{k=0}^{n-1}\left\{\left[|\phi'(\mathbf{u}_{n,k})|^q+\left(2^{s+1}-1\right)|\phi'(\mathbf{u}_{n,k+1})|^q\right]^{\frac{1}{q}}+.\right.$$
$$\left.+\left[\left(2^{s+1}-1\right)|\phi'(\mathbf{u}_{n,k})|^q+|\phi'(\mathbf{u}_{n,k+1})|^q\right]^{\frac{1}{q}}\right\}$$

This ends our proof. □

Corollary 3.1. *Choosing $s=1$ in Theorem 3, we get the following inequality for convex function:*

$$|\mathcal{S}_n(\phi;\zeta_1,\zeta_2)|\leq\frac{(\zeta_2-\zeta_1)}{n}\left(\frac{1}{4}\right)^{1+\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\tag{2.11}$$
$$\times\sum_{k=0}^{n-1}\left\{[|\phi'(\boldsymbol{u}_{n,k})|^q+3|\phi'(\boldsymbol{u}_{n,k+1})|^q]^{\frac{1}{q}}+[3|\phi'(\boldsymbol{u}_{n,k})|^q+|\phi'(\boldsymbol{u}_{n,k+1})|^q]^{\frac{1}{q}}\right\}.$$

Corollary 3.2. *Taking $n = 1$ in Theorem 3, we obtain*

$$\left|\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(x)dx-\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right| \leq \tag{2.12}$$

$$\leq (\zeta_2-\zeta_1)\left(\frac{1}{2}\right)^{2+\frac{s}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\times$$

$$\times\left\{\left[|\phi'(\zeta_1)|^q+\left(2^{s+1}-1\right)|\phi'(\zeta_2)|^q\right]^{\frac{1}{q}}+\left[\left(2^{s+1}-1\right)|\phi'(\zeta_1)|^q+|\phi'(\zeta_2)|^q\right]^{\frac{1}{q}}\right\}.$$

Theorem 4. *Let $\phi : \mathcal{I} \subseteq [0,+\infty) \to \Re$ be a differentiable function on $\mathcal{I}^\circ$ and $\zeta_1, \zeta_2 \in \mathcal{I}^\circ$ with $\zeta_1 < \zeta_2$ such that $\phi' \in \mathcal{L}[\zeta_1, \zeta_2]$. Also, let $n \in \mathbb{N}$ and $s \in (0,1]$ be fixed. If $|\phi'|^q$ is s–convex function on $[\zeta_1, \zeta_2]$, then for $q \geq 1$, the following inequality holds:*

$$|\mathcal{S}_n(\phi;\zeta_1,\zeta_2)| \leq \frac{(\zeta_2-\zeta_1)}{n}\left(\frac{1}{2}\right)^{3+\frac{s-1}{q}}\times \tag{2.13}$$

$$\sum_{k=0}^{n-1}\left\{\left[\frac{1}{s+2}|\phi'(\boldsymbol{u}_{n,k})|^q+Q(s)|\phi'(\boldsymbol{u}_{n,k+1})|^q\right]^{\frac{1}{q}}+\right.$$

$$\left.+\left[Q(s)|\phi'(\boldsymbol{u}_{n,k})|^q+\frac{1}{s+2}|\phi'(\boldsymbol{u}_{n,k+1})|^q\right]^{\frac{1}{q}}\right\},$$

where

$$Q(s) := \int_0^1 \varrho(2-\varrho)^s d\varrho = \frac{2}{s+1}\left(2^{s+1}-1\right)-\frac{1}{s+2}\left(2^{s+2}-1\right).$$

Proof. By using Lemma 1, the well–known power mean inequality, s–convexity of $|\phi'|^q$ and properties of modulus, we have

$$|\mathcal{S}_n(\phi;\zeta_1,\zeta_2)| \leq \left(\frac{\zeta_2-\zeta_1}{4n}\right)\times$$

$$\sum_{k=0}^{n-1}\left\{\int_0^1 \varrho\left|\phi'\left(\frac{\varrho}{2}\mathbf{u}_{n,k}+\frac{(2-\varrho)}{2}\mathbf{u}_{n,k+1}\right)\right|d\varrho+\right.$$

$$\left.+\int_0^1 \varrho\left|\phi'\left(\frac{(2-\varrho)}{2}\mathbf{u}_{n,k}+\frac{\varrho}{2}\mathbf{u}_{n,k+1}\right)\right|d\varrho\right\}$$

$$\leq \left(\frac{\zeta_2-\zeta_1}{4n}\right)\left(\int_0^1 \varrho d\varrho\right)^{1-\frac{1}{q}}\times\sum_{k=0}^{n-1}\left\{\left(\int_0^1 \varrho\left|\phi'\left(\frac{\varrho}{2}\mathbf{u}_{n,k}+\frac{(2-\varrho)}{2}\mathbf{u}_{n,k+1}\right)\right|^q d\varrho\right)^{\frac{1}{q}}+\right.$$

$$\left.+\left(\int_0^1 \varrho\left|\phi'\left(\frac{(2-\varrho)}{2}\mathbf{u}_{n,k}+\frac{\varrho}{2}\mathbf{u}_{n,k+1}\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right\} \leq$$

$$\leq \left(\frac{\zeta_2-\zeta_1}{4n}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\times$$

$$\times\sum_{k=0}^{n-1}\left\{\left(\int_0^1 \varrho\left[\left(\frac{\varrho}{2}\right)^s|\phi'(\mathbf{u}_{n,k})|^q+\left(1-\frac{\varrho}{2}\right)^s|\phi'(\mathbf{u}_{n,k+1})|\right]^q d\varrho\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 \varrho\left[\left(1-\frac{\varrho}{2}\right)^s|\phi'(\mathbf{u}_{n,k})|^q+\left(\frac{\varrho}{2}\right)^s|\phi'(\mathbf{u}_{n,k+1})|\right]^q d\varrho\right)^{\frac{1}{q}}\Bigg\}$$

$$=\frac{(\zeta_2-\zeta_1)}{n}\left(\frac{1}{2}\right)^{3+\frac{s-1}{q}}\times\sum_{k=0}^{n-1}\left\{\left[\frac{1}{s+2}|\phi'(\mathbf{u}_{n,k})|^q+Q(s)|\phi'(\mathbf{u}_{n,k+1})|^q\right]^{\frac{1}{q}}\right\}+$$

$$+\left\{\left[Q(s)|\phi'(\mathbf{u}_{n,k})|^q+\frac{1}{s+2}|\phi'(\mathbf{u}_{n,k+1})|^q\right]^{\frac{1}{q}}\right\},$$

which completes the proof. □

Corollary 4.1. *Choosing $s=1$ in Theorem 4, we get the following inequality for convex function:*

$$|\mathcal{S}_n(\phi;\zeta_1,\zeta_2)|\leq\left(\frac{\zeta_2-\zeta_1}{8n}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}} \tag{2.14}$$

$$\times\sum_{k=0}^{n-1}\left\{[|\phi'(\boldsymbol{u}_{n,k})|^q+2|\phi'(\boldsymbol{u}_{n,k+1})|^q]^{\frac{1}{q}}+[2|\phi'(\boldsymbol{u}_{n,k})|^q+|\phi'(\boldsymbol{u}_{n,k+1})|^q]^{\frac{1}{q}}\right\}.$$

Corollary 4.2. *Taking $n=1$ in Theorem 4, we obtain*

$$\left|\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(x)dx-\phi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right|\leq(\zeta_2-\zeta_1)\left(\frac{1}{2}\right)^{3+\frac{s-1}{q}} \tag{2.15}$$

$$\times\left\{\left[\frac{1}{s+2}|\phi'(\zeta_1)|^q+Q(s)|\phi'(\zeta_2)|^q\right]^{\frac{1}{q}}+\left[Q(s)|\phi'(\zeta_1)|^q+\frac{1}{s+2}|\phi'(\zeta_2)|^q\right]^{\frac{1}{q}}\right\}.$$

3. Examples and Applications

3.1. **Bessel Functions.** Consider the function $\mathfrak{B}_\tau:(0,+\infty)\to[1,+\infty)$ given by

$$\mathfrak{B}_\tau(x)=2^\tau\Gamma(\tau+1)x^{-\tau}\mathcal{N}_\tau(x),$$

where $\mathcal{N}_\tau$ is the modified Bessel function of the first kind defined by (see [19, (2) on page 77]):

$$\mathcal{N}_\tau(x)=\sum_{n=0}^{+\infty}\frac{\left(\frac{x}{2}\right)^{\tau+2n}}{n!\,\Gamma(\tau+1+n)},\quad x\in\Re.$$

The first order derivative formula of $\mathfrak{B}_\tau(x)$ is given by [19]:

$$\mathfrak{B}'_\tau(x)=\frac{x}{2(\tau+1)}\mathfrak{B}_{\tau+1}(x), \tag{3.1}$$

and the second derivative can be easily calculated from (3.1) to be

$$\mathfrak{B}''_\tau(x)=\frac{x^2\mathfrak{B}_{\tau+2}(x)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(x)}{2(\tau+1)}. \tag{3.2}$$

Example 3.1. *Let $0 < \zeta_1 < \zeta_2$ and $\tau > -1$. Then, by applying Corollary 2.2 (note that all assumptions are satisfied) and the identities* (3.1) *and* (3.2)*, we have*

$$\left|\frac{\mathfrak{B}_\tau(\zeta_2)-\mathfrak{B}_\tau(\zeta_1)}{\zeta_2-\zeta_1}-\frac{(\zeta_1+\zeta_2)}{4(\tau+1)}\mathfrak{B}_{\tau+1}\left(\frac{\zeta_1+\zeta_2}{2}\right)\right|$$
$$\leq (\zeta_2-\zeta_1)\frac{\left(2^{s+1}-1\right)}{2^{s+1}(s+1)(s+2)}\Big[\left(\frac{\zeta_1^2\mathfrak{B}_{\tau+2}(\zeta_1)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\zeta_1)}{2(\tau+1)}\right)$$
$$+\left(\frac{\zeta_2^2\mathfrak{B}_{\tau+2}(\zeta_2)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\zeta_2)}{2(\tau+1)}\right)\Big].$$

Example 3.2. *Let $0 < \zeta_1 < \zeta_2$ and $\tau > -1$. Then, by applying Corollary 3.2 (note that all assumptions are satisfied) and the identities* (3.1) *and* (3.2)*, we get*

$$\left|\frac{\mathfrak{B}_\tau(\zeta_2)-\mathfrak{B}_\tau(\zeta_1)}{\zeta_2-\zeta_1}-\frac{(\zeta_1+\zeta_2)}{4(\tau+1)}\mathfrak{B}_{\tau+1}\left(\frac{\zeta_1+\zeta_2}{2}\right)\right|$$
$$\leq (\zeta_2-\zeta_1)\left(\frac{1}{2}\right)^{2+\frac{s}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$
$$\times\Big\{\Big[\left(\frac{\zeta_1^2\mathfrak{B}_{\tau+2}(\zeta_1)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\zeta_1)}{2(\tau+1)}\right)^q+\left(2^{s+1}-1\right)\left(\frac{\zeta_2^2\mathfrak{B}_{\tau+2}(\zeta_2)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\zeta_2)}{2(\tau+1)}\right)^q\Big]^{\frac{1}{q}}$$
$$+\Big[\left(2^{s+1}-1\right)\left(\frac{\zeta_1^2\mathfrak{B}_{\tau+2}(\zeta_1)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\zeta_1)}{2(\tau+1)}\right)^q+\left(\frac{\zeta_2^2\mathfrak{B}_{\tau+2}(\zeta_2)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\zeta_2)}{2(\tau+1)}\right)^q\Big]^{\frac{1}{q}}\Big\}.$$

Example 3.3. *Let $0 < \zeta_1 < \zeta_2$ and $\tau > -1$. Then, by applying Corollary 4.2 (note that all assumptions are satisfied) and the identities* (3.1) *and* (3.2)*, we obtain*

$$\left|\frac{\mathfrak{B}_\tau(\zeta_2)-\mathfrak{B}_\tau(\zeta_1)}{\zeta_2-\zeta_1}-\frac{(\zeta_1+\zeta_2)}{4(\tau+1)}\mathfrak{B}_{\tau+1}\left(\frac{\zeta_1+\zeta_2}{2}\right)\right|$$
$$\leq (\zeta_2-\zeta_1)\left(\frac{1}{2}\right)^{3+\frac{s-1}{q}}$$
$$\times\Big\{\Big[Q(s)\left(\frac{\zeta_1^2\mathfrak{B}_{\tau+2}(\zeta_1)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\zeta_1)}{2(\tau+1)}\right)^q+\frac{1}{s+2}\left(\frac{\zeta_2^2\mathfrak{B}_{\tau+2}(\zeta_2)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\zeta_2)}{2(\tau+1)}\right)^q\Big]^{\frac{1}{q}}$$
$$+\Big[\frac{1}{s+2}\left(\frac{\zeta_1^2\mathfrak{B}_{\tau+2}(\zeta_1)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\zeta_1)}{2(\tau+1)}\right)^q+Q(s)\left(\frac{\zeta_2^2\mathfrak{B}_{\tau+2}(\zeta_2)}{4(\tau+1)(\tau+2)}+\frac{\mathfrak{B}_{\tau+1}(\zeta_2)}{2(\tau+1)}\right)^q\Big]^{\frac{1}{q}}\Big\}.$$

Remark 3.1: The above application is given for special cases of our main results. The general case is omitted here and interested readers can find them.

3.2. **Special means.** We consider the following two special means for positive real numbers ζ_1 and ζ_2, where $\zeta_1 < \zeta_1$:

(1) The arithmetic mean:

$$\mathcal{A}(\zeta_1,\zeta_2)=\frac{\zeta_1+\zeta_2}{2},$$

(2) The generalized logarithmic mean:

$$\mathcal{L}_r(\zeta_1,\zeta_2)=\left[\frac{\zeta_2^{r+1}-\zeta_1^{r+1}}{(r+1)(\zeta_2-\zeta_1)}\right]^{\frac{1}{r}},\quad r\in\Re\setminus\{-1,0\}.$$

Proposition 3.1. *Let $n\in\mathbb{N}$ and $0<\zeta_1<\zeta_2$, where $s\in(0,1]$ is fixed. Then the following inequality holds:*

$$\left|n\mathcal{L}_s^s(\zeta_1,\zeta_2)-\sum_{k=0}^{n-1}\frac{1}{n^s}\mathcal{A}^s\left((2(n-k)-1)\zeta_1,(2k+1)\zeta_2\right)\right|\leq \tag{3.3}$$
$$\leq\frac{(\zeta_2-\zeta_1)}{4}\frac{\left(2^{s+1}-1\right)s}{n^s(s+1)(s+2)}$$
$$\times\sum_{k=0}^{n-1}\left[\mathcal{A}^{s-1}\left((n-k)\zeta_1,k\zeta_2\right)+\mathcal{A}^{s-1}\left((n-k-1)\zeta_1,(k+1)\zeta_2\right)\right].$$

Proof. Taking $\phi(x)=x^s$, $x\in[\zeta_1,\zeta_2]$ with $s\in(0,1]$ and using Theorem 2, we obtain the desired result (3.3). □

Proposition 3.2. *Let $n\in\mathbb{N}$ and $0<\zeta_1<\zeta_2$, where $s\in(0,1]$ is fixed. Then for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, the following inequality holds:*

$$\left|n\mathcal{L}_s^s(\zeta_1,\zeta_2)-\sum_{k=0}^{n-1}\frac{1}{n^s}\mathcal{A}^s\left((2(n-k)-1)\zeta_1,(2k+1)\zeta_2\right)\right|\leq \tag{3.4}$$
$$\leq\frac{(\zeta_2-\zeta_1)s}{8n^s}\left(\frac{1}{2}\right)^{s\left(\frac{1}{q}-1\right)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$
$$\times\sum_{k=0}^{n-1}\left\{\left[\mathcal{A}^{q(s-1)}\left((n-k)\zeta_1,k\zeta_2\right)+\left(2^{s+1}-1\right)\mathcal{A}^{q(s-1)}\left((n-k-1)\zeta_1,(k+1)\zeta_2\right)\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\left(2^{s+1}-1\right)\mathcal{A}^{q(s-1)}\left((n-k)\zeta_1,k\zeta_2\right)+\mathcal{A}^{q(s-1)}\left((n-k-1)\zeta_1,(k+1)\zeta_2\right)\right]^{\frac{1}{q}}\right\}.$$

Proof. Choosing $\phi(x)=x^s$, $x\in[\zeta_1,\zeta_2]$ with $s\in(0,1]$ and using Theorem 3, we get the desired result (3.4). □

Proposition 3.3. *Let $n\in\mathbb{N}$ and $0<\zeta_1<\zeta_2$, where $s\in(0,1]$ is fixed. Then for $q\geq1$, the following inequality holds:*

$$\left|n\mathcal{L}_s^s(\zeta_1,\zeta_2)-\sum_{k=0}^{n-1}\frac{1}{n^s}\mathcal{A}^s\left((2(n-k)-1)\zeta_1,(2k+1)\zeta_2\right)\right|\leq\frac{(\zeta_2-\zeta_1)s}{16n^s}\left(\frac{1}{2}\right)^{\frac{s-1}{q}-s}$$
$$\times\sum_{k=0}^{n-1}\left\{\left[\frac{1}{s+2}\mathcal{A}^{q(s-1)}\left((n-k)\zeta_1,k\zeta_2\right)+Q(s)\mathcal{A}^{q(s-1)}\left((n-k-1)\zeta_1,(k+1)\zeta_2\right)\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[Q(s)\mathcal{A}^{q(s-1)}\left((n-k)\zeta_1,k\zeta_2\right)+\frac{1}{s+2}\mathcal{A}^{q(s-1)}\left((n-k-1)\zeta_1,(k+1)\zeta_2\right)\right]^{\frac{1}{q}}\right\}, \tag{3.5}$$

where $Q(s)$ is defined as in Theorem 4.

Proof. Taking $\phi(x) = x^s$, $x \in [\zeta_1, \zeta_2]$ with $s \in (0,1]$ and using Theorem 4, we capture the desired result (3.5). $\square$

3.3. **Midpoint quadrature formula.** Let $\mathcal{O}$ be the partition of the points $\zeta_1 = \pi_0 < \pi_1 < \ldots < \pi_n = \zeta_2$ of the interval $[\zeta_1, \zeta_2]$ and consider the quadrature formula

$$\int_{\zeta_1}^{\zeta_2} \phi(x)\, dx = \mathcal{M}_n^{(i)}(\phi, \mathcal{O}) + \mathcal{E}(\phi, \mathcal{O}), \tag{3.6}$$

where

$$\mathcal{M}_n^{(i)}(\phi, \mathcal{O}) := \frac{(\pi_{i+1} - \pi_i)}{n} \sum_{k=0}^{n-1} \phi\left(\frac{(2(n-k)-1)\pi_i + (2k+1)\pi_{i+1}}{2n}\right), \tag{3.7}$$

is the midpoint version and $\mathcal{E}(\phi, \mathcal{O})$ denotes the associated approximation error. In this last section, we are going to derive some new estimates for the midpoint quadrature formula. The following numerical results are given to illustrative the implementation of above midpoint quadrature formula.

Proposition 3.4. *Let $n \in \mathbb{N}$, $0 < \zeta_1 < \zeta_2$ and $s \in (0,1]$ be fixed. Assume that $\phi : [\zeta_1, \zeta_2] \to \Re$ be a differentiable function on (ζ_1, ζ_2). If $\phi' \in \mathcal{L}[\zeta_1, \zeta_2]$ and $|\phi'|$ is s–convex function on $[\zeta_1, \zeta_2]$, then we have*

$$|\mathcal{E}(\phi, \mathcal{O})| \leq \frac{\left(2^{s+1} - 1\right)}{2^{s+1}(s+1)(s+2)} \sum_{i=0}^{l-1} \left(\frac{\pi_{i+1} - \pi_i}{n}\right)^2 \sum_{k=0}^{n-1} \left[|\phi'(\boldsymbol{e}_{n,k}^{(i)})| + |\phi'(\boldsymbol{e}_{n,k+1}^{(i)})|\right], \tag{3.8}$$

where

$$\boldsymbol{e}_{n,k}^{(i)} := \frac{(n-k)\pi_i + k\pi_{i+1}}{n} \quad \textit{and} \quad \boldsymbol{e}_{n,k+1}^{(i)} := \frac{(n-k-1)\pi_i + (k+1)\pi_{i+1}}{n}.$$

Proof. Applying Theorem 2 on the subintervals $[\pi_i, \pi_{i+1}]$ $(i = 0, \ldots, l-1)$ of the partition $\mathcal{O}$, we get

$$\begin{aligned} &\Big| \int_{\pi_i}^{\pi_{i+1}} \phi(x) dx - \frac{(\pi_{i+1} - \pi_i)}{n} \sum_{k=0}^{n-1} \phi\left(\frac{(2(n-k)-1)\pi_i + (2k+1)\pi_{i+1}}{2n}\right) \Big| \\ &\leq \left(\frac{\pi_{i+1} - \pi_i}{n}\right)^2 \frac{\left(2^{s+1} - 1\right)}{2^{s+1}(s+1)(s+2)} \sum_{k=0}^{n-1} \left[|\phi'(\mathbf{e}_{n,k}^{(i)})| + |\phi'(\mathbf{e}_{n,k+1}^{(i)})|\right]. \end{aligned} \tag{3.9}$$

Hence from (3.9), we obtain

$$|\mathcal{E}(\phi, \mathcal{O})| = \Big| \int_{\zeta_1}^{\zeta_2} \phi(x)\, dx - \mathcal{M}_n^{(i)}(\phi, \mathcal{O}) \Big|$$

$$\leq \sum_{i=0}^{l-1} \Big| \int_{\pi_i}^{\pi_{i+1}} \phi(x)\, dx - \frac{(\pi_{i+1} - \pi_i)}{n} \sum_{k=0}^{n-1} \phi\left(\frac{(2(n-k)-1)\pi_i + (2k+1)\pi_{i+1}}{2n}\right) \Big|$$

$$\leq \frac{\left(2^{s+1}-1\right)}{2^{s+1}(s+1)(s+2)} \sum_{i=0}^{l-1} \left(\frac{\pi_{i+1}-\pi_i}{n}\right)^2 \sum_{k=0}^{n-1} \left[|\phi'(\mathbf{e}_{n,k}^{(i)})|+|\phi'(\mathbf{e}_{n,k+1}^{(i)})|\right].$$

The proof of Proposition 3.4 is completed. □

Proposition 3.5. *Let $n \in \mathbb{N}$, $0 < \zeta_1 < \zeta_2$ and $s \in (0,1]$ be fixed. Assume that $\phi : [\zeta_1, \zeta_2] \to \Re$ be a differentiable function on (ζ_1, ζ_2). If $\phi' \in \mathcal{L}[\zeta_1, \zeta_2]$ and $|\phi'|^q$ is s–convex function on $[\zeta_1, \zeta_2]$, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$|\mathcal{E}(\phi, \mathcal{O})| \leq \left(\frac{1}{2}\right)^{2+\frac{s}{q}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{s+1}\right)^{\frac{1}{q}} \sum_{i=0}^{l-1} \left(\frac{\pi_{i+1}-\pi_i}{n}\right)^2 \tag{3.10}$$

$$\times \sum_{k=0}^{n-1} \left\{ \left[|\phi'(\boldsymbol{e}_{n,k}^{(i)})|^q + \left(2^{s+1}-1\right) |\phi'(\boldsymbol{e}_{n,k+1}^{(i)})|^q\right]^{\frac{1}{q}} + \right.$$

$$\left. + \left[\left(2^{s+1}-1\right) |\phi'(\boldsymbol{e}_{n,k}^{(i)})|^q + |\phi'(\boldsymbol{e}_{n,k+1}^{(i)})|^q\right]^{\frac{1}{q}} \right\},$$

where $\boldsymbol{e}_{n,k}^{(i)}$ and $\boldsymbol{e}_{n,k+1}^{(i)}$ are as defined in Proposition 3.4.

Proof. The proof is similarly as Proposition 3.4 but using Theorem 3. □

Proposition 3.6. *Let $n \in \mathbb{N}$, $0 < \zeta_1 < \zeta_2$ and $s \in (0,1]$ be fixed. Assume that $\phi : [\zeta_1, \zeta_2] \to \Re$ be a differentiable function on (ζ_1, ζ_2). If $\phi' \in \mathcal{L}[\zeta_1, \zeta_2]$ and $|\phi'|^q$ is s–convex function on $[\zeta_1, \zeta_2]$, then for $q \geq 1$, we have*

$$|\mathcal{E}(\phi, \mathcal{O})| \leq \left(\frac{1}{2}\right)^{3+\frac{s-1}{q}} \sum_{i=0}^{l-1} \left(\frac{\pi_{i+1}-\pi_i}{n}\right)^2 \tag{3.11}$$

$$\times \sum_{k=0}^{n-1} \left\{ \left[\frac{1}{s+2} |\phi'(\boldsymbol{e}_{n,k}^{(i)})|^q + Q(s) |\phi'(\boldsymbol{e}_{n,k+1}^{(i)})|^q\right]^{\frac{1}{q}} + \right.$$

$$\left. + \left[Q(s) |\phi'(\boldsymbol{e}_{n,k}^{(i)})|^q + \frac{1}{s+2} |\phi'(\boldsymbol{e}_{n,k+1}^{(i)})|^q\right]^{\frac{1}{q}} \right\},$$

where $\boldsymbol{e}_{n,k}^{(i)}$ and $\boldsymbol{e}_{n,k+1}^{(i)}$ are as defined in Proposition 3.4 and $Q(s)$ is defined as in Theorem 4.

Proof. The proof is similarly as Proposition 3.4 but applying Theorem 4. □

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