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SOME CLASSIFICATIONS OF SUBRINGS OF THE RING OF CONTINUOUS FUNCTIONS

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Abstract. In this work, for a topological space (X, τ) and the ring of functions $C(X,\tau) = \{f \mid f : (X, \tau) \to (\mathbb{R}, \tau_{st})$ and f is continuous $\}$, where τ_{st} is the standard topology on the real line, for a closed, sublattice and subring $\mathcal{A}$ of $C(X, \tau)$ we investigate sufficient conditions under which is not possible to find a topology $\tau' \subset \tau$ on X such that $\mathcal{A} = C(X, \tau')$.

1. Introduction

Let $\mathbb{R}^X$ is the collection of all real valued functions on a set X. Let f and g be functions defined on a set X, and let $(f+g)(x) = f(x)+g(x)$ and $(f.g)(x) = f(x)\cdot g(x)$ for all $x \in X$. Then $\mathbb{R}^x$ is a ring under these operations of addition and multiplication. Let X is non empty set and τ be a topology on it. We denote the ring of real valued continuous functions from X to $\mathbb{R}$ by $C(X, \tau)$ and the ring of bounded, real valued, continuous functions from X to $\mathbb{R}$ by $C^*(X,\tau)$. We simply write $C(X)$, (resp. $C^*(X)$) instead of $C(X, \tau)$, (resp. $C^*(X, \tau)$) in case there is no change for confusion. The partial ordering on $\mathbb{R}^X$ is defined by; $f \geq g$ if and only if $f(x) \geq g(x)$ for all $x \in X$. For any f, g in $\mathbb{R}^X$, $f \vee g$ and $f \wedge g$ exist in $\mathbb{R}^X$. In fact $(f \vee g)(x) = \max(f(x), g(x))$ for all $x \in X$ and $(f \wedge g)(x) = \min(f(x), g(x))$ for all $x \in X$. Thus $\mathbb{R}^X$ is lattice-ordered ring. For every $f \in \mathbb{R}^X$, since $|f|(x) = \max(f(x), -f(x)) = |f(x)|$ for all $x \in X$, that $|f| = f \vee (-f) \in \mathbb{R}^X$. Since $(f \vee g)(x) = \max\{f(x), g(x)\} = \frac{1}{2}[f(x) + g(x) + |f(x) - g(x)|]$, f, g $C(X)$ that $f \vee g \in C(X)$. Similarly, $f \wedge g \in C(X)$ is obtained. Therefore $C(X)$ is a subalgebra and sublattice of $\mathbb{R}^X$. We know from [3] $C^*(X)$ is a subalgebra and sublattice of $C(X)$. If $(\mathbb{R}, d)$ is a metric space (bounded metric) and X is a topological space, then $\bar{d}(f, g) = sup\{d(f(x), g(x)), x \in X\}$ determines a metric on $\mathbb{R}^X$. The topology determined by this metric is the uniform convergence topology. It is clear that, with this topology, $C(X)$ is closed subring of $\mathbb{R}^X$

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Let X be any set and $\mathcal{A}$ be a family of functions from X to $\mathbb{R}$. We will investigate the necessary and sufficient conditions for a $\tau' \subset \tau$ topology to exist such that $\mathcal{A} = C(X, \tau') \subset C(X, \tau)$.

In this article for a closed, sublattice and subring $\mathcal{A}$ of $C(X, \tau)$ we examine the existence of a topology τ' such that $\mathcal{A} = C(X, \tau')$. In the second section, we give some basic definitions and theorems. In the third section, by defining a special subring, we show some basic properties of this subring. In fourth section we found sufficient conditions under which it is not possible to find a topology $\tau' \subset \tau$ on X such that $\mathcal{A} = C(X, \tau')$.

2. Preliminaries

Throughout this paper, we will define φ as follows, let $i : X \to Y$ be continuous and the map $\varphi : C(Y) \to C(X)$, $f \in C(Y)$ with $\varphi(f) = f \circ i$ is a ring homomorphism.

Theorem 1. *If $\varphi : C(Y) \to C(X)$ is a ring homomorphism, then φ is the lattice homomorphism.*

Proof. The proof is trivial. □

Theorem 2. *If $f : (X, \tau_1) \to (Y, \tau_2)$ is continuous, $\varphi : C(Y) \to C(X)$ is ring homomorphism and $f(X)$ is dense subset of Y, then φ is one to one.*

Proof. The proof is trivial. □

Definition 2.1. [4, 5] Let A be a nonempty subset of topological space (X, τ). The collection $\tau_{|A} = \{A \cap U : U \in \tau\}$ of subsets of A is a topology on A called the *subspace topology.* The topological space (A, τ_A) is said to be a *subspace* of (X, τ).

Let (X, τ) is a topological space and $A \subseteq X$. Then the closure of A is denoted by $\bar{A}$.

Let (X, τ) be a topological space and $x \in X$. Then the open neighborhood system at x is denoted by $V(x) = \{V \in \tau : x \in V\}$.

3. Characteristics of the special subring

In this section, we will investigate and identify the subring of the ring of continuous functions $C(X)$ that possess the properties of being a subalgebra, sublattice, and closed.

Lemma 1. *Let Y is bounded metric space, $X \subset Y$ and $i : X \to Y$ be continuous. If $\overline{i(X)} = Y$ and $\varphi(C(Y)) \subset C(X)$, then $\varphi(C(Y))$ is closed in $C(X)$*

Proof. Firstly we show that φ is isometric. We know that

$$\bar{d}_X(foi, goi) = sup\{d((foi)(x), (goi)(x)), x \in X\}$$

since

$$\{d((foi)(x), (goi)(x)), x \in X\} \subseteq \{d(f(y), g(y)), y \in Y\}$$

then

$$sup\{d((foi)(x),(goi)(x)),x\in X\}\leq sup\{d(f(y),g(y)),y\in Y\}$$

Now let's take $h=|f-g|\wedge 1$, here $\bar{d}_Y(f,g)=sup\{h(y),y\in Y\}$ and so

$$\bar{d}_X((foi)(x),(goi)(x),x\in X)=sup\{(hoi)(x),x\in X\}$$

Suppose that

$$sup\{(hoi)(x),x\in X\}<sup\{(h(y):y\in Y\}$$

Let $sup\{(hoi)(x),x\in X\}=a$. Then, since

$$\bar{d}_Y(f,g)=sup\{h(y),y\in Y\}>a$$

there must exist at least one element, y_0, in the set Y such that $h(y_0)>a$. Since the function $h:Y\to\mathbb{R}$ is continuous at the point $y_0\in Y$, then for each open set V containing $h(y_0)$ there exist an open set U containing Y such that $h(U)\subset V$. Since $i:X\to Y$ is continuous and $\overline{i(X)}=Y$, then for every open set $U\subset Y$, $U\cap i(X)\neq\emptyset$. Let's take a point $y'\in U\cap i(X)$ and $V=(a,+\infty)$. Since $h(U)\subset V$, then $h(y')>a$ for $y'\in U$. Since $i:X\to Y$ is continuous, then there is a $x'\in X$ such that $i(x')=y'$. Since

$$sup\{(hoi)(x'),x'\in X\}=a$$

then $(hoi)(x')\leq a$. However, this contradicts the fact that $h(y')>a$, as $(hoi)(x')=h(i(x'))=h(y')$. Thus, it is shown that

$$sup\{(h(y):y\in Y\}=sup\{(hoi)(x),x\in X\}$$

In other words,

$$\bar{d}_Y(f,g)=\bar{d}_X(foi,goi)$$

and so φ is an isometry.

Now let's show that $\varphi(C(Y))$ is closed. Let's take a sequence $(g_n)\in\varphi(C(Y))$. Since $g_n\to g$ is a convergent sequence, then it is a Cauchy sequence. Then there exists an $n\in\mathbb{N}$ such that $\bar{d}_X(g_m,g_n)<\varepsilon$ for every $\varepsilon>0$ and every $m,n>n_0$. Also, since $(f_noi)=(g_n)$ and $(f_moi)=(g_m)$, then $\bar{d}_X(f_moi,f_noi)<\varepsilon$. Due to the isometry, $\bar{d}_Y(f_m,f_n)<\varepsilon$ is provided. Then, (f_n) is a Cauchy sequence. Since $C(Y)$ is complete, then there is a $f\in C(Y)$ such that $f_n\to f$. Because of the isometry,

$$f_n\to f\Leftrightarrow\overline{d_Y}(f_n,f)\to 0$$

and as a result $\bar{d}_X(f_noi,foi)\to 0$. Since $f_noi\to foi$ and $(f_noi)=(g_n)$, then $g_n\to foi$. Simultaneously, since $g_n\to g$, then $g=foi$. Thus, $\varphi(f)=foi=g$, $g\in\varphi(C(Y))$. This show that $\varphi(C(Y))$ is closed. □

Theorem 3. *Let X and Y be two topological spaces, and let $i:X\to Y$ be a continuous function such that $i(X)$ is a dense subset of Y and $Y\neq i(X)$. Additionally, let*

$$i:X\to i(X)$$

be a homeomorphism and let

$$I = \{f \circ i : f(y) = 0, \forall y \in (Y - i(X))\} \subseteq \varphi(C(Y)) \subseteq C(X)$$

then,

(1) $I \cap C^*(X)$ is ideal of $C^*(X)$.
(2) If $g \in I, h \in C^*(X)$ then $g.h \in I$.

Proof.
(1) Let take $f \circ i \in I \cap C^*(X)$ and $g \in C^*(X)$. Assume that $(f \circ i).g = h$. We defined as

$$\overline{h}(y) = \begin{cases} 0, & y \in Y - i(X) \\ ((f \circ i).g)(x), & i(x) = y, y \in i(X) \end{cases}$$

Let us show the continuity of the $\overline{h}$ for three different cases.
Case 1 Let $y_0 \in Y - i(X)$. Then, $\overline{h}(y_0) = 0$. Let's take $(-\varepsilon, \varepsilon)$ as a neighborhood of $0 \in \mathbb{R}$ for every $\varepsilon > 0$. Since $g \in C^*(X)$, there exists a $r \in \mathbb{R}$ such that $r > 0$ and $|g(x)| \leq r$ for all $x \in X$. Simultaneously, since the function f is continuous, there exists a neighborhood U of y_0 such that $f(U) \subseteq \left(\frac{-\varepsilon}{r}, \frac{\varepsilon}{r}\right)$. For $y \in U$, if $y \in i(X)$, then $\overline{h}(y) = ((f \circ i).g)(x)$, where $i(x) = y$, and thus $f(y) \cdot g(x) \in (-\varepsilon, \varepsilon)$. Therefore, $\overline{h}(U) \subseteq (-\varepsilon, \varepsilon)$ becomes true. Thus, $\overline{h}$ is continuous.
Case 2 Let $y_0 \in i(X))$ and $f(y_0) = 0$. Since $g \in C^*(X)$, there exists a $r \in \mathbb{R}$ such that $r > 0$ and $|g(x)| \leq r$ for all $x \in X$. Now let's show that $\overline{h}(U) \subseteq (-\varepsilon, \varepsilon)$. For $y_0 \in U$, if $y_0 \in Y - i(X)$, then $\overline{h}(y_0) = 0 \in (-\varepsilon, \varepsilon)$. For $y_0 \in U$, if $y_0 \in i(X)$, then $\overline{h}(y_0) = ((f \circ i).g)(x)$ and $i(x) = y_0$, and thus $f(y_0) \cdot g(x) \in (-\varepsilon, \varepsilon)$. Therefore, $\overline{h}(U) \subseteq (-\varepsilon, \varepsilon)$ holds true. Thus, the function $\overline{h}$ is continuous at the point y_0.
Case 3 Let $f(y_0) \neq 0$. Assume that $f(y_0) = m$. If $i : X \to i(X)$ is a bijective homeomorphism and $i(x_0) = y_0$, and $g(x_0) = n$, then $\overline{h}(y_0) = ((f \circ i)g)(x) = f(y).g(x) = m.n$ holds true. Now we can show that function $\overline{h}$ is continuous at point y_0. Since $g \in C^*(X)$, there exists a $r \in \mathbb{R}$ such that $r > 0$ and $|g(x)| \leq r$ for all $x \in X$. Additionally, since the function g is continuous at point x_0, there exists a neighborhood of x_0 called V such that $g(V) \subseteq \left(n - \frac{\varepsilon}{2(|m|+1)}, n + \frac{\varepsilon}{2(|m|+1)}\right)$ for each $\varepsilon > 0$. Simultaneously, since the function f is also continuous at the point y_0, then $y_0 \in i(X)$ and there is an open set $U \subseteq Y$ such that $|m| < \frac{\varepsilon}{r}$, $f(U) \subseteq \left(m - \frac{\varepsilon}{r}, m + \frac{\varepsilon}{r}\right)$ and $0 \notin \left(m - \frac{\varepsilon}{r}, m + \frac{\varepsilon}{r}\right)$. Therefore, $y_0 \in U$. Now we show that for any $\varepsilon > 0$, $\overline{h}(U) \subseteq (m.n - \varepsilon, m.n + \varepsilon)$. Since i is continuous, then for $x_0 \in X$ there is a open set $U \subseteq i(X)$ such that $i(x_0) = y_o \in U$. We know that the function $i^{-1} : i(X) \to X$ is also continuous. Let's take a W neighborhood of the point y_0 such that $i^{-1}(W) = M$. If $W = U$ is chosen, then $i(M \cap V) = i(M) \cap i(V) = U \cap i(V) = N$ and $N \subseteq U$ is an open neighborhood of the point y_0. Let's take a $y \in N$. Since $N \subseteq i(M \cap V)$, there is a $x \in M \cap V$ such that $i(x) = y$. Then, since $f(U) \subseteq \left(m - \frac{\varepsilon}{2r}, m + \frac{\varepsilon}{2r}\right)$, then $f(y) \in \left(m - \frac{\varepsilon}{2r}, m + \frac{\varepsilon}{2r}\right)$. Since $g(x) \in \left(n - \frac{\varepsilon}{2(|m|+1)}, n + \frac{\varepsilon}{2(|m|+1)}\right)$ then,

$$\begin{aligned} |g(x).f(y) - m.n| &= |g(x).(f(x) - m) + (g(x) - n).m| \\ &\leq |g(x)| . |f(y) - m| + |g(x) - n| \, |m| \end{aligned}$$

$$\begin{aligned} &< r.\left|f(y)-m\right|+\left|g(x)-n\right|.(|m|+1) \\ &\leq r.\frac{\varepsilon}{2r}+\frac{\varepsilon}{2.\left(|m|+1\right)}.\left(|m|+1\right) \\ &= \varepsilon \end{aligned}$$

Thus, $\overline{h}(N) \subseteq (m.n-\varepsilon, m.n+\varepsilon)$ and so $\overline{h}$ is continuous.

Now we show that N is a open subset of Y. We know $N = U \cap i(V)$ and i is a homomorphism. If $|m|< \frac{\varepsilon}{r}$, then there exists an open set $U \subseteq Y$, with $U \subseteq i(X)$, such that the image of f over U is contained in the interval $(m-\frac{\varepsilon}{r}, m+\frac{\varepsilon}{r})$ for any point $y_0 \in U$. For $i(x_0) = y_0 \in U$ since both $i^{-1}(U)$ and $V \subseteq X$ are an open neighborhood of the point x_0, then $V \cap i^{-1}(U)$ is also an open neighborhood of point x_0. Now, let's take an image of $V \cap i^{-1}(U)$.

$$\begin{aligned} &i\left(V \cap i^{-1}(U)\right) \\ &= i(V) \cap i(i^{-1}(U) \\ &= i(V) \cap U \\ &= (i(V) \cap U) \\ &\subseteq U^{open} \\ &\subseteq Y \end{aligned}$$

Thus, $i(V) \cap U$ is open subset of Y.
(2) Since the boundedness of the function $f \circ i$ was not used in (1), the proof is the same as before. □

Example 1. Let X be a locally compact but non-compact space, and let X^t be a one-point compactification of X. Let $\varphi(C(X^t))$ be defined as follows:

$\varphi(C(X^t)) = \{g : g \in C(X), \exists L \in \mathbb{R}$ such that $\forall \varepsilon > 0, g(X-K) \subseteq (L-\varepsilon, L+\varepsilon)$,

for some compact set $K \subset X$.
Then:

(1) $I = \{\varphi(f) : f(\infty) = 0\}$ is ideal of $\varphi(C(X^t))$.
(2) I is an ideal of $C^*(X)$.

Theorem 4. *Let X and Y be two topological space, $i : X \to Y$ is continuous and $i(X)$ is a dense subset of Y. If $Y \neq i(X)$, $i : X \to i(X)$ is a homeomorphism and $I = \{f \circ i : f(y) = 0, \forall y \in (Y - i(X))\} \subseteq \varphi(C(Y)) \subseteq C(X)$, then I is closed in $C(X)$.*

Proof. Let's take a sequence $(g_n) \in I$. Since $g_n \to g$ is a convergent sequence, then it is a Cauchy sequence. Then, there is a $n \in \mathbb{N}$ such that $\overline{d}_X(g_m, g_n) < \varepsilon$ for every $\varepsilon > 0$ and every $m, n > n_0$. Since $(f_n oi) = (g_n)$ and $(f_m oi) = (g_m)$, then $\overline{d}_X(f_m oi, f_n oi) < \varepsilon$. Because of the isometry in Lemma 1, the inequality $\overline{d}_Y(f_m, f_n) < \varepsilon$ holds. Thus, (f_n) is a Cauchy sequence. Since $C(Y)$ is complete, then there is a $f \in C(Y)$ such that $f_n \to f$. Since $(f_n \circ i) \in I$, it follows that $f_n(y) = 0$ for all $y \in Y - i(X)$. Thus $f_n \to f \Leftrightarrow \overline{d}_y(f_n, f) \to 0$. Conversely, suppose that $f(y) = m$, where $m \neq 0$ and there exists $y \in Y$ such that $y \notin i(X)$.

Then, we have $|f_n(y) - f(y)| = |0 - m| = |-m|$. If we choose $\varepsilon = \frac{|m|}{2}$, the inequality $|f_n(y) - f(y)| < \varepsilon$ is not satisfied. As a result, $f(y) = 0$ for all $y \in Y$ such that $y \notin i(X)$. Because of the isometry, $\overline{d}_x(f_n \circ i, f \circ i) \to 0$. Since $f_n \circ i \to f \circ i$ and $(f_n \circ i) = (g_n)$, we have that $g_n \to f \circ i$. Simultaneously, since $g_n \to g$, we have that $g = f \circ i$. As $f \circ i \in I$, it follows that g also belongs to I. Hence, I is a closed subspace of $C(X)$. □

Theorem 5. *Let X and Y be two topological spaces, $i : X \to Y$ is continuous and $i(X)$ is a dense subset of Y. If $Y \neq i(X)$, $i : X \to i(X)$ is a homeomorphism and $I = \{f \circ i : f(y) = 0, \forall y \in (Y - i(X))\} \subseteq \varphi(C(Y)) \subseteq C(X)$, then I is sublattice of $C(X)$.*

Proof. Let take $f \circ i, g \circ i \in I$.

$$(f \circ i) \vee (g \circ i) = \frac{1}{2}\left((f \circ i) + (g \circ i) + |(f \circ i) - (g \circ i)|\right)$$

We know that for every $y \in Y - i(X)$, $f(y) = g(y) = 0$. Since

$$(f \circ i)(x) + (g \circ i)(x) = ((f + g) \circ i)(x)$$

and

$$(f \circ i)(x) - (g \circ i)(x) = ((f - g) \circ i)(x)$$

then $(f + g)(y) = 0$ and $(f - g)(y) = 0$ for every $y \in Y - i(X)$. Thus,

$$(f \circ i) \vee (g \circ i) \in I$$

Similarly, we can show that $(f \circ i) \wedge (g \circ i) \in I$. Hence, I is sub-lattice of $C(X)$. □

Lemma 2. *If $C(X, \tau) - C(X, \tau') \neq \emptyset$ then $C^*(X, \tau) - C^*(X, \tau') \neq \emptyset$*

Proof. Let $f \in C(X, \tau) - C(X, \tau')$. There is at least one point $x_0 \in X$ such that f is not $\tau' - \tau_{st}$- continuous at this point. Let $f(x_0) = a$, $a \in \mathbb{R}$. For every neighborhood U of x_0, there exists $\varepsilon_0 > 0$ such that $f(U) \not\subset (a - \varepsilon_0, a + \varepsilon_0)$. By defining

$$g = (f \vee (a - \varepsilon_0)) \wedge (a + \varepsilon_0)$$

the function g becomes $\tau' - \tau_{st}$ discontinuous and bounded at the point x_0. Now, let's show this. Suppose that the function g is $\tau' - \tau_{st}$-continuous at the point x_0. Let us set $g(x_0) = a$ and consider $(a - \varepsilon_0, a + \varepsilon_0)$ as a neighborhood of a. Then, there exists a neighborhood U_1 of x_0 such that $g(U_1) \subset (a - \varepsilon_0, a + \varepsilon_0)$. Thus,

$$\begin{aligned} g(U_1) &= (f(U_1) \vee (a - \varepsilon_0)) \wedge (a + \varepsilon_0) \\ &\subset (a - \varepsilon_0, a + \varepsilon_0). \end{aligned}$$

For every $x \in U_1$,

$$(f(x) \vee (a - \varepsilon_0)) \wedge (a + \varepsilon_0) \in (a - \varepsilon_0, a + \varepsilon_0)$$

and so

$$\min\{\max\{(f(x), (a - \varepsilon_0))\}, (a + \varepsilon_0)\} \in (a - \varepsilon_0, a + \varepsilon_0)$$

This can only be achieved by ensuring;

$$\max\{(f(x), (a - \varepsilon_0))\} < (a + \varepsilon_0)$$

and

$$\max\{(f(x),(a-\varepsilon_0))\} > (a-\varepsilon_0)$$

From this, we conclude that $f(x) < a+\varepsilon_0$ and $f(x) > a-\varepsilon_0$, which implies $f(x) \in (a-\varepsilon_0, a+\varepsilon_0)$ for all x. Hence, we reach a contradiction because $f(U_1) \subset (a-\varepsilon_0, a+\varepsilon_0)$ for every $x \in U_1$.

□

4. ∗−IDEAL

In this section, we will define a specific ideal and establish the conditions under which it is not possible to find a topology $\tau' \subset \tau$ such that $\mathcal{A} = C(X,\tau')$, by using this ideal.

Definition 4.1. Let $I \subset C(X)$ be a subalgebra, sub-lattice and closed. If for every $f \in I$ and $g \in C^*(X)$, $f, g \in I$, then I is called an $*-$ *ideal.*

Definition 4.2. For a function $f : A \to B$ to be an into function there will be one or more elements in set B that do not have a pre-image in set A.

Lemma 3. *If X is a proper open subset of Y, $i : X \to Y$ is an into function, and Y is a T_4 space, then there exists a bounded function $f \in C(Y)$ such that $f(x) \neq 0$ for all $x \in X$ and $I = \{f \circ i : f \in C(Y), f(Y-X) = 0\}$.*

Proof. Let $A = \{x\}$ be a singleton set for a $x \in X$. A and $Y - X$ are closed. Furthermore, let $g : (Y-X) \cup A \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} 1, & t = x \\ 0, & t \in Y - X \end{cases}$$

Now let us show that g is continuous. Let F be a closed subset of $\mathbb{R}$. Since

$$g^{-1}(F) = \begin{cases} A \cup (Y-X), & 0,1 \in F \\ Y-X, & 0 \in F, 1 \notin F \\ A, & 0 \notin F, 1 \in F \\ \emptyset, & 0,1 \notin F \end{cases}$$

then $g^{-1}(F)$ is closed. Thus g is continuous. Since the function g is bounded and $A \cup (Y-X)$ is closed, then according to Tietze's Expansion Theorem g has a continuous and bounded expansion of $f : Y \to \mathbb{R}$. Thus, for every $x \in X$ there is a bounded function f such that $f(x) \neq 0$. □

Theorem 6. *Let $\mathcal{A}$ be a subalgebra, sublattice, and closed subset of $C(X,\tau)$ with $\mathcal{A} \neq C(X,\tau)$. If $I \subseteq \mathcal{A}$ is a $*$-ideal and there exists $f \in I$ such that $f(x) \neq 0$ for all $x \in X$, then it is not possible to find a topology $\tau' \subset \tau$ such that $\mathcal{A} = C(X,\tau')$.*

Proof. Suppose that, there exists a topology $\tau' \subset \tau$ such that $\mathcal{A} = C(X,\tau')$. Let's take a $g \in C^*(X,\tau) - C^*(X,\tau')$. Then there is at least one point $x_0 \in X$ such that g is not $\tau' - \tau_{st}$-continuous at this point. Let $g(x_0) = n$, $n \in \mathbb{R}$. Let $f \in I$ be a function such that f is τ'-τ-continuous and $f(x_0) \neq 0$. We can choose $f(x_0) = m > 0$, since if $f(x_0) < 0$, we can select $-f$ instead of f. There exists an $\varepsilon_0 > 0$ such that $m - \varepsilon_0 > 0$. Since f is $\tau' - \tau_{st}$- continuous, there is a neighborhood $U \in \tau'$ of x_0 such that $f(U) \subset (m - \varepsilon_0, m + \varepsilon_0)$. Then, $f(x) > 0$ for all x. The function $\frac{1}{f}$ is $\tau' - \tau_{std}$-continuous on the set U. The product of the functions $f.g$ and $\frac{1}{f}$ is $\tau' - \tau_{std}$-continuous on the set U. However, since g is not continuous at x_0, there exists a neighborhood V of $g(x_0)$ such that $g(U) \not\subset V$ for any neighborhood U of x_0. Since the restriction $\left(f.g.\frac{1}{f}\right)_{|U} = g_{|U}$ is continuous at point x_0 and $U \in \tau'$, then the function $\left(f.g.\frac{1}{f}\right)_{|U} = g_{|U}$ is $\tau' - \tau_{std}$-continuous at point x_0. Then $\left(f.g.\frac{1}{f}\right)(x_0) = g(x_0) = n$. There is a neighborhood W of x_0 such that $g(W) \subset V$ for the V neighborhood of n. Since $x_0 \in W \in \tau'$ and $x_0 \in U \in \tau'$, it follows that $x_0 \in U \cap W = U_2$. So, x_0 has a neighborhood $U_2 \in \tau'$ such that $g(U_2) \subset V$, which contradicts the fact that $g(U) \not\subset V$. □

The following example shows that it is not possible to find a $\tau' \subset \tau$ such that $\mathcal{A} = C(X, \tau')$.

Example 2. Let $\mathcal{A} = C^*(X) \subset C(X)$. In this case, there is no $\tau' \subset \tau$ such that $C^*(X) = C(X, \tau')$.

5. Question

If a topology τ' does not exist such that $\mathcal{A} = C(X, \tau')$, is there a larger space Y containing X such that $\mathcal{A} = C(Y)$?

This problem is open.

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