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SOME NEW ESTIMATES USING GENERALIZED QUANTUM MONTGOMERY IDENTITY VIA STRONGLY PREINVEX FUNCTIONS OF HIGHER ORDER

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Abstract. In this paper, we obtain a new generalized version of the quantum Montgomery identity. Using this identity, some new estimates, for the class of strongly preinvex functions of higher order, are established. Furthermore, novel inequalities are deduced from our main results as special cases and in addition, recapture some known results as well. We anticipate that results presented herein will trigger further interest in this direction.

1. Introduction

Quantum calculus or q-calculus has received much attention in the last years and is served as bridge between mathematics and physics. Recently, Tariboon et al. in [18], defined q-derivative and q-integral as follows:

Definition 1.1. Let $\Psi : [e_1, e_2] \to \mathbb{R}$ be a continuous function and let $x \in [e_1, e_2]$ and $0 < \mathrm{q} < 1$ be a constant. Then the q-derivative on $[e_1, e_2]$ of function $\Psi(x)$ is defined as

$$ {}_{e_1}D_{\mathrm{q}}\Psi(x) = \frac{\Psi(x) - \Psi(qx + (1-\mathrm{q})e_1)}{(1-\mathrm{q})(x-e_1)}, \ \ x \neq e_1. \tag{1.1}$$

We say that $\Psi(x)$ is q-*differentiable* on $[e_1, e_2]$ provided ${}_{e_1}D_{\mathrm{q}}\Psi(x)$ exists for all $x \in [e_1, e_2]$.

Definition 1.2. Let $\Psi : [e_1, e_2] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-*integral* on $[e_1, e_2]$ is defined as

$$\int_{e_1}^{x} \Psi(\ell)\, {}_{e_1}d_{\mathrm{q}}\ell = (1-\mathrm{q})(x-e_1)\sum_{n=0}^{\infty} \mathrm{q}^n \Psi(\mathrm{q}^n x + (1-\mathrm{q}^n)e_1) \tag{1.2}$$

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for $x \in [e_1, e_2]$.

For more details on q-calculus and certain q-analogues of classical inequalities, see [1],[3]-[5],[7],[10]-[13],[17]-[21],[23].

The following famous identity given in [[9]], is called Montgomery identity:

$$\Psi(x) = \frac{1}{e_2 - e_1} \int_{e_1}^{e_2} \Psi(\ell)\, d\ell + \frac{1}{e_2 - e_1} \int_{e_1}^{x} (\ell - e_1) \Psi'(\ell)\, d\ell + \frac{1}{e_2 - e_1} \int_{x}^{e_2} (\ell - e_2) \Psi'(\ell)\, d\ell, \tag{1.3}$$

where the function $\Psi(x)$ is continuous on $[e_1, e_2]$ with a continuous first derivative in (e_1, e_2).

By changing variable, the Montgomery identity (1.3) could be expressed as follows:

$$\Psi(x) - \frac{1}{e_2 - e_1} \int_{e_1}^{e_2} \Psi(\ell)\, d\ell = (e_2 - e_1) \int_0^1 H(\ell)\, \Psi'((1 - \ell) e_1 + \ell e_2)\, d\ell, \tag{1.4}$$

where

$$H(\ell) := \begin{cases} \ell, & \ell \in \left[0, \frac{x - e_1}{e_2 - e_1}\right]; \\ \ell - 1, & \ell \in \left(\frac{x - e_1}{e_2 - e_1}, 1\right]. \end{cases}$$

We recall now some basic definitions for our study as follows:

Let $\mathbf{E}$ be a non–empty set, $\Psi : \mathbf{E} \to \mathbb{R}$ be a continuous functions and $\zeta : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$ be a continuous bifunction.

Definition 1.3. [4] *A set $\mathbf{E} \subset \mathbb{R}$ is said to be invex with respect to bifunction $\zeta(.,.)$, if*

$$e_1 + \ell\zeta(e_2, e_1) \in \mathbf{E}, \quad \forall\, e_1, e_2 \in \mathbf{E},\ \ell \in [0, 1].$$

Definition 1.4. [22] *A function $\Psi : \mathbf{E} \to \mathbb{R}$ is said to be preinvex with respect to bifunction $\zeta(.,.)$, if*

$$\Psi(e_1 + \ell\zeta(e_2, e_1)) \le (1 - \ell)\Psi(e_1) + \ell\Psi(e_2), \quad \forall\, e_1, e_2 \in \mathbf{E},\ \ell \in [0, 1].$$

The notion of strongly convex functions was introduced by Karamardian [6] and Polyak [15].

Definition 1.5. *A function $\Psi : \mathbf{E} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be strongly convex with modulus $\mu > 0$, if*

$$\Psi((1 - \ell)e_1 + \ell e_2) \le (1 - \ell)\Psi(e_1) + \ell\Psi(e_2) - \mu\ell(1 - \ell)(e_2 - e_1)^2$$

for all $e_1, e_2 \in \mathbf{E}$ and $\ell \in [0, 1]$.

In [6], Karamardian noticed that every strongly monotone has a gradient map if and only if all differentiable function is strongly convex. Higher order strongly convex functions introduced by Lin *et al.* in [8], to abridge the research of linear programming with equilibrium constraints.

Definition 1.6. *A function $\Psi : \mathbf{E} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be strongly convex with modulus $\mu > 0$ and order $\sigma > 0$, if*

$$\Psi((1-\ell)e_1 + \ell e_2) \leq (1-\ell)\Psi(e_1) + \ell\Psi(e_2) - \mu\ell(1-\ell)(e_2 - e_1)^{\sigma}$$

for all $e_1, e_2 \in \mathbf{E}$ and $\ell \in [0,1]$.

Recently, Awan *et al.* in [2], defined the following class of strongly preinvex functions of higher order.

Definition 1.7. *A function $\Psi : \mathbf{E} \longrightarrow \mathbb{R}$ is said to be strongly preinvex with modulus $\mu > 0$, order $\sigma > 0$ and with respect to $\zeta : \mathbf{E} \times \mathbf{E} \longrightarrow \mathbb{R}$, if*

$$\Psi(e_1 + \ell\zeta(e_2, e_1)) \leq (1-\ell)\Psi(e_1) + \ell\Psi(e_2) - \mu\ell(1-\ell)\zeta^{\sigma}(e_2, e_1) \tag{1.5}$$

for all $e_1, e_2 \in \mathbf{E}$ and $\ell \in [0,1]$.

Motivated by the above literatures, the main objective of this article is to obtain a generalization of the Montgomery identity given in (1.4) using the concepts of q-calculus. From this identity, several new and known q-analogues of integral inequalities involving strongly preinvex functions of higher order will be obtain. We also will discuss some new special cases of the main results. Finally, a brief conclusion will be provided.

2. Main results

Throughout this section, we shall let $P = [e_1, e_1 + \zeta(e_2, e_1)]$ and P° denote the interior of P. For the sake of brevity, we define the following function $\wp_{\zeta} : P \to [0,1]$ by

$$\wp_{\zeta}(x) := \frac{x - e_1}{\zeta(e_2, e_1)}, \quad \text{where} \quad \zeta(e_2, e_1) > 0, \quad \text{and} \quad x \in P.$$

Lemma 1 (Generalized quantum Montgomery identity)**.** *If $\Psi : P \to \mathbb{R}$ is a* q*-differentiable function such that ${}_{e}D_{\mathrm{q}}\Psi$ is quantum integrable on P°, then the following identity holds:*

$$\Psi(x) - \frac{1}{\zeta(e_2, e_1)} \int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell)\, {}_{e_1}d_{\mathrm{q}}\ell = \zeta(e_2, e_1) \tag{2.1}$$

$$\times \int_0^1 T_{\mathrm{q}}(\ell)\, {}_{e_1}D_{\mathrm{q}}\Psi(e_1 + \ell\zeta(e_2, e_1))\, {}_0d_{\mathrm{q}}\ell,$$

where

$$T_{\mathrm{q}}(\ell) := \begin{cases} \mathrm{q}\ell, & \ell \in [0, \wp_{\zeta}(x)]\,; \\ \mathrm{q}\ell - 1, & \ell \in (\wp_{\zeta}(x), 1]\,. \end{cases}$$

Proof. By using Definitions 1.1 and 1.2, we have

$$\zeta(e_2,e_1)\int_0^1 T_{\mathrm{q}}(\ell)\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\,{}_0d_{\mathrm{q}}\ell$$

$$=\zeta(e_2,e_1)\left[\int_0^{\wp_\zeta(x)}\mathrm{q}\ell\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\,{}_0d_{\mathrm{q}}\ell+\right.$$

$$\left.+\int_{\wp_\zeta(x)}^1(\mathrm{q}\ell-1)\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\,{}_0d_{\mathrm{q}}\ell\right]$$

$$=\zeta(e_2,e_1)\left[\int_0^{\wp_\zeta(x)}\mathrm{q}\ell\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\,{}_0d_{\mathrm{q}}\ell+\right.$$

$$\left.\int_0^1(\mathrm{q}\ell-1)\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\,{}_0d_{\mathrm{q}}\ell-\int_0^{\wp_\zeta(x)}(\mathrm{q}\ell-1)\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\,{}_0d_{\mathrm{q}}\ell\right]$$

$$=\zeta(e_2,e_1)\left[\int_0^1(\mathrm{q}\ell-1)\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\,{}_0d_{\mathrm{q}}\ell+\int_0^{\wp_\zeta(x)}{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\,{}_0d_{\mathrm{q}}\ell\right]$$

$$=\zeta(e_2,e_1)\left[\int_0^1\mathrm{q}\ell\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\,{}_0d_{\mathrm{q}}\ell-\int_0^1{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\,{}_0d_{\mathrm{q}}\ell\right.$$

$$\left.+\int_0^{\wp_\zeta(x)}{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\,{}_0d_{\mathrm{q}}\ell\right]$$

$$=\frac{1}{1-\mathrm{q}}\left[\begin{array}{l}\mathrm{q}(1-\mathrm{q})\left[\begin{array}{c}\sum_{n=0}^{\infty}\mathrm{q}^n\Psi(e_1+\mathrm{q}^n\zeta(e_2,e_1))\\-\sum_{n=0}^{\infty}\mathrm{q}^n\Psi(e_1+\mathrm{q}^{n+1}\zeta(e_2,e_1))\end{array}\right]\\-(1-\mathrm{q})\left[\begin{array}{c}\sum_{n=0}^{\infty}\mathrm{q}^n\frac{\Psi(e_1+\mathrm{q}^n\zeta(e_2,e_1))}{\mathrm{q}^n}\\-\sum_{n=0}^{\infty}\mathrm{q}^n\frac{\Psi(e_1+\mathrm{q}^{n+1}\zeta(e_2,e_1))}{\mathrm{q}^n}\end{array}\right]\\+(1-\mathrm{q})\wp_\zeta(x)\left[\begin{array}{c}\sum_{n=0}^{\infty}\mathrm{q}^n\frac{\Psi(e_1+\mathrm{q}^n\wp_\zeta(x)\zeta(e_2,e_1))}{\mathrm{q}^n\wp_\zeta(x)}\\-\sum_{n=0}^{\infty}\mathrm{q}^n\frac{\Psi\left(e_1+\mathrm{q}^{n+1}\wp_\zeta(x)\zeta(e_2,e_1)\right)}{\mathrm{q}^n\wp_\zeta(x)}\end{array}\right]\end{array}\right]$$

$$= \left[\begin{array}{l} \mathrm{q}\left[\begin{array}{l} \sum\limits_{n=0}^{\infty} \mathrm{q}^n \Psi(e_1 + \mathrm{q}^n \zeta(e_2, e_1)) \\ -\sum\limits_{n=0}^{\infty} \mathrm{q}^n \Psi(e_1 + \mathrm{q}^{n+1} \zeta(e_2, e_1)) \end{array}\right] \\ -\left[\begin{array}{l} \sum\limits_{n=0}^{\infty} \Psi(e_1 + \mathrm{q}^n \zeta(e_2, e_1)) \\ -\sum\limits_{n=0}^{\infty} \Psi(e_1 + \mathrm{q}^{n+1} \zeta(e_2, e_1)) \end{array}\right] \\ +\left[\begin{array}{l} \sum\limits_{n=0}^{\infty} \Psi\left(e_1 + \mathrm{q}^n \wp_\zeta(x) \zeta(e_2, e_1)\right) \\ -\sum\limits_{n=0}^{\infty} \Psi\left(e_1 + \mathrm{q}^{n+1} \wp_\zeta(x) \zeta(e_2, e_1)\right) \end{array}\right] \end{array}\right]$$

$$= \left[\begin{array}{l} \mathrm{q}\left[\begin{array}{l} \sum\limits_{n=0}^{\infty} \mathrm{q}^n \Psi(e_1 + \mathrm{q}^n \zeta(e_2, e_1)) \\ -\frac{1}{\mathrm{q}}\sum\limits_{n=1}^{\infty} \mathrm{q}^n \Psi(e_1 + \mathrm{q}^n \zeta(e_2, e_1)) \end{array}\right] \\ -\left[\begin{array}{l} \sum\limits_{n=0}^{\infty} \Psi(e_1 + \mathrm{q}^n \zeta(e_2, e_1)) \\ -\sum\limits_{n=1}^{\infty} \Psi(e_1 + \mathrm{q}^n \zeta(e_2, e_1)) \end{array}\right] \\ +\left[\begin{array}{l} \sum\limits_{n=0}^{\infty} \Psi\left(e_1 + \mathrm{q}^n \wp_\zeta(x) \zeta(e_2, e_1)\right) \\ -\sum\limits_{n=1}^{\infty} \Psi\left(e_1 + \mathrm{q}^n \wp_\zeta(x) \zeta(e_2, e_1)\right) \end{array}\right] \end{array}\right]$$

$$= \left[\begin{array}{l} \mathrm{q}\left[(1-\frac{1}{\mathrm{q}}) \sum\limits_{n=0}^{\infty} \mathrm{q}^n \Psi(e_1 + \mathrm{q}^n \zeta(e_2, e_1)) + \frac{\Psi(e_1+\zeta(e_2,e_1))}{\mathrm{q}} \right] \\ -\Psi(e_1 + \zeta(e_2, e_1)) + \Psi\left(e_1 + \wp_\zeta(x) \zeta(e_2, e_1)\right) \end{array}\right]$$

$$= \Psi(x) - (1-\mathrm{q}) \sum_{n=0}^{\infty} \mathrm{q}^n \Psi(e_1 + \mathrm{q}^n \zeta(e_2, e_1))$$

$$= \Psi(x) - \frac{1}{\zeta(e_2, e_1)} \int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell)\, {}_{e_1}d_{\mathrm{q}}\ell.$$

The proof of our lemma is completed. □

Remark 2.1: Taking $\mathrm{q} \to 1^-$ in Lemma 1, we have

$$\Psi(x) - \frac{1}{\zeta(e_2, e_1)} \int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell) d\ell = \zeta(e_2, e_1)$$

$$\times \left[\int_0^{\wp_\zeta(x)} \ell \Psi'(e_1 + \ell\zeta(e_2, e_1)) d\ell + \int_{\wp_\zeta(x)}^1 (\ell - 1) \Psi'(e_1 + \ell\zeta(e_2, e_1)) d\ell \right].$$

Remark 2.2: Taking $\zeta(e_2, e_1) = e_2 - e_1$ in Lemma 1, we get [7, Lemma 3].

Remark 2.3: Taking $\mathrm{q} \to 1^-$ and $\zeta(e_2, e_1) = e_2 - e_1$ in Lemma 1, we obtain the Montgomery identity given in (1.4).

Remark 2.4: Taking $x = \frac{2e_1+\zeta(e_2,e_1)}{2}$ in Remark 2.1, we get [14, Lemma 3.10].

$$\Psi\left(\frac{2e_1+\zeta(e_2,e_1)}{2}\right) - \frac{1}{\zeta(e_2,e_1)} \int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell)d\ell = \zeta(e_2,e_1)$$
$$\times \left[\int_0^{\frac{1}{2}} \ell\Psi'(e_1+\ell\zeta(e_2,e_1))d\ell + \int_{\frac{1}{2}}^1 (\ell-1)\Psi'(e_1+\ell\zeta(e_2,e_1))d\ell\right].$$

Remark 2.5: Taking $x = \frac{qe_1+e_2}{1+\mathrm{q}}$ and $\zeta(e_2, e_1) = e_2 - e_1$ in Lemma 1, we obtain equality (4.1) of [1].

$$\Psi\left(\frac{qe_1+e_2}{1+\mathrm{q}}\right) - \frac{1}{e_2-e_1} \int_{e_1}^{e_2} \Psi(\ell)\, {}_{e_1}d_{\mathrm{q}}\ell$$
$$= (e_2-e_1)\left[\begin{array}{l} \int_0^{\frac{1}{1+\mathrm{q}}} \mathrm{q}\ell\, {}_{e_1}D_{\mathrm{q}}\Psi((1-\ell)e_1+\ell e_2)\, {}_0d_{\mathrm{q}}\ell \\ + \int_{\frac{1}{1+\mathrm{q}}}^1 (\mathrm{q}\ell-1)\, {}_{e_1}D_{\mathrm{q}}\Psi((1-\ell)e_1+\ell e_2)\, {}_0d_{\mathrm{q}}\ell \end{array}\right].$$

Remark 2.6: Taking $x = \frac{e_1+\mathrm{q}(e_1+\zeta(e_2,e_1))}{1+\mathrm{q}}$ in Lemma 1, we have

$$\Psi\left(\frac{e_1+\mathrm{q}(e_1+\zeta(e_2,e_1))}{1+\mathrm{q}}\right) - \frac{1}{\zeta(e_2,e_1)} \int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell)\, {}_{e_1}d_{\mathrm{q}}\ell$$
$$= \zeta(e_2,e_1)\left[\begin{array}{l} \int_0^{\frac{\mathrm{q}}{1+\mathrm{q}}} \mathrm{q}\ell\, {}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\, {}_0d_{\mathrm{q}}\ell \\ + \int_{\frac{\mathrm{q}}{1+\mathrm{q}}}^1 (\mathrm{q}\ell-1)\, {}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))\, {}_0d_{\mathrm{q}}\ell \end{array}\right].$$

Now, using Lemma 1, we can derive our main results for the class of higher order strongly preinvex functions.

Theorem 1. *Let $\Psi : P \to \mathbb{R}$ be a function such that ${}_{e_1}D_{\mathrm{q}}\Psi$ is* q*-integrable on P°. If $|{}_{e_1}D_{\mathrm{q}}\Psi|^r$ is strongly preinvex functions of order $\sigma > 0$ with modulus $\mu > 0$ on P, then for $r > 1$ and $p^{-1} + r^{-1} = 1$, the following inequality holds:*

$$\left|\Psi(x) - \frac{1}{\zeta(e_2,e_1)} \int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell)\, {}_{e_1}d_{\mathrm{q}}\ell\right| \le \mathrm{q}\zeta(e_2,e_1)$$

$$\times\left[\begin{array}{c}[L_1(\mathrm{q},e_1,e_2,x)]^{\frac{1}{p}}\left[\begin{array}{c}|{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r L_2(\mathrm{q},e_1,e_2,x)\\+|{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r L_3(\mathrm{q},e_1,e_2,x)-\\-\mu\zeta^{\sigma}(e_2,e_1)P_1(\mathrm{q},e_1,e_2,x)\end{array}\right]^{\frac{1}{r}}\\+[L_4(\mathrm{q},e_1,e_2,x)]^{\frac{1}{p}}\left[\begin{array}{c}|{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r L_5(\mathrm{q},e_1,e_2,x)\\+|{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r L_6(\mathrm{q},e_1,e_2,x)-\\-\mu\zeta^{\sigma}(e_2,e_1)P_2(\mathrm{q},e_1,e_2,x)\end{array}\right]^{\frac{1}{r}}\end{array}\right],$$

where

$$L_1(\mathrm{q},e_1,e_2,x):=\int_0^{\wp_\zeta(x)}\ell^p\,{}_0d_{\mathrm{q}}\ell=[\wp_\zeta(x)]^p\frac{(1-\mathrm{q})}{1-\mathrm{q}^{p+1}},$$

$$L_2(\mathrm{q},e_1,e_2,x):=\int_0^{\wp_\zeta(x)}(1-\ell)\,{}_0d_{\mathrm{q}}\ell=\wp_\zeta(x)-\frac{1}{1+\mathrm{q}}[\wp_\zeta(x)]^2,$$

$$L_3(\mathrm{q},e_1,e_2,x):=\int_0^{\wp_\zeta(x)}\ell\,{}_0d_{\mathrm{q}}\ell=\frac{1}{1+\mathrm{q}}[\wp_\zeta(x)]^2,$$

$$L_4(\mathrm{q},e_1,e_2,x):=\int_{\wp_\zeta(x)}^{1}\left(\ell-\frac{1}{\mathrm{q}}\right)^p{}_0d_{\mathrm{q}}\ell=\ (1-\mathrm{q})\left[\begin{array}{c}\sum\limits_{n=0}^{\infty}\mathrm{q}^n\left(\mathrm{q}^n-\frac{1}{\mathrm{q}}\right)^p-\\ \wp_\zeta(x)\sum\limits_{n=0}^{\infty}\mathrm{q}^n\left(\mathrm{q}^n\wp_\zeta(x)-\frac{1}{\mathrm{q}}\right)^p\end{array}\right],$$

$$L_5(\mathrm{q},e_1,e_2,x):=\int_{\wp_\zeta(x)}^{1}(1-\ell)\,{}_0d_{\mathrm{q}}\ell=\frac{\mathrm{q}}{1+\mathrm{q}}-\wp_\zeta(x)+\frac{1}{1+\mathrm{q}}[\wp_\zeta(x)]^2,$$

$$L_6(\mathrm{q},e_1,e_2,x):=\int_{\wp_\zeta(x)}^{1}\ell\,{}_0d_{\mathrm{q}}\ell=\frac{1}{1+\mathrm{q}}\left(1-[\wp_\zeta(x)]^2\right),$$

and

$$P_1(\mathrm{q},e_1,e_2,x):=\int_0^{\wp_\zeta(x)}\ell(1-\ell)\,{}_0d_{\mathrm{q}}\ell=\frac{1}{1+\mathrm{q}}[\wp_\zeta(x)]^2-\frac{1}{1+\mathrm{q}+\mathrm{q}^2}[\wp_\zeta(x)]^3,$$

$$P_2(\mathrm{q},e_1,e_2,x):=\int_{\wp_\zeta(x)}^{1}\ell(1-\ell)\,{}_0d_{\mathrm{q}}\ell=L_6(\mathrm{q},e_1,e_2,x)-\frac{1}{1+\mathrm{q}+\mathrm{q}^2}\left(1-[\wp_\zeta(x)]^3\right).$$

Proof. Using Lemma 1, strongly preinvexity of order $\sigma>0$ with modulus $\mu>0$ of $|{}_{e_1}D_{\mathrm{q}}\Psi|^r$ and Hölder's inequality, we get

$$\left|\Psi(x)-\frac{1}{\zeta(e_2,e_1)}\int_{e_1}^{e_1+\zeta(e_2,e_1)}\Psi(\ell)\,{}_{e_1}d_{\mathrm{q}}\ell\right|$$

$$\leq \zeta(e_2,e_1)\left[\begin{array}{c}\int\limits_0^{\wp_\zeta(x)} \mathrm{q}\ell|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))|\,{}_0d_{\mathrm{q}}\ell \\ +\int\limits_{\wp_\zeta(x)}^{1}(\mathrm{q}\ell-1)|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))|\,{}_0d_{\mathrm{q}}\ell\end{array}\right]$$

$$\leq \zeta(e_2,e_1)\left[\left(\int\limits_0^{\wp_\zeta(x)}(\mathrm{q}\ell)^p\,{}_0d_{\mathrm{q}}\ell\right)^{\frac{1}{p}}\left(\int\limits_0^{\wp_\zeta(x)}|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))|^r\,{}_0d_{\mathrm{q}}\ell\right)^{\frac{1}{r}}\right.$$

$$\left.+\left(\int\limits_{\wp_\zeta(x)}^{1}(\mathrm{q}\ell-1)^p\,{}_0d_{\mathrm{q}}\ell\right)^{\frac{1}{p}}\left(\int\limits_{\wp_\zeta(x)}^{1}|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))|^r\,{}_0d_{\mathrm{q}}\ell\right)^{\frac{1}{r}}\right]$$

$$\leq \mathrm{q}\zeta(e_2,e_1)\Bigg\{\left(\int\limits_0^{\wp_\zeta(x)}\ell^p\,{}_0d_{\mathrm{q}}\ell\right)^{\frac{1}{p}}$$

$$\times\left(|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r\int\limits_0^{\wp_\zeta(x)}(1-\ell)\,{}_0d_{\mathrm{q}}\ell+|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r\int\limits_0^{\wp_\zeta(x)}\ell\,{}_0d_{\mathrm{q}}\ell-\right.$$

$$\left.-\mu\zeta^\sigma(e_2,e_1)\int\limits_0^{\wp_\zeta(x)}\ell(1-\ell)\,{}_0d_{\mathrm{q}}\ell\right)^{\frac{1}{r}}+\left(\int\limits_{\wp_\zeta(x)}^{1}\left(\ell-\frac{1}{\mathrm{q}}\right)^p\,{}_0d_{\mathrm{q}}\ell\right)^{\frac{1}{p}}$$

$$\times\left(|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r\int\limits_{\wp_\zeta(x)}^{1}(1-\ell)\,{}_0d_{\mathrm{q}}\ell+\right.$$

$$\left.+|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r\int\limits_{\wp_\zeta(x)}^{1}\ell\,{}_0d_{\mathrm{q}}\ell-\mu\zeta^\sigma(e_2,e_1)\int\limits_{\wp_\zeta(x)}^{1}\ell(1-\ell)\,{}_0d_{\mathrm{q}}\ell\right)^{\frac{1}{r}}\Bigg\}.$$

The proof of Theorem 1 is completed. □

We point out some special cases of Theorem 1.

Corollary 1.1. I. *Taking* $\mathrm{q}\to 1^-$ *in Theorem 1, we have*

$$\left|\Psi(x)-\frac{1}{\zeta(e_2,e_1)}\int\limits_{e_1}^{e_1+\zeta(e_2,e_1)}\Psi(\ell)d\ell\right|\leq\zeta(e_2,e_1)$$

$$\left[\begin{array}{c} [L_7(e_1,e_2,x)]^{\frac{1}{p}}\left[\begin{array}{c} |\Psi'(e_1)|^r L_8(e_1,e_2,x)+|\Psi'(e_2)|^r L_9(e_1,e_2,x)- \\ -\mu\zeta^\sigma(e_2,e_1)P_3(e_1,e_2,x)\end{array}\right]^{\frac{1}{r}} \\ +[L_{10}(e_1,e_2,x)]^{\frac{1}{p}}\left[\begin{array}{c} |\Psi'(e_1)|^r L_{11}(e_1,e_2,x)+|\Psi'(e_2)|^r L_{12}(e_1,e_2,x)- \\ -\mu\zeta^\sigma(e_2,e_1)P_4(e_1,e_2,x)\end{array}\right]^{\frac{1}{r}}\end{array}\right],$$

where

$$L_7(e_1,e_2,x) := \int_0^{\wp_\zeta(x)} \ell^p d\ell = \frac{1}{p+1}[\wp_\zeta(x)]^{p+1},$$

$$L_8(e_1,e_2,x) := \int_0^{\wp_\zeta(x)} (1-\ell)d\ell = \wp_\zeta(x) - \frac{1}{2}[\wp_\zeta(x)]^2,$$

$$L_9(e_1,e_2,x) := \int_0^{\wp_\zeta(x)} \ell d\ell = \frac{1}{2}[\wp_\zeta(x)]^2,$$

$$L_{10}(e_1,e_2,x) := \int_{\wp_\zeta(x)}^1 (1-\ell)^p d\ell = \frac{1}{p+1}\left(\frac{e_1+\zeta(e_2,e_1)-x}{\zeta(e_2,e_1)}\right)^{p+1},$$

$$L_{11}(e_1,e_2,x) := \int_{\wp_\zeta(x)}^1 (1-\ell)d\ell = \frac{e_1+\zeta(e_2,e_1)-x}{\zeta(e_2,e_1)} - \frac{1}{2}\left(1-[\wp_\zeta(x)]^2\right),$$

$$L_{12}(e_1,e_2,x) := \int_{\wp_\zeta(x)}^1 \ell d\ell = \frac{1}{2}\left(1-[\wp_\zeta(x)]^2\right),$$

and

$$P_3(e_1,e_2,x) := \int_0^{\wp_\zeta(x)} \ell(1-\ell)d\ell = \frac{1}{2}[\wp_\zeta(x)]^2 - \frac{1}{3}[\wp_\zeta(x)]^3,$$

$$P_4(e_1,e_2,x) := \int_{\wp_\zeta(x)}^1 \ell(1-\ell)d\ell = L_{12}(e_1,e_2,x) - \frac{1}{3}\left(1-[\wp_\zeta(x)]^3\right).$$

II. *Taking* q $\to 1^-$, $\mu \to 0^+$ *and* $x = \frac{2e_1+\zeta(e_2,e_1)}{2}$ *in Theorem 1, we get* [16, Theorem 6].

$$\left|\Psi\left(\frac{2e_1+\zeta(e_2,e_1)}{\zeta(e_2,e_1)}\right) - \frac{1}{\zeta(e_2,e_1)}\int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell)d\ell\right|$$

$$\le \frac{\zeta(e_2,e_1)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[(3|\Psi'(e_1)|^r+|\Psi'(e_2)|^r)^{\frac{1}{r}} + (|\Psi'(e_1)|^r+3|\Psi'(e_2)|^r)^{\frac{1}{r}}\right].$$

III. *Taking* $x = \frac{qe_1+e_2}{1+\mathrm{q}}$, $\mu \to 0^+$ *and* $\zeta(e_2, e_1) = e_2 - e_1$ *in Theorem 1, we obtain* [1, Theorem 18].

$$\left|\Psi\left(\frac{qe_1+e_2}{1+\mathrm{q}}\right) - \frac{1}{e_2-e_1}\int_{e_1}^{e_2} \Psi(\ell)\, {}_{e_1}d_{\mathrm{q}}\ell\right| \leq$$

$$\mathrm{q}(e_2-e_1)\left[\left(\frac{1}{(1+\mathrm{q})^{p+1}}\frac{(1-\mathrm{q})}{1-\mathrm{q}^{p+1}}\right)\left[\frac{(\mathrm{q}^2+2\mathrm{q})|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r}{(1+\mathrm{q})^3} + \frac{\mathrm{q}^2|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r}{(1+\mathrm{q})^3}\right]^{\frac{1}{r}} + \right.$$

$$\left.\left(\int_{\frac{1}{1+\mathrm{q}}}^{1}\left(\ell - \frac{1}{\mathrm{q}}\right)^p {}_{e_1}d_{\mathrm{q}}\ell\right)^{\frac{1}{p}}\left[\frac{(\mathrm{q}^3+\mathrm{q}^2-\mathrm{q})|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r}{(1+\mathrm{q})^3} + \frac{(\mathrm{q}^2+2\mathrm{q})|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r}{(1+\mathrm{q})^3}\right]^{\frac{1}{r}}\right],$$

where

$$\int_{\frac{1}{1+\mathrm{q}}}^{1}\left(\ell-\frac{1}{\mathrm{q}}\right)^p {}_0d_{\mathrm{q}}\ell = (1-\mathrm{q})\left[\begin{array}{c}\sum\limits_{n=0}^{\infty}\mathrm{q}^n\left(\mathrm{q}^n-\frac{1}{\mathrm{q}}\right)^p \\ -\frac{1}{1+\mathrm{q}}\sum\limits_{n=0}^{\infty}\mathrm{q}^n\left(\mathrm{q}^n\left(\frac{1}{1+\mathrm{q}}\right)-\frac{1}{\mathrm{q}}\right)^p\end{array}\right].$$

IV. *Taking* $\mu \to 0^+$ *and* $x = \frac{e_1+\mathrm{q}(e_1+\zeta(e_2,e_1))}{1+\mathrm{q}}$ *in Theorem 1, we get*

$$\left|\Psi\left(\frac{e_1+\mathrm{q}(e_1+\zeta(e_2,e_1))}{1+\mathrm{q}}\right) - \frac{1}{\zeta(e_2,e_1)}\int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell)\, {}_{e_1}d_{\mathrm{q}}\ell\right| \leq \mathrm{q}\zeta(e_2,e_1)$$

$$\times\Bigg\{\left(\frac{\mathrm{q}^p}{(1+\mathrm{q})^{p+1}}\frac{(1-\mathrm{q})}{1-\mathrm{q}^{p+1}}\right)\left[\frac{(\mathrm{q}^3+\mathrm{q}^2+\mathrm{q})|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r}{(1+\mathrm{q})^3} + \frac{\mathrm{q}^2|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r}{(1+\mathrm{q})^3}\right]^{\frac{1}{r}}$$

$$+\left(\int_{\frac{\mathrm{q}}{1+\mathrm{q}}}^{1}\left(\ell-\frac{1}{\mathrm{q}}\right)^p {}_0d_{\mathrm{q}}\ell\right)^{\frac{1}{p}}\left[\frac{\mathrm{q}^2|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r}{(1+\mathrm{q})^3} + \frac{(1+2\mathrm{q})|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r}{(1+\mathrm{q})^3}\right]^{\frac{1}{r}}\Bigg\},$$

where

$$\int_{\frac{\mathrm{q}}{1+\mathrm{q}}}^{1}\left(\ell-\frac{1}{\mathrm{q}}\right)^p {}_0d_{\mathrm{q}}\ell = (1-\mathrm{q})\left[\begin{array}{c}\sum\limits_{n=0}^{\infty}\mathrm{q}^n\left(\mathrm{q}^n-\frac{1}{\mathrm{q}}\right)^p \\ -\frac{\mathrm{q}}{1+\mathrm{q}}\sum\limits_{n=0}^{\infty}\mathrm{q}^n\left(\mathrm{q}^n\left(\frac{\mathrm{q}}{1+\mathrm{q}}\right)-\frac{1}{\mathrm{q}}\right)^p\end{array}\right].$$

Theorem 2. *Let* $\Psi : P \to \mathbb{R}$ *be a function such that* ${}_{e_1}D_{\mathrm{q}}\Psi$ *is* q*-integrable on* P°. *If* $|\,{}_{e_1}D_{\mathrm{q}}\Psi|^r$ *is strongly preinvex functions of order* $\sigma > 0$ *with modulus* $\mu > 0$ *on* P, *then for* $r \geq 1$, *the following inequality holds:*

$$\left|\Psi(x) - \frac{1}{\zeta(e_2,e_1)}\int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell)\, {}_{e_1}d_{\mathrm{q}}\ell\right| \leq \zeta(e_2,e_1)$$

$$\times\left[\begin{array}{c} [J_1(\mathrm{q},e_1,e_2,x)]^{1-\frac{1}{r}}\left[\begin{array}{c} |{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r J_2(\mathrm{q},e_1,e_2,x) \\ +|{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r J_3(\mathrm{q},e_1,e_2,x)- \\ -\mu\zeta^{\sigma}(e_2,e_1)Q_1(\mathrm{q},e_1,e_2,x) \end{array}\right]^{\frac{1}{r}} \\ +[J_4(\mathrm{q},e_1,e_2,x)]^{1-\frac{1}{r}}\left[\begin{array}{c} |{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r J_5(\mathrm{q},e_1,e_2,x) \\ +|{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r J_6(\mathrm{q},e_1,e_2,x)- \\ -\mu\zeta^{\sigma}(e_2,e_1)Q_2(\mathrm{q},e_1,e_2,x) \end{array}\right]^{\frac{1}{r}} \end{array}\right],$$

where

$$J_1(\mathrm{q},e_1,e_2,x) := \int_0^{\wp_\zeta(x)} \mathrm{q}\ell \,{}_0d_{\mathrm{q}}\ell = \frac{\mathrm{q}}{1+\mathrm{q}}[\wp_\zeta(x)]^2,$$

$$J_2(\mathrm{q},e_1,e_2,x) := \int_0^{\wp_\zeta(x)} (\mathrm{q}\ell-\mathrm{q}\ell^2) \,{}_0d_{\mathrm{q}}\ell = J_1(\mathrm{q},e_1,e_2,x) - J_3(\mathrm{q},e_1,e_2,x),$$

$$J_3(\mathrm{q},e_1,e_2,x) := \int_0^{\wp_\zeta(x)} \mathrm{q}\ell^2 \,{}_0d_{\mathrm{q}}\ell = \frac{\mathrm{q}}{1+\mathrm{q}+\mathrm{q}^2}[\wp_\zeta(x)]^3,$$

$$J_4(\mathrm{q},e_1,e_2,x) := \int_{\wp_\zeta(x)}^1 (\mathrm{q}\ell-1) \,{}_0d_{\mathrm{q}}\ell = \frac{\mathrm{q}}{1+\mathrm{q}}\left(\frac{e_1+\zeta(e_2,e_1)-x}{\zeta(e_2,e_1)}\right)^2,$$

$$J_5(\mathrm{q},e_1,e_2,x) := \int_{\wp_\zeta(x)}^1 (1-\mathrm{q}\ell-\ell+\mathrm{q}\ell^2) \,{}_0d_{\mathrm{q}}\ell = J_4(\mathrm{q},e_1,e_2,x) - J_6(\mathrm{q},e_1,e_2,x),$$

$$J_6(\mathrm{q},e_1,e_2,x) := \int_{\wp_\zeta(x)}^1 (\mathrm{q}\ell^2-\ell) \,{}_0d_{\mathrm{q}}\ell =$$

$$= \frac{1}{(1+\mathrm{q})(1+\mathrm{q}+\mathrm{q}^2)} - \frac{1}{1+\mathrm{q}}[\wp_\zeta(x)]^2 + \frac{\mathrm{q}}{1+\mathrm{q}+\mathrm{q}^2}[\wp_\zeta(x)]^3,$$

and

$$Q_1(\mathrm{q},e_1,e_2,x) := \int_0^{\wp_\zeta(x)} \mathrm{q}\ell^2(1-\ell) \,{}_0d_{\mathrm{q}}\ell =$$

$$= \frac{\mathrm{q}}{1+\mathrm{q}+\mathrm{q}^2}[\wp_\zeta(x)]^3 - \frac{\mathrm{q}}{1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3}[\wp_\zeta(x)]^4,$$

$$Q_2(\mathrm{q},e_1,e_2,x) := \int_{\wp_\zeta(x)}^1 \ell(\mathrm{q}\ell-1)(1-\ell) \,{}_0d_{\mathrm{q}}\ell = -\frac{1}{1+\mathrm{q}}\left(1-[\wp_\zeta(x)]^2\right) +$$

$$+\frac{(1+\mathrm{q})}{1+\mathrm{q}+\mathrm{q}^2}\left(1-[\wp_\zeta(x)]^3\right) - \frac{\mathrm{q}}{1+\mathrm{q}+\mathrm{q}^2+\mathrm{q}^3}\left(1-[\wp_\zeta(x)]^4\right).$$

Proof. Using Lemma 1, strongly preinvexity of order $\sigma > 0$ with modulus $\mu > 0$ of $|{}_{e_1}D_{\mathrm{q}}\Psi|^r$ and the well known power mean inequality, we have

$$\left|\Psi(x) - \frac{1}{\zeta(e_2,e_1)} \int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell)\, {}_{e_1}d_{\mathrm{q}}\ell\right|$$

$$\le \zeta(e_2,e_1)\left[\begin{array}{c} \int_0^{\wp_\zeta(x)} \mathrm{q}\ell|\, {}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))|\, {}_0d_{\mathrm{q}}\ell \\ + \int_{\wp_\zeta(x)}^1 (\mathrm{q}\ell-1)|\, {}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))|\, {}_0d_{\mathrm{q}}\ell \end{array}\right]$$

$$\le \zeta(e_2,e_1)\left[\left(\int_0^{\wp_\zeta(x)} \mathrm{q}\ell\, {}_0d_{\mathrm{q}}\ell\right)^{1-\frac{1}{r}} \left(\int_0^{\wp_\zeta(x)} \mathrm{q}\ell|\, {}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))|^r\, {}_0d_{\mathrm{q}}\ell\right)^{\frac{1}{r}}\right.$$

$$\left. + \left(\int_{\wp_\zeta(x)}^1 (\mathrm{q}\ell-1)\, {}_0d_{\mathrm{q}}\ell\right)^{1-\frac{1}{r}} \left(\int_{\wp_\zeta(x)}^1 (\mathrm{q}\ell-1)|\, {}_{e_1}D_{\mathrm{q}}\Psi(e_1+\ell\zeta(e_2,e_1))|^r\, {}_0d_{\mathrm{q}}\ell\right)^{\frac{1}{r}}\right]$$

$$\le \zeta(e_2,e_1)\Bigg\{\left(\int_0^{\wp_\zeta(x)} \mathrm{q}\ell\, {}_0d_{\mathrm{q}}\ell\right)^{1-\frac{1}{r}}$$

$$\times\left(|\, {}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r \int_0^{\wp_\zeta(x)} \mathrm{q}\ell(1-\ell)\, {}_0d_{\mathrm{q}}\ell + |\, {}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r \int_0^{\wp_\zeta(x)} \mathrm{q}\ell^2\, {}_0d_{\mathrm{q}}\ell - \mu\zeta^\sigma(e_2,e_1)\right.$$

$$\left.\int_0^{\wp_\zeta(x)} \mathrm{q}\ell^2(1-\ell)\, {}_0d_{\mathrm{q}}\ell\right)^{\frac{1}{r}} + \left(\int_{\wp_\zeta(x)}^1 (\mathrm{q}\ell-1)\, {}_0d_{\mathrm{q}}\ell\right)^{1-\frac{1}{r}}$$

$$\times\Big[|\, {}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r \int_{\wp_\zeta(x)}^1 (\mathrm{q}\ell-1)(1-\ell)\, {}_0d_{\mathrm{q}}\ell + |\, {}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r \int_{\wp_\zeta(x)}^1 (\mathrm{q}\ell-1)\,\ell\, {}_0d_{\mathrm{q}}\ell$$

$$-\mu\zeta^\sigma(e_2,e_1)\int_{\wp_\zeta(x)}^1 \ell(\mathrm{q}\ell-1)(1-\ell)\, {}_0d_{\mathrm{q}}\ell\Big]^{\frac{1}{r}}\Bigg\}.$$

The proof of Theorem 2 is completed. ∎

We point out some special cases of Theorem 2.

Corollary 2.1. I. *Taking $r=1$ in Theorem 2, we have*

$$\left|\Psi(x)-\frac{1}{\zeta(e_2,e_1)}\int_{e_1}^{e_1+\zeta(e_2,e_1)}\Psi(\ell)\,{}_{e_1}d_{\mathrm{q}}\ell\right|\leq\zeta(e_2,e_1)\times$$

$$\left[\begin{array}{c}|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|\;\;[J_2(\mathrm{q},e_1,e_2,x)+J_5(\mathrm{q},e_1,e_2,x)]+\\+|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|[J_3(\mathrm{q},e_1,e_2,x)+J_6(\mathrm{q},e_1,e_2,x)]-\\-\mu\zeta^{\sigma}(e_2,e_1)\left(Q_1(\mathrm{q},e_1,e_2,x)+Q_2(\mathrm{q},e_1,e_2,x)\right)\end{array}\right].$$

II. *Taking $r=1$ and $\mathrm{q}\to1^-$ in Theorem 2, we get*

$$\left|\Psi(x)-\frac{1}{\zeta(e_2,e_1)}\int_{e_1}^{e_1+\zeta(e_2,e_1)}\Psi(\ell)d\ell\right|\leq\zeta(e_2,e_1)\times$$

$$\left[\begin{array}{c}|\Psi'(e_1)|\;\;[J_7(e_1,e_2,x)+J_9(e_1,e_2,x)]+\\+|\Psi'(e_2)|[J_8(e_1,e_2,x)+J_{10}(e_1,e_2,x)]-\\-\mu\zeta^{\sigma}(e_2,e_1)\left(Q_3(e_1,e_2,x)+Q_4(e_1,e_2,x)\right)\end{array}\right],$$

where

$$J_7(e_1,e_2,x):=\int_0^{\wp_\zeta(x)}\ell(1-\ell)d\ell=\frac{1}{2}[\wp_\zeta(x)]^2-\frac{1}{3}[\wp_\zeta(x)]^3,$$

$$J_8(e_1,e_2,x):=\int_0^{\wp_\zeta(x)}\ell^2d\ell=\frac{1}{3}[\wp_\zeta(x)]^3,$$

$$J_9(e_1,e_2,x):=\int_{\wp_\zeta(x)}^1(1-2\ell+\ell^2)d\ell=\frac{1}{3}-\wp_\zeta(x)+[\wp_\zeta(x)]^2-\frac{1}{3}[\wp_\zeta(x)]^3,$$

$$J_{10}(e_1,e_2,x):=\int_{\wp_\zeta(x)}^1(\ell-\ell^2)d\ell=\frac{1}{6}-\frac{1}{2}[\wp_\zeta(x)]^2+\frac{1}{3}[\wp_\zeta(x)]^3,$$

and

$$Q_3(e_1,e_2,x):=\int_0^{\wp_\zeta(x)}\ell^2(1-\ell)d\ell=\frac{1}{3}[\wp_\zeta(x)]^3-\frac{1}{4}[\wp_\zeta(x)]^4,$$

$$Q_4(e_1,e_2,x):=\int_{\wp_\zeta(x)}^1\ell(\ell-1)(1-\ell)d\ell$$

$$=-\frac{1}{2}\left(1-[\wp_\zeta(x)]^2\right)+\frac{2}{3}\left(1-[\wp_\zeta(x)]^3\right)-\frac{1}{4}\left(1-[\wp_\zeta(x)]^4\right).$$

III. *Taking* $\mathrm{q} \to 1^-$, $\mu \to 0^+$ *and* $x = \frac{2e_1+\zeta(e_2,e_1)}{2}$ *in Theorem 2, we obtain* [16, Theorem 8].

$$\left|\Psi\left(\frac{2e_1+\zeta(e_2,e_1)}{\zeta(e_2,e_1)}\right) - \frac{1}{\zeta(e_2,e_1)}\int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell)d\ell\right|$$
$$\leq \frac{\zeta(e_2,e_1)}{8}\left[\left(\frac{2|\Psi'(e_1)|^r+|\Psi'(e_2)|^r}{3}\right)^{\frac{1}{r}} + \left(\frac{|\Psi'(e_1)|^r+2|\Psi'(e_2)|^r}{3}\right)^{\frac{1}{r}}\right].$$

IV. *Taking* $r = 1$, $\mathrm{q} \to 1^-$, $\mu \to 0^+$ *and* $x = \frac{2e_1+\zeta(e_2,e_1)}{2}$ *in Theorem 2, we get* [16, Theorem 5].

$$\left|\Psi\left(\frac{2e_1+\zeta(e_2,e_1)}{\zeta(e_2,e_1)}\right) - \frac{1}{\zeta(e_2,e_1)}\int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell)d\ell\right| \leq \frac{\zeta(e_2,e_1)}{8}\left[|\Psi'(e_1)|+|\Psi'(e_2)|\right].$$

IV. *Taking* $\mu \to 0^+$ *and* $x = \frac{qe_1+e_2}{1+\mathrm{q}}$ *and* $\zeta(e_2,e_1) = e_2 - e_1$ *in Theorem 2, we have the following inequalities, for more details, see* [7].

$$\left|\Psi\left(\frac{qe_1+e_2}{1+\mathrm{q}}\right) - \frac{1}{e_2-e_1}\int_{e_1}^{e_2} \Psi(\ell)\,{}_{e_1}d_{\mathrm{q}}\ell\right| \leq (e_2-e_1)\times$$
$$\left[\begin{array}{l} \frac{1}{(1+\mathrm{q})^{3-\frac{3}{r}}}\left[|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r\frac{\mathrm{q}^2(1+\mathrm{q})}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3} + |\,{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r\frac{\mathrm{q}}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3}\right]^{\frac{1}{r}} + \\ \left(\frac{\mathrm{q}}{1+\mathrm{q}}\right)^{3-\frac{3}{r}}\left[|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r\frac{(\mathrm{q}^5+\mathrm{q}^4+\mathrm{q}^3-2\mathrm{q})}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3} + |\,{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r\frac{2\mathrm{q}}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3}\right]^{\frac{1}{r}} \end{array}\right].$$

V. *Taking* $r = 1$, $\mu \to 0^+$, $x = \frac{qe_1+e_2}{1+\mathrm{q}}$ *and* $\zeta(e_2,e_1) = e_2 - e_1$ *in Theorem 2, we obtain* [1, Theorem 13].

$$\left|\Psi\left(\frac{qe_1+e_2}{1+\mathrm{q}}\right) - \frac{1}{e_2-e_1}\int_{e_1}^{e_2} \Psi(\ell)\,{}_{e_1}d_{\mathrm{q}}\ell\right| \leq (e_2-e_1)\times$$
$$\left[|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|\frac{(\mathrm{q}^5+\mathrm{q}^4+2\mathrm{q}^3+\mathrm{q}^2-2\mathrm{q})}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3} + |\,{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|\frac{3\mathrm{q}}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3}\right].$$

VI. *Taking* $\mu \to 0^+$ *and* $x = \frac{e_1+\mathrm{q}(e_1+\zeta(e_2,e_1))}{1+\mathrm{q}}$ *in Theorem 2, we get*

$$\left|\Psi\left(\frac{e_1+\mathrm{q}(e_1+\zeta(e_2,e_1))}{1+\mathrm{q}}\right) - \frac{1}{\zeta(e_2,e_1)}\int_{e_1}^{e_1+\zeta(e_2,e_1)} \Psi(\ell)\,{}_{e_1}d_{\mathrm{q}}\ell\right| \leq \zeta(e_2,e_1)\times$$
$$\left[\begin{array}{l} \left(\frac{\mathrm{q}}{1+\mathrm{q}}\right)^{3-\frac{3}{r}}\left[|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r\frac{\mathrm{q}^3(1+\mathrm{q}^2)}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3} + |\,{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r\frac{\mathrm{q}^4}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3}\right]^{\frac{1}{r}} + \\ \left(\frac{\mathrm{q}}{(1+\mathrm{q})^3}\right)^{1-\frac{1}{r}}\left[|\,{}_{e_1}D_{\mathrm{q}}\Psi(e_1)|^r\frac{(2\mathrm{q}^3+\mathrm{q}^2-\mathrm{q}-1)}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3} + |\,{}_{e_1}D_{\mathrm{q}}\Psi(e_2)|^r\frac{(1+2\mathrm{q}-\mathrm{q}^3)}{(1+\mathrm{q}+\mathrm{q}^2)(1+\mathrm{q})^3}\right]^{\frac{1}{r}} \end{array}\right].$$

3. Conclusion

It is expected that from the results obtained, and following the methodology applied, additional special functions may also be evaluated. Future works can be developed in the area of numerical analysis and even contributions using quantum algorithms, using the theorems and corollaries presented. Finally, our results can be applied to derive some inequalities using special means and error estimations. The authors hope that the ideas and techniques of this paper will inspire interested readers working in this fascinating field.

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