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TRAPEZIUM–TYPE INEQUALITIES VIA GENERALIZED INTEGRAL OPERATORS FOR STRONGLY CONVEX FUNCTIONS AND THEIR APPLICATIONS

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Abstract. The aim of this paper is to introduce an identity for a generalized integral operator via differentiable function. By using this integral equation, some new bounds on Hermite–Hadamard type integral inequalities for differentiable functions are derived that are in absolute value at certain powers strongly convex with positive modulus c. By taking suitable choices of function, some interesting results are obtained. At the end, some applications of presented results to special means and new error estimates for the trapezium formula have been analyzed. The ideas and techniques of this paper may stimulate further research in different areas of pure and applied sciences.

1. Introduction and Preliminaries

The following inequality, known as Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $\zeta_1, \zeta_2 \in I$ with $\zeta_1 < \zeta_2$. Then the following inequality holds:*

$$f\left(\frac{\zeta_1+\zeta_2}{2}\right) \leq \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(x)dx \leq \frac{f(\zeta_1)+f(\zeta_2)}{2}. \tag{1.1}$$

This inequality (1.1) is also known as trapezium inequality.

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis and related disciplines. Authors of recent decades have studied (1.1) in the premises of newly introduced concepts

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due to a significant role of convex functions in optimization theory. For more results in this direction, we refer to [1]–[24] and references stated therein.

The aim of this paper is to establish trapezium type generalized integral inequalities for strongly convex functions with positive modulus c via generalized integral operators. Interestingly, the special cases of presented results, are fractional integral inequalities. Therefore, it is important to summarize and unify the study of fractional integrals.

Let us recall some special functions and evoke some basic definitions as follows:

Definition 1.1. [8] *A function $f : [0, \zeta] \to \mathbb{R}$ is called m-convex with $m \in (0, 1]$, if for any $x, y \in [0, \zeta]$ and $t \in [0, 1]$, we have*

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y).$$

Definition 1.2. [8] *A function $f : I \to \mathbb{R}$ is called strongly m-convex with $m \in (0, 1]$ and positive modulus c, if for any $x, y \in I$ and $t \in [0, 1]$, we have*

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y) - cmt(1 - t)(y - x)^2.$$

Strongly convex functions have been introduced by Polyak. Strongly m–convex functions play an important role for proving the convergence of a gradient type algorithm for minimizing a function. This class of functions has a significant role solving several problems that arise in in optimization theory and mathematical economics, see [8] and references therein.

Taking $m = 1$ in Definition 1.2, we get the following useful definition for our main results.

Definition 1.3. *A function $f : I \to \mathbb{R}$ is called strongly convex with positive modulus c, if for any $x, y \in I$ and $t \in [0, 1]$, we have*

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) - ct(1 - t)(y - x)^2.$$

Definition 1.4. *Let $f \in L[\zeta_1, \zeta_2]$ (the set of all integrable functions on $[\zeta_1, \zeta_2]$). Then Riemann–Liouville fractional integrals of order $\alpha > 0$ with $\zeta_1 \geq 0$ are defined by*

$$J^{\alpha}_{\zeta_1^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_{\zeta_1}^{x} (x - t)^{\alpha-1} f(t)dt, \quad x > \zeta_1$$

and

$$J^{\alpha}_{\zeta_2^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\zeta_2} (t - x)^{\alpha-1} f(t)dt, \quad \zeta_2 > x.$$

For $\alpha = 1$, fractional integrals become classical integrals.

Define a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\phi(t)}{t} dt < \infty, \tag{1.2}$$

$$\frac{1}{\mathbf{A}} \leq \frac{\phi(s)}{\phi(r)} \leq \mathbf{A} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2, \tag{1.3}$$

$$\frac{\phi(r)}{r^2} \leq \mathbf{B}\frac{\phi(s)}{s^2} \text{ for } s \leq r, \tag{1.4}$$

$$\left|\frac{\phi(r)}{r^2} - \frac{\phi(s)}{s^2}\right| \leq \mathbf{C}|r-s|\frac{\phi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2, \tag{1.5}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C} > 0$ are independent of $r, s > 0$. If $\phi(r)r^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\phi(r)}{r^\beta}$ is decreasing for some $\beta \geq 0$, then ϕ satisfies (1.2)–(1.5), see [12]. Therefore, the left-sided and right-sided generalized integral operators are defined as follows:

$${}_{\zeta_1^+}I_\phi f(x) = \int_{\zeta_1}^{x} \frac{\phi(x-t)}{x-t} f(t)dt, \quad x > \zeta_1$$

and

$${}_{\zeta_2^-}I_\phi f(x) = \int_{x}^{\zeta_2} \frac{\phi(t-x)}{t-x} f(t)dt, \quad x < \zeta_2.$$

The most important feature of generalized integrals is that; they produce Riemann–Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals, etc., see [25].

Motivated by the above literatures, the main objective of this paper is to introduce in Section 2 an identity for a generalized integral operator via differentiable function. By using the established identity as an auxiliary result, some new estimates on trapezium-type integral inequalities for differentiable functions that are in absolute value at certain powers strongly convex with positive modulus c are obtained. It is worth mentioning that some new fractional integral inequalities have been deduced by taking various suitable choices of function. In Section 3, some applications to special means and new error estimates for the trapezium formula are given. The ideas and techniques of this paper may stimulate further research in different areas of pure and applied sciences.

2. Main results

Throughout this paper, let $P := [\zeta_1, \zeta_2]$ with $\zeta_1 < \zeta_2$ and c a positive real number. Also for sake of brevity, we define

$$\Upsilon(t) := \int_0^t \frac{\phi\left(\frac{(\zeta_2-\zeta_1)u}{4}\right)}{u} du < +\infty, \quad \forall\, t \in [0,1].$$

For establishing some new results regarding general fractional integrals we need to prove the following lemma.

Lemma 1. *Let $f : P \to \mathbb{R}$ be a differentiable mapping on (ζ_1, ζ_2). If $f' \in L(P)$, then the following identity for generalized fractional integrals holds:*

$$\frac{1}{4}\Big[f\left(\frac{3\zeta_1+\zeta_2}{4}\right) + f\left(\frac{\zeta_1+\zeta_2}{2}\right) + f\left(\frac{\zeta_1+3\zeta_2}{4}\right) + f(\zeta_2)\Big]$$

$$-\frac{1}{4\Upsilon(1)}\Big[{}_{\left(\frac{3\zeta_1+\zeta_2}{4}\right)^-}I_\phi f(\zeta_1) + {}_{\left(\frac{\zeta_1+\zeta_2}{2}\right)^-}I_\phi f\left(\frac{3\zeta_1+\zeta_2}{4}\right)$$
$$+ {}_{\left(\frac{\zeta_1+3\zeta_2}{4}\right)^-}I_\phi f\left(\frac{\zeta_1+\zeta_2}{2}\right) + {}_{\zeta_2^-}I_\phi f\left(\frac{\zeta_1+3\zeta_2}{4}\right)\Big]$$
$$= \frac{1}{4\Upsilon(1)}\int_{\zeta_1}^{\zeta_2} p_\Upsilon(t) f'(t)dt,$$

where

$$p_\Upsilon(t) := \begin{cases} \Upsilon\left(\frac{4(t-\zeta_1)}{\zeta_2-\zeta_1}\right), & t\in\left[\zeta_1, \frac{3\zeta_1+\zeta_2}{4}\right); \\ \Upsilon\left(\frac{4\left(t-\frac{3\zeta_1+\zeta_2}{4}\right)}{\zeta_2-\zeta_1}\right), & t\in\left[\frac{3\zeta_1+\zeta_2}{4}, \frac{\zeta_1+\zeta_2}{2}\right); \\ \Upsilon\left(\frac{4\left(t-\frac{\zeta_1+\zeta_2}{2}\right)}{\zeta_2-\zeta_1}\right), & t\in\left[\frac{\zeta_1+\zeta_2}{2}, \frac{\zeta_1+3\zeta_2}{4}\right); \\ \Upsilon\left(\frac{4\left(t-\frac{\zeta_1+3\zeta_2}{4}\right)}{\zeta_2-\zeta_1}\right), & t\in\left[\frac{\zeta_1+3\zeta_2}{4}, \zeta_2\right]. \end{cases}$$

We denote

$$T_{f,\Upsilon}(\zeta_1,\zeta_2) := \frac{1}{4\Upsilon(1)}\int_{\zeta_1}^{\zeta_2} p_\Upsilon(t) f'(t)dt. \tag{2.1}$$

Proof. Integrating by parts (2.1) and changing the variables of integration, we have

$$T_{f,\Upsilon}(\zeta_1,\zeta_2) = \frac{1}{4\Upsilon(1)}$$
$$\times\Big[\int_{\zeta_1}^{\frac{3\zeta_1+\zeta_2}{4}} \Upsilon\left(\frac{4(t-\zeta_1)}{\zeta_2-\zeta_1}\right) f'(t)dt + \int_{\frac{3\zeta_1+\zeta_2}{4}}^{\frac{\zeta_1+\zeta_2}{2}} \Upsilon\left(\frac{4\left(t-\frac{3\zeta_1+\zeta_2}{4}\right)}{\zeta_2-\zeta_1}\right) f'(t)dt$$
$$+\int_{\frac{\zeta_1+\zeta_2}{2}}^{\frac{\zeta_1+3\zeta_2}{4}} \Upsilon\left(\frac{4\left(t-\frac{\zeta_1+\zeta_2}{2}\right)}{\zeta_2-\zeta_1}\right) f'(t)dt + \int_{\frac{\zeta_1+3\zeta_2}{4}}^{\zeta_2} \Upsilon\left(\frac{4\left(t-\frac{\zeta_1+3\zeta_2}{4}\right)}{\zeta_2-\zeta_1}\right) f'(t)dt\Big]$$
$$= \frac{1}{4\Upsilon(1)}\Big[\Upsilon\left(\frac{4(t-\zeta_1)}{\zeta_2-\zeta_1}\right) f(t)\Big|_{\zeta_1}^{\frac{3\zeta_1+\zeta_2}{4}} - \int_{\zeta_1}^{\frac{3\zeta_1+\zeta_2}{4}} \frac{\phi(t-\zeta_1)}{t-\zeta_1} f(t)dt$$
$$+\Upsilon\left(\frac{4\left(t-\frac{3\zeta_1+\zeta_2}{4}\right)}{\zeta_2-\zeta_1}\right) f(t)\Big|_{\frac{3\zeta_1+\zeta_2}{4}}^{\frac{\zeta_1+\zeta_2}{2}} - \int_{\frac{3\zeta_1+\zeta_2}{4}}^{\frac{\zeta_1+\zeta_2}{2}} \frac{\phi\left(t-\frac{3\zeta_1+\zeta_2}{4}\right)}{t-\frac{3\zeta_1+\zeta_2}{4}} f(t)dt$$
$$+\Upsilon\left(\frac{4\left(t-\frac{\zeta_1+\zeta_2}{2}\right)}{\zeta_2-\zeta_1}\right) f(t)\Big|_{\frac{\zeta_1+\zeta_2}{2}}^{\frac{\zeta_1+3\zeta_2}{4}} - \int_{\frac{\zeta_1+\zeta_2}{2}}^{\frac{\zeta_1+3\zeta_2}{4}} \frac{\phi\left(t-\frac{\zeta_1+\zeta_2}{2}\right)}{t-\frac{\zeta_1+\zeta_2}{2}} f(t)dt$$
$$+\Upsilon\left(\frac{4\left(t-\frac{\zeta_1+3\zeta_2}{4}\right)}{\zeta_2-\zeta_1}\right) f(t)\Big|_{\frac{\zeta_1+3\zeta_2}{4}}^{\zeta_2} - \int_{\frac{\zeta_1+3\zeta_2}{4}}^{\zeta_2} \frac{\phi\left(t-\frac{\zeta_1+3\zeta_2}{4}\right)}{t-\frac{\zeta_1+3\zeta_2}{4}} f(t)dt\Big]$$
$$= \frac{1}{4}\Big[f\left(\frac{3\zeta_1+\zeta_2}{4}\right) + f\left(\frac{\zeta_1+\zeta_2}{2}\right) + f\left(\frac{\zeta_1+3\zeta_2}{4}\right) + f(\zeta_2)\Big]$$

$$-\frac{1}{4\Upsilon(1)}\Big[{}_{\left(\frac{3\zeta_1+\zeta_2}{4}\right)^-}I_\phi f(\zeta_1)+{}_{\left(\frac{\zeta_1+\zeta_2}{2}\right)^-}I_\phi f\left(\frac{3\zeta_1+\zeta_2}{4}\right)$$

$$+{}_{\left(\frac{\zeta_1+3\zeta_2}{4}\right)^-}I_\phi f\left(\frac{\zeta_1+\zeta_2}{2}\right)+{}_{\zeta_2^-}I_\phi f\left(\frac{\zeta_1+3\zeta_2}{4}\right)\Big].$$

This completes the proof of Lemma 1 . □

Remark 2.1: If we take $\phi(t)=t$ in Lemma 1, then we get the following new trapezium type integral identity:

$$T_f(\zeta_1,\zeta_2)=\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}q(t)f'(t)dt$$

$$=\frac{1}{4}\Big[f\left(\frac{3\zeta_1+\zeta_2}{4}\right)+f\left(\frac{\zeta_1+\zeta_2}{2}\right)+f\left(\frac{\zeta_1+3\zeta_2}{4}\right)+f(\zeta_2)\Big]-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt,$$

where

$$q(t):=\begin{cases} t-\zeta_1, & t\in\left[\zeta_1,\frac{3\zeta_1+\zeta_2}{4}\right);\\ t-\frac{3\zeta_1+\zeta_2}{4}, & t\in\left[\frac{3\zeta_1+\zeta_2}{4},\frac{\zeta_1+\zeta_2}{2}\right);\\ t-\frac{\zeta_1+\zeta_2}{2}, & t\in\left[\frac{\zeta_1+\zeta_2}{2},\frac{\zeta_1+3\zeta_2}{4}\right);\\ t-\frac{\zeta_1+3\zeta_2}{4}, & t\in\left[\frac{\zeta_1+3\zeta_2}{4},\zeta_2\right].\end{cases}$$

Theorem 2. *Let $f:P\to\mathbb{R}$ be a differentiable mapping on (ζ_1,ζ_2). If $|f'|^q$ is strongly convex on P for $q>1$ and $p^{-1}+q^{-1}=1$, then the following inequality for generalized fractional integrals holds:*

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\le\frac{(\zeta_2-\zeta_1)}{16\sqrt[q]{2}\Upsilon(1)}\sqrt[p]{B_\Upsilon(p)}$$

$$\times\Big\{\sqrt[q]{|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+|f'(\zeta_1)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$

$$+\sqrt[q]{|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$

$$+\sqrt[q]{|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$

$$+\sqrt[q]{|f'(\zeta_2)|^q+|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}\Big\},$$

where

$$B_\Upsilon(p):=\int_0^1[\Upsilon(t)]^p dt.$$

Proof. From Lemma 1, strongly convexity with positive modulus c of $|f'|^q$, Hölder inequality and properties of the modulus, we obtain that

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)| \leq \frac{1}{4\Upsilon(1)}\int_{\zeta_1}^{\zeta_2} p_\Upsilon(t)|f'(t)|dt$$

$$\times\Big[\int_{\zeta_1}^{\frac{3\zeta_1+\zeta_2}{4}}\Upsilon\left(\frac{4(t-\zeta_1)}{\zeta_2-\zeta_1}\right)|f'(t)|dt+\int_{\frac{3\zeta_1+\zeta_2}{4}}^{\frac{\zeta_1+\zeta_2}{2}}\Upsilon\left(\frac{4\left(t-\frac{3\zeta_1+\zeta_2}{4}\right)}{\zeta_2-\zeta_1}\right)|f'(t)|dt$$

$$+\int_{\frac{\zeta_1+\zeta_2}{2}}^{\frac{\zeta_1+3\zeta_2}{4}}\Upsilon\left(\frac{4\left(t-\frac{\zeta_1+\zeta_2}{2}\right)}{\zeta_2-\zeta_1}\right)|f'(t)|dt+\int_{\frac{\zeta_1+3\zeta_2}{4}}^{\zeta_2}\Upsilon\left(\frac{4\left(t-\frac{\zeta_1+3\zeta_2}{4}\right)}{\zeta_2-\zeta_1}\right)|f'(t)|dt\Big]$$

$$=\frac{(\zeta_2-\zeta_1)}{16\Upsilon(1)}$$

$$\times\int_0^1\Upsilon(t)\Big[\Big|f'\left(\left(\frac{3\zeta_1+\zeta_2}{4}\right)t+(1-t)\zeta_1\right)\Big|+\Big|f'\left(\left(\frac{\zeta_1+\zeta_2}{2}\right)t+\left(\frac{3\zeta_1+\zeta_2}{4}\right)(1-t)\right)\Big|$$

$$+\Big|f'\left(\left(\frac{\zeta_1+3\zeta_2}{4}\right)t+\left(\frac{\zeta_1+\zeta_2}{2}\right)(1-t)\right)\Big|+\Big|f'\left(\zeta_2 t+\left(\frac{\zeta_1+3\zeta_2}{4}\right)(1-t)\right)\Big|\Big]dt$$

$$\leq\frac{(\zeta_2-\zeta_1)}{16\Upsilon(1)}\left(\int_0^1[\Upsilon(t)]^p dt\right)^{\frac{1}{p}}$$

$$\times\Big\{\left(\int_0^1\Big|f'\left(\left(\frac{3\zeta_1+\zeta_2}{4}\right)t+(1-t)\zeta_1\right)\Big|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\Big|f'\left(\left(\frac{\zeta_1+\zeta_2}{2}\right)t+\left(\frac{3\zeta_1+\zeta_2}{4}\right)(1-t)\right)\Big|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\Big|f'\left(\left(\frac{\zeta_1+3\zeta_2}{4}\right)t+\left(\frac{\zeta_1+\zeta_2}{2}\right)(1-t)\right)\Big|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\Big|f'\left(\zeta_2 t+\left(\frac{\zeta_1+3\zeta_2}{4}\right)(1-t)\right)\Big|^q dt\right)^{\frac{1}{q}}\Big\}$$

$$\leq\frac{(\zeta_2-\zeta_1)}{16\Upsilon(1)}\sqrt[p]{B_\Upsilon(p)}$$

$$\times\Big\{\Big[\int_0^1\left(t\Big|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)\Big|^q+(1-t)|f'(\zeta_1)|^q-ct(1-t)\left(\frac{\zeta_2-\zeta_1}{4}\right)^2\right)dt\Big]^{\frac{1}{q}}$$

$$+\Big[\int_0^1\left(t\Big|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)\Big|^q+(1-t)\Big|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)\Big|^q-ct(1-t)\left(\frac{\zeta_2-\zeta_1}{4}\right)^2\right)dt\Big]^{\frac{1}{q}}$$

$$+\Big[\int_0^1\left(t\Big|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)\Big|^q+(1-t)\Big|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)\Big|^q-ct(1-t)\left(\frac{\zeta_2-\zeta_1}{4}\right)^2\right)dt\Big]^{\frac{1}{q}}$$

$$+\Big[\int_0^1\left(t|f'(\zeta_2)|^q+(1-t)\Big|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)\Big|^q-ct(1-t)\left(\frac{\zeta_2-\zeta_1}{4}\right)^2\right)dt\Big]^{\frac{1}{q}}\Big\}$$

$$= \frac{(\zeta_2-\zeta_1)}{16\sqrt[q]{2}\Upsilon(1)}\sqrt[p]{B_\Upsilon(p)}$$
$$\times\Bigg\{\sqrt[q]{|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+|f'(\zeta_1)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'(\zeta_2)|^q+|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}\Bigg\}.$$

The proof of Theorem 2 is completed. □

We now state some special cases of Theorem 2.

Corollary 2.1. *Taking $c\to 0^+$ in Theorem 2, we have the following inequality for convex functions:*

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\le\frac{(\zeta_2-\zeta_1)}{16\sqrt[q]{2}\Upsilon(1)}\sqrt[p]{B_\Upsilon(p)}$$
$$\times\Bigg\{\sqrt[q]{|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+|f'(\zeta_1)|^q}+\sqrt[q]{|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q}$$
$$+\sqrt[q]{|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q}+\sqrt[q]{|f'(\zeta_2)|^q+|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q}\Bigg\}.$$

Corollary 2.2. *Choosing $K=\|f'\|_\infty$ in Theorem 2, we have*

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\le\frac{(\zeta_2-\zeta_1)}{4\sqrt[q]{2}\Upsilon(1)}\sqrt[q]{2K^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}\sqrt[p]{B_\Upsilon(p)}.$$

Corollary 2.3. *Taking $\phi(t)=t$ in Theorem 2, we obtain*

$$|T_f(\zeta_1,\zeta_2)|\le\frac{(\zeta_2-\zeta_1)}{16\sqrt[q]{2}\sqrt[p]{p+1}}$$
$$\times\Bigg\{\sqrt[q]{|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+|f'(\zeta_1)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$

$$+\sqrt[q]{|f'(\zeta_2)|^q+|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}\Bigg\}.$$

Corollary 2.4. *Taking $\phi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2, we get*

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\le\frac{(\zeta_2-\zeta_1)}{16\sqrt[q]{2}\sqrt[p]{p\alpha+1}}$$
$$\times\Bigg\{\sqrt[q]{|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+|f'(\zeta_1)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'(\zeta_2)|^q+|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}\Bigg\}.$$

Corollary 2.5. *Taking $\phi(t)=t(\zeta_2-t)^{\alpha-1}$ for $\alpha\in(0,1)$ in Theorem 2, we obtain*

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\le\frac{\alpha(\zeta_2-\zeta_1)}{16\sqrt[q]{2}[\zeta_2^\alpha-\left(\frac{\zeta_1+3\zeta_2}{4}\right)^\alpha]}\sqrt[p]{B^\star_\Upsilon(p)}$$
$$\times\Bigg\{\sqrt[q]{|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+|f'(\zeta_1)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'(\zeta_2)|^q+|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}\Bigg\},$$

where

$$B^\star_\Upsilon(p):=\frac{4}{\alpha^p(\zeta_2-\zeta_1)}\int_{\frac{\zeta_1+3\zeta_2}{4}}^{\zeta_2}(\zeta_2^\alpha-t^\alpha)^p\,dt.$$

Corollary 2.6. *Taking $\phi(t)=\frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha\in(0,1)$ in Theorem 2, we get*

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\le\frac{(\alpha-1)(\zeta_2-\zeta_1)}{16\sqrt[q]{2}\Big\{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{(\zeta_2-\zeta_1)}{4}\right]-1\Big\}}\sqrt[p]{B^\diamond_\Upsilon(p)}$$
$$\times\Bigg\{\sqrt[q]{|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+|f'(\zeta_1)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$

$$+\sqrt[q]{|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{|f'(\zeta_2)|^q+|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q-\frac{c}{3}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}\Bigg\},$$

where

$$B^{\diamond}_{\Upsilon}(p):=\frac{4\alpha}{(\alpha-1)^{p+1}(\zeta_2-\zeta_1)}\int_0^{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{(\zeta_2-\zeta_1)}{4}\right]-1}\frac{t^p}{t+1}dt. \tag{2.2}$$

Theorem 3. *Let $f:P\to\mathbb{R}$ be a differentiable mapping on (ζ_1,ζ_2). If $|f'|^q$ is strongly convex on P for $q\geq 1$, then the following inequality for generalized fractional integrals holds:*

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\leq\frac{(\zeta_2-\zeta_1)}{16\Upsilon(1)}[B_\Upsilon(1)]^{1-\frac{1}{q}}$$
$$\times\Bigg\{\sqrt[q]{C_\Upsilon|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+(B_\Upsilon(1)-C_\Upsilon)|f'(\zeta_1)|^q-cD_\Upsilon\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{C_\Upsilon|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+(B_\Upsilon(1)-C_\Upsilon)|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q-cD_\Upsilon\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{C_\Upsilon|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+(B_\Upsilon(1)-C_\Upsilon)|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q-cD_\Upsilon\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}$$
$$+\sqrt[q]{C_\Upsilon|f'(\zeta_2)|^q+(B_\Upsilon(1)-C_\Upsilon)|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q-cD_\Upsilon\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}\Bigg\},$$

where

$$C_\Upsilon:=\int_0^1 t\Upsilon(t)dt,\quad D_\Upsilon:=\int_0^1 t(1-t)\Upsilon(t)dt$$

and $B_\Upsilon(1)$ is defined in Theorem 2.

Proof. From Lemma 1, strongly convexity with positive modulus c of $|f'|^q$, the well–known power mean inequality and properties of the modulus, we have

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\leq\frac{1}{4\Upsilon(1)}\int_{\zeta_1}^{\zeta_2}p_\Upsilon(t)|f'(t)|dt$$
$$\times\Big[\int_{\zeta_1}^{\frac{3\zeta_1+\zeta_2}{4}}\Upsilon\left(\frac{4(t-\zeta_1)}{\zeta_2-\zeta_1}\right)|f'(t)|dt+\int_{\frac{3\zeta_1+\zeta_2}{4}}^{\frac{\zeta_1+\zeta_2}{2}}\Upsilon\left(\frac{4\left(t-\frac{3\zeta_1+\zeta_2}{4}\right)}{\zeta_2-\zeta_1}\right)|f'(t)|dt$$
$$+\int_{\frac{\zeta_1+\zeta_2}{2}}^{\frac{\zeta_1+3\zeta_2}{4}}\Upsilon\left(\frac{4\left(t-\frac{\zeta_1+\zeta_2}{2}\right)}{\zeta_2-\zeta_1}\right)|f'(t)|dt+\int_{\frac{\zeta_1+3\zeta_2}{4}}^{\zeta_2}\Upsilon\left(\frac{4\left(t-\frac{\zeta_1+3\zeta_2}{4}\right)}{\zeta_2-\zeta_1}\right)|f'(t)|dt\Big]$$

$$
\begin{aligned}
&= \frac{(\zeta_2-\zeta_1)}{16\Upsilon(1)}\\
&\times \int_0^1 \Upsilon(t)\Big[\Big|f'\left(\left(\frac{3\zeta_1+\zeta_2}{4}\right)t+(1-t)\zeta_1\right)\Big|+\Big|f'\left(\left(\frac{\zeta_1+\zeta_2}{2}\right)t+\left(\frac{3\zeta_1+\zeta_2}{4}\right)(1-t)\right)\Big|\\
&+\Big|f'\left(\left(\frac{\zeta_1+3\zeta_2}{4}\right)t+\left(\frac{\zeta_1+\zeta_2}{2}\right)(1-t)\right)\Big|+\Big|f'\left(\zeta_2 t+\left(\frac{\zeta_1+3\zeta_2}{4}\right)(1-t)\right)\Big|\Big]dt\\
&\leq \frac{(\zeta_2-\zeta_1)}{16\Upsilon(1)}\left(\int_0^1\Upsilon(t)dt\right)^{1-\frac{1}{q}}\\
&\times\Big\{\left(\int_0^1\Upsilon(t)\Big|f'\left(\left(\frac{3\zeta_1+\zeta_2}{4}\right)t+(1-t)\zeta_1\right)\Big|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\Upsilon(t)\Big|f'\left(\left(\frac{\zeta_1+\zeta_2}{2}\right)t+\left(\frac{3\zeta_1+\zeta_2}{4}\right)(1-t)\right)\Big|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\Upsilon(t)\Big|f'\left(\left(\frac{\zeta_1+3\zeta_2}{4}\right)t+\left(\frac{\zeta_1+\zeta_2}{2}\right)(1-t)\right)\Big|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\Upsilon(t)\Big|f'\left(\zeta_2 t+\left(\frac{\zeta_1+3\zeta_2}{4}\right)(1-t)\right)\Big|^q dt\right)^{\frac{1}{q}}\Big\}\\
&\leq \frac{(\zeta_2-\zeta_1)}{16\Upsilon(1)}[B_\Upsilon(1)]^{1-\frac{1}{q}}\\
&\times\Big\{\Big[\int_0^1\Upsilon(t)\left(t\Big|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)\Big|^q+(1-t)|f'(\zeta_1)|^q-ct(1-t)\left(\frac{\zeta_2-\zeta_1}{4}\right)^2\right)dt\Big]^{\frac{1}{q}}\\
&+\Big[\int_0^1\Upsilon(t)\left(t\Big|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)\Big|^q+(1-t)\Big|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)\Big|^q-ct(1-t)\left(\frac{\zeta_2-\zeta_1}{4}\right)^2\right)dt\Big]^{\frac{1}{q}}\\
&+\Big[\int_0^1\Upsilon(t)\left(t\Big|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)\Big|^q+(1-t)\Big|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)\Big|^q-ct(1-t)\left(\frac{\zeta_2-\zeta_1}{4}\right)^2\right)dt\Big]^{\frac{1}{q}}\\
&+\Big[\int_0^1\Upsilon(t)\left(t|f'(\zeta_2)|^q+(1-t)\Big|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)\Big|^q-ct(1-t)\left(\frac{\zeta_2-\zeta_1}{4}\right)^2\right)dt\Big]^{\frac{1}{q}}\Big\}\\
&= \frac{(\zeta_2-\zeta_1)}{16\Upsilon(1)}[B_\Upsilon(1)]^{1-\frac{1}{q}}\\
&\times\Big\{\sqrt[q]{C_\Upsilon|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+\left(B_\Upsilon(1)-C_\Upsilon\right)|f'(\zeta_1)|^q-cD_\Upsilon\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}\\
&+\sqrt[q]{C_\Upsilon|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+\left(B_\Upsilon(1)-C_\Upsilon\right)|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q-cD_\Upsilon\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}\\
&+\sqrt[q]{C_\Upsilon|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+\left(B_\Upsilon(1)-C_\Upsilon\right)|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q-cD_\Upsilon\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}
\end{aligned}
$$

$$+\sqrt[q]{C_{\Upsilon}|f'(\zeta_2)|^q+(B_{\Upsilon}(1)-C_{\Upsilon})\,|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q-cD_{\Upsilon}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}\Bigg\}.$$

This completes the proof of Theorem 3. □

We now state some special cases of Theorem 3.

Corollary 3.1. *Taking $c\to 0^+$ in Theorem 3, we get the following inequality for convex functions:*

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\leq\frac{(\zeta_2-\zeta_1)}{16\Upsilon(1)}[B_{\Upsilon}(1)]^{1-\frac{1}{q}}$$
$$\times\Bigg\{\sqrt[q]{C_{\Upsilon}|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+(B_{\Upsilon}(1)-C_{\Upsilon})\,|f'(\zeta_1)|^q}$$
$$+\sqrt[q]{C_{\Upsilon}|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+(B_{\Upsilon}(1)-C_{\Upsilon})\,|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q}$$
$$+\sqrt[q]{C_{\Upsilon}|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+(B_{\Upsilon}(1)-C_{\Upsilon})\,|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q}$$
$$+\sqrt[q]{C_{\Upsilon}|f'(\zeta_2)|^q+(B_{\Upsilon}(1)-C_{\Upsilon})\,|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q}\Bigg\}.$$

Corollary 3.2. *Taking $K=\|f'\|_{\infty}$ in Theorem 3, we have*

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\leq\frac{(\zeta_2-\zeta_1)}{4\Upsilon(1)}[B_{\Upsilon}(1)]^{1-\frac{1}{q}}\sqrt[q]{K^qB_{\Upsilon}(1)-cD_{\Upsilon}\left(\frac{\zeta_2-\zeta_1}{4}\right)^2}.$$

Corollary 3.3. *Taking $\phi(t)=t$ and $c\to 0^+$ in Theorem 3, we obtain*

$$|T_f(\zeta_1,\zeta_2)|\leq\frac{(\zeta_2-\zeta_1)}{32\sqrt[q]{3}}$$
$$\times\Bigg\{\sqrt[q]{2|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+|f'(\zeta_1)|^q}+\sqrt[q]{2|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q}$$
$$+\sqrt[q]{2|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q}+\sqrt[q]{2|f'(\zeta_2)|^q+|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q}\Bigg\}.$$

Corollary 3.4. *Taking $\phi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ and $c\to 0^+$ in Theorem 3, we get*

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\leq\frac{(\zeta_2-\zeta_1)}{16\sqrt[q]{\alpha+2}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)}$$
$$\times\Bigg\{\sqrt[q]{(\alpha+1)|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+|f'(\zeta_1)|^q}$$
$$+\sqrt[q]{(\alpha+1)|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q}$$

$$+\sqrt[q]{(\alpha+1)|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q}$$
$$+\sqrt[q]{(\alpha+1)|f'(\zeta_2)|^q+|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q}\Big\}.$$

Corollary 3.5. *Taking $\phi(t)=t(\zeta_2-t)^{\alpha-1}$ for $\alpha\in(0,1)$ and $c\to 0^+$ in Theorem 3, we obtain*

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\leq \frac{(\zeta_2-\zeta_1)}{16\Upsilon^*(1)}[B_\Upsilon(1)]^{1-\frac{1}{q}}$$
$$\times\Big\{\sqrt[q]{C_{\Upsilon^*}|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+\left(B_\Upsilon(1)-C_{\Upsilon^*}\right)|f'(\zeta_1)|^q}$$
$$+\sqrt[q]{C_{\Upsilon^*}|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+\left(B_\Upsilon(1)-C_{\Upsilon^*}\right)|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q}$$
$$+\sqrt[q]{C_{\Upsilon^*}|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+\left(B_\Upsilon(1)-C_{\Upsilon^*}\right)|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q}$$
$$+\sqrt[q]{C_{\Upsilon^*}|f'(\zeta_2)|^q+\left(B_\Upsilon(1)-C_{\Upsilon^*}\right)|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q}\Big\},$$

where

$$\Upsilon^*(t):=\frac{\zeta_2^\alpha-\left(\zeta_2-\frac{(\zeta_2-\zeta_1)t}{4}\right)^\alpha}{\alpha},\quad C_{\Upsilon^*}:=\int_0^1 t\Upsilon^*(t)dt$$

and

$$B_\Upsilon(1):=\frac{4}{\alpha(\zeta_2-\zeta_1)}\Big\{\zeta_2^\alpha\left(\frac{\zeta_1+3\zeta_2}{4}\right)-\frac{\zeta_2^{\alpha+1}-\left(\frac{\zeta_1+3\zeta_2}{4}\right)^{\alpha+1}}{\alpha+1}\Big\}.$$

Corollary 3.6. *Taking $\phi(t)=\frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha\in(0,1)$ and $c\to 0^+$ in Theorem 3, we get*

$$|T_{f,\Upsilon}(\zeta_1,\zeta_2)|\leq \frac{(\zeta_2-\zeta_1)}{16\Upsilon^\diamond(1)}[B_\Upsilon^\diamond(1)]^{1-\frac{1}{q}}$$
$$\times\Big\{\sqrt[q]{C_{\Upsilon^\diamond}|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q+\left(B_\Upsilon^\diamond(1)-C_{\Upsilon^\diamond}\right)|f'(\zeta_1)|^q}$$
$$+\sqrt[q]{C_{\Upsilon^\diamond}|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q+\left(B_\Upsilon^\diamond(1)-C_{\Upsilon^\diamond}\right)|f'\left(\frac{3\zeta_1+\zeta_2}{4}\right)|^q}$$
$$+\sqrt[q]{C_{\Upsilon^\diamond}|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q+\left(B_\Upsilon^\diamond(1)-C_{\Upsilon^\diamond}\right)|f'\left(\frac{\zeta_1+\zeta_2}{2}\right)|^q}$$
$$+\sqrt[q]{C_{\Upsilon^\diamond}|f'(\zeta_2)|^q+\left(B_\Upsilon^\diamond(1)-C_{\Upsilon^\diamond}\right)|f'\left(\frac{\zeta_1+3\zeta_2}{4}\right)|^q}\Big\},$$

where

$$\Upsilon^{\diamond}(t) := \frac{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{(\zeta_2-\zeta_1)t}{4}\right]-1}{\alpha-1}, \quad C_{\Upsilon^{\diamond}} := \int_0^1 t\Upsilon^{\diamond}(t)dt$$

and $B^{\diamond}_{\Upsilon}(1)$ *is defined by (2.2) for value* $p=1$.

Remark 2.2: Applying the corollaries of our Theorems 2 and 3, interested readers can calculate the expression of $|T_{f,\Upsilon}(\zeta_1,\zeta_2)|$ for different choices of function ϕ, so we omit here the details.

3. Applications

Let us consider the following special means for positive real numbers ζ_1,ζ_2, where $\zeta_1<\zeta_2$:

(1) The arithmetic mean:
$$\mathcal{A}(\zeta_1,\zeta_2)=\frac{\zeta_1+\zeta_2}{2},$$
(2) The harmonic mean:
$$\mathcal{H}(\zeta_1,\zeta_2)=\frac{2}{\frac{1}{\zeta_1}+\frac{1}{\zeta_2}},$$
(3) The logarithmic mean:
$$\mathcal{L}(\zeta_1,\zeta_2)=\frac{\zeta_2-\zeta_1}{\ln\zeta_2-\ln\zeta_1},$$
(4) The generalized log-mean:
$$\mathcal{L}_r(\zeta_1,\zeta_2)=\left[\frac{\zeta_2^{r+1}-\zeta_1^{r+1}}{(r+1)(\zeta_2-\zeta_1)}\right]^{\frac{1}{r}};\ \ r\in\mathbb{R}\setminus\{-1,0\}.$$

Now, using the results obtained in Section 2, we present some applications to special means.

Proposition 3.1. *Let* $0<\zeta_1<\zeta_2$. *Then for* $r\in\mathbb{R}$ *and* $r\geq 2$, *where* $q>1$ *and* $p^{-1}+q^{-1}=1$, *the following inequality holds:*

$$\left|\mathcal{A}\left(\frac{\mathcal{A}\left(\mathcal{A}^r(3\zeta_1,\zeta_2),\mathcal{A}^r(\zeta_1,3\zeta_2)\right)}{2^r},\mathcal{A}\left(\mathcal{A}(\zeta_1,\zeta_2),\zeta_2^r\right)\right)-\mathcal{L}_r^r(\zeta_1,\zeta_2)\right|\leq\frac{r(\zeta_2-\zeta_1)}{16\sqrt[p]{p+1}}$$
$$\times\left\{\sqrt[q]{\mathcal{A}\left(\zeta_1^{q(r-1)},\left(\frac{3\zeta_1+\zeta_2}{4}\right)^{q(r-1)}\right)}+\sqrt[q]{\mathcal{A}\left(\left(\frac{\zeta_1+\zeta_2}{2}\right)^{q(r-1)},\left(\frac{3\zeta_1+\zeta_2}{4}\right)^{q(r-1)}\right)}\right.$$
$$\left.+\sqrt[q]{\mathcal{A}\left(\left(\frac{\zeta_1+3\zeta_2}{4}\right)^{q(r-1)},\left(\frac{\zeta_1+\zeta_2}{2}\right)^{q(r-1)}\right)}+\sqrt[q]{\mathcal{A}\left(\left(\frac{\zeta_1+3\zeta_2}{4}\right)^{q(r-1)},\zeta_2^{q(r-1)}\right)}\right\}.$$

Proof. Taking $c\to0^+$, $f(t)=t^r$ and $\phi(t)=t$, in Theorem 2, one can obtain the result immediately. □

Proposition 3.2. *Let $0 < \zeta_1 < \zeta_2$. Then for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality hold:*

$$\left|\frac{1}{\mathcal{H}\left(\mathcal{H}\left(\frac{3\zeta_1+\zeta_2}{4}, \frac{\zeta_1+\zeta_2}{2}\right), \mathcal{H}\left(\frac{\zeta_1+3\zeta_2}{4}, \zeta_2\right)\right)} - \frac{1}{\mathcal{L}(\zeta_1, \zeta_2)}\right| \leq \frac{(\zeta_2 - \zeta_1)}{16\sqrt[p]{p+1}}$$

$$\times\left\{\frac{1}{\sqrt[q]{\mathcal{H}\left(\zeta_1^{2q}, \left(\frac{3\zeta_1+\zeta_2}{4}\right)^{2q}\right)}} + \frac{1}{\sqrt[q]{\mathcal{H}\left(\left(\frac{\zeta_1+\zeta_2}{2}\right)^{2q}, \left(\frac{3\zeta_1+\zeta_2}{4}\right)^{2q}\right)}}\right.$$

$$\left.+\frac{1}{\sqrt[q]{\mathcal{H}\left(\left(\frac{\zeta_1+3\zeta_2}{4}\right)^{2q}, \left(\frac{\zeta_1+\zeta_2}{2}\right)^{2q}\right)}} + \frac{1}{\sqrt[q]{\mathcal{H}\left(\left(\frac{\zeta_1+3\zeta_2}{4}\right)^{2q}, \zeta_2^{2q}\right)}}\right\}.$$

Proof. Taking $c \to 0^+$, $f(t) = \frac{1}{t}$ and $\phi(t) = t$, in Theorem 2, the result follows immediately. □

Proposition 3.3. *Let $0 < \zeta_1 < \zeta_2$. Then for $r \in \mathbb{R}$ and $r \geq 2$, where $q \geq 1$, the following inequality hold:*

$$\left|A\left(\frac{\mathcal{A}\left(\mathcal{A}^r(3\zeta_1, \zeta_2), \mathcal{A}^r(\zeta_1, 3\zeta_2)\right)}{2^r}, \mathcal{A}\left(\mathcal{A}(\zeta_1, \zeta_2), \zeta_2^r\right)\right) - \mathcal{L}_r^r(\zeta_1, \zeta_2)\right| \leq \sqrt[q]{\frac{2}{3}}\frac{r(\zeta_2 - \zeta_1)}{32}$$

$$\times\left\{\sqrt[q]{\mathcal{A}\left(\zeta_1^{q(r-1)}, 2\left(\frac{3\zeta_1+\zeta_2}{4}\right)^{q(r-1)}\right)} + \sqrt[q]{\mathcal{A}\left(2\left(\frac{\zeta_1+\zeta_2}{2}\right)^{q(r-1)}, \left(\frac{3\zeta_1+\zeta_2}{4}\right)^{q(r-1)}\right)}\right.$$

$$\left.+\sqrt[q]{\mathcal{A}\left(2\left(\frac{\zeta_1+3\zeta_2}{4}\right)^{q(r-1)}, \left(\frac{\zeta_1+\zeta_2}{2}\right)^{q(r-1)}\right)} + \sqrt[q]{\mathcal{A}\left(\left(\frac{\zeta_1+3\zeta_2}{4}\right)^{q(r-1)}, 2\zeta_2^{q(r-1)}\right)}\right\}.$$

Proof. Taking $c \to 0^+$, $f(t) = t^r$ and $\phi(t) = t$, in Theorem 3, one can obtain the desired result. □

Proposition 3.4. *Let $0 < \zeta_1 < \zeta_2$. Then for $q \geq 1$, the following inequality hold:*

$$\left|\frac{1}{\mathcal{H}\left(\mathcal{H}\left(\frac{3\zeta_1+\zeta_2}{4}, \frac{\zeta_1+\zeta_2}{2}\right), \mathcal{H}\left(\frac{\zeta_1+3\zeta_2}{4}, \zeta_2\right)\right)} - \frac{1}{\mathcal{L}(\zeta_1, \zeta_2)}\right| \leq \sqrt[q]{\frac{2}{3}}\frac{(\zeta_2 - \zeta_1)}{32}$$

$$\times\left\{\frac{1}{\sqrt[q]{\mathcal{H}\left(2\zeta_1^{2q}, \left(\frac{3\zeta_1+\zeta_2}{4}\right)^{2q}\right)}} + \frac{1}{\sqrt[q]{\mathcal{H}\left(\left(\frac{\zeta_1+\zeta_2}{2}\right)^{2q}, 2\left(\frac{3\zeta_1+\zeta_2}{4}\right)^{2q}\right)}}\right.$$

$$\left.+\frac{1}{\sqrt[q]{\mathcal{H}\left(\left(\frac{\zeta_1+3\zeta_2}{4}\right)^{2q}, 2\left(\frac{\zeta_1+\zeta_2}{2}\right)^{2q}\right)}} + \frac{1}{\sqrt[q]{\mathcal{H}\left(2\left(\frac{\zeta_1+3\zeta_2}{4}\right)^{2q}, \zeta_2^{2q}\right)}}\right\}.$$

Proof. If we take $c \to 0^+$, $f(t) = \dfrac{1}{t}$ and $\phi(t) = t$, in Theorem 3, then the result follows immediately. □

Next, we provide some new error estimates for the trapezium formula. Let $\mathcal{Q}$ be the partition of the points $\zeta_1 = \ell_0 < \ell_1 < \ldots < \ell_k = \zeta_2$ of the interval P. Let us consider the following quadrature formula:

$$\int_{\zeta_1}^{\zeta_2} f(x)dx = T(f,\mathcal{Q}) + E(f,\mathcal{Q}),$$

where

$$T(f,\mathcal{Q}) := \sum_{i=0}^{k-1}\left[f\left(\frac{3\ell_i+\ell_{i+1}}{4}\right)+f\left(\frac{\ell_i+\ell_{i+1}}{2}\right)+f\left(\frac{\ell_i+3\ell_{i+1}}{4}\right)+f(\ell_{i+1})\right]\frac{(\ell_{i+1}-\ell_i)}{4}$$

is the trapezium version and $E(f,\mathcal{Q})$ denotes their associated approximation error.

Proposition 3.5. *Let $f : P \to \mathbb{R}$ be a differentiable function on (ζ_1,ζ_2), where $\zeta_1 < \zeta_2$. If $|f'|^q$ is strongly convex on P for $q > 1$ and $p^{-1}+q^{-1} = 1$, then the following inequality holds:*

$$|E(f,\mathcal{Q})| \le \frac{1}{16\sqrt[q]{2}\sqrt[p]{p+1}}\sum_{i=0}^{k-1}(\ell_{i+1}-\ell_i)^2$$
$$\times\Bigg\{\sqrt[q]{|f'\left(\frac{3\ell_i+\ell_{i+1}}{4}\right)|^q+|f'(\ell_i)|^q-\frac{c}{3}\left(\frac{\ell_{i+1}-\ell_i}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\ell_i+\ell_{i+1}}{2}\right)|^q+|f'\left(\frac{3\ell_i+\ell_{i+1}}{4}\right)|^q-\frac{c}{3}\left(\frac{\ell_{i+1}-\ell_i}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\ell_i+3\ell_{i+1}}{4}\right)|^q+|f'\left(\frac{\ell_i+\ell_{i+1}}{2}\right)|^q-\frac{c}{3}\left(\frac{\ell_{i+1}-\ell_i}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\ell_i+3\ell_{i+1}}{4}\right)|^q+|f'(\ell_{i+1})|^q-\frac{c}{3}\left(\frac{\ell_{i+1}-\ell_i}{4}\right)^2}\Bigg\}.$$

Proof. Applying Theorem 2 for $\phi(t) = t$ on the subintervals $[\ell_i,\ell_{i+1}]$ $(i = 0,\ldots,k-1)$ of the partition $\mathcal{Q}$, we have

$$\left|f\left(\frac{3\ell_i+\ell_{i+1}}{4}\right)+f\left(\frac{\ell_i+\ell_{i+1}}{2}\right)+f\left(\frac{\ell_i+3\ell_{i+1}}{4}\right)+f(\ell_{i+1})-\frac{4}{\ell_{i+1}-\ell_i}\int_{\ell_i}^{\ell_{i+1}} f(x)dx\right|$$
$$\le \frac{(\ell_{i+1}-\ell_i)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \tag{3.1}$$
$$\times\Bigg\{\sqrt[q]{|f'\left(\frac{3\ell_i+\ell_{i+1}}{4}\right)|^q+|f'(\ell_i)|^q-\frac{c}{3}\left(\frac{\ell_{i+1}-\ell_i}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\ell_i+\ell_{i+1}}{2}\right)|^q+|f'\left(\frac{3\ell_i+\ell_{i+1}}{4}\right)|^q-\frac{c}{3}\left(\frac{\ell_{i+1}-\ell_i}{4}\right)^2}$$

$$+\sqrt[q]{|f'\left(\frac{\ell_i+3\ell_{i+1}}{4}\right)|^q+|f'\left(\frac{\ell_i+\ell_{i+1}}{2}\right)|^q-\frac{c}{3}\left(\frac{\ell_{i+1}-\ell_i}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\ell_i+3\ell_{i+1}}{4}\right)|^q+|f'(\ell_{i+1})|^q-\frac{c}{3}\left(\frac{\ell_{i+1}-\ell_i}{4}\right)^2}\Bigg\}.$$

Hence from (3.1), we obtain that

$$|E(f,\mathcal{Q})|=\Big|\int_{\zeta_1}^{\zeta_2}f(x)dx-T(f,\mathcal{Q})\Big|$$
$$\leq\Big|\sum_{i=0}^{k-1}\Big\{\int_{\ell_i}^{\ell_{i+1}}f(x)dx$$
$$-\Big[f\left(\frac{3\ell_i+\ell_{i+1}}{4}\right)+f\left(\frac{\ell_i+\ell_{i+1}}{2}\right)+f\left(\frac{\ell_i+3\ell_{i+1}}{4}\right)+f(\ell_{i+1})\Big]\frac{(\ell_{i+1}-\ell_i)}{4}\Big\}\Big|$$
$$\leq\sum_{i=0}^{k-1}\Big|\Big\{\int_{\ell_i}^{\ell_{i+1}}f(x)dx$$
$$-\Big[f\left(\frac{3\ell_i+\ell_{i+1}}{4}\right)+f\left(\frac{\ell_i+\ell_{i+1}}{2}\right)+f\left(\frac{\ell_i+3\ell_{i+1}}{4}\right)+f(\ell_{i+1})\Big]\frac{(\ell_{i+1}-\ell_i)}{4}\Big\}\Big|$$
$$\leq\frac{1}{16\sqrt[q]{2}\sqrt[p]{p+1}}\sum_{i=0}^{k-1}(\ell_{i+1}-\ell_i)^2$$
$$\times\Bigg\{\sqrt[q]{|f'\left(\frac{3\ell_i+\ell_{i+1}}{4}\right)|^q+|f'(\ell_i)|^q-\frac{c}{3}\left(\frac{\ell_{i+1}-\ell_i}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\ell_i+\ell_{i+1}}{2}\right)|^q+|f'\left(\frac{3\ell_i+\ell_{i+1}}{4}\right)|^q-\frac{c}{3}\left(\frac{\ell_{i+1}-\ell_i}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\ell_i+3\ell_{i+1}}{4}\right)|^q+|f'\left(\frac{\ell_i+\ell_{i+1}}{2}\right)|^q-\frac{c}{3}\left(\frac{\ell_{i+1}-\ell_i}{4}\right)^2}$$
$$+\sqrt[q]{|f'\left(\frac{\ell_i+3\ell_{i+1}}{4}\right)|^q+|f'(\ell_{i+1})|^q-\frac{c}{3}\left(\frac{\ell_{i+1}-\ell_i}{4}\right)^2}\Bigg\}.$$

The proof of Proposition 3.5 is completed. □

Proposition 3.6. *Let $f:P\to\mathbb{R}$ be a differentiable function on (ζ_1,ζ_2), where $\zeta_1<\zeta_2$. If $|f'|^q$ is convex on P for $q\geq 1$, then the following inequality holds:*

$$|E(f,\mathcal{Q})|\leq\frac{1}{32\sqrt[q]{3}}\sum_{i=0}^{k-1}(\ell_{i+1}-\ell_i)^2$$
$$\times\Bigg\{\sqrt[q]{2|f'\left(\frac{3\ell_i+\ell_{i+1}}{4}\right)|^q+|f'(\ell_i)|^q}+\sqrt[q]{2|f'\left(\frac{\ell_i+\ell_{i+1}}{2}\right)|^q+|f'\left(\frac{3\ell_i+\ell_{i+1}}{4}\right)|^q}$$

$$+\sqrt[q]{2|f'\left(\frac{\ell_i+3\ell_{i+1}}{4}\right)|^q+|f'\left(\frac{\ell_i+\ell_{i+1}}{2}\right)|^q}+\sqrt[q]{|f'\left(\frac{\ell_i+3\ell_{i+1}}{4}\right)|^q+2|f'(\ell_{i+1})|^q}\Bigg\}.$$

Proof. The proof is analogous as to that of Proposition 3.5 but use Theorem 3 and taking $c \to 0^+$. □

Remark 3.1: Applying our Theorems 2 and 3, for special positive values c and various suitable choices of function such as $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, $\phi(t) = t(\zeta_2 - t)^{\alpha-1}$, and $\phi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0,1]$, such that $|f'|^q$ to be strongly convex with positive modulus c, we can deduce some new bounds for special means and trapezium formula using above ideas and techniques. We omit their proofs and the details are left to the interested readers.

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