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NEWTON'S TYPE QUANTUM FRACTIONAL INTEGRAL INEQUALITIES PERTAINING TO n-POLYNOMIAL CONVEX FUNCTIONS AND APPLICATION

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Abstract. In this paper, we consider a new Newton's type quantum fractional integral identity. Following that as an auxiliary result, we established in our main results some integral inequalities of Newton's type including n-polynomial convex functions. From our main results, we discuss in detail several special cases. Finally, an example and application of a special means of positive real numbers are presented to support our theoretical results.

1. Introduction

Integral inequalities are very useful tools for finding estimations. They can be applied in different fields of mathematics such as fractional calculus and discrete fractional calculus etc., see [20, 11, 23, 31, 32].

Convexity study is crucial regarding theoretical behavior of mathematical inequalities, e.g., [25]. For some other theoretical studies of inequalities on different types of convex functions, see, e.g., GA-convex [43], MT-convex [24], (α, m)-convex [36], F-convex [26], η_ψ-convex [6], a generalized class of convexity [29], and many other types can be found in [12].

Definition 1.1. [12] A function $\hbar : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *convex* if

$$\hbar(\jmath\widetilde{\chi}_1 + (1-\jmath)\widetilde{\chi}_2) \leq \jmath\hbar(\widetilde{\chi}_1) + (1-\jmath)\hbar(\widetilde{\chi}_2)$$

holds for all $\widetilde{\chi}_1, \widetilde{\chi}_2 \in \mathbb{I}$ and $\jmath \in [0,1]$. Likewise, $\hbar$ is concave if $-\hbar$ is convex.

Definition 1.2. [34, 40] Let $n \in \mathbb{N}$. A nonnegative function $\hbar : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ is called *n-polynomial convex* if

$$\hbar(\jmath\widetilde{\chi}_1 + (1-\jmath)\widetilde{\chi}_2) \leq \frac{1}{n}\sum_{\ell=1}^{n}\left(1-(1-\jmath)^\ell\right)\hbar(\widetilde{\chi}_1) + \frac{1}{n}\sum_{\ell=1}^{n}\left(1-\jmath^\ell\right)\hbar(\widetilde{\chi}_2)$$

holds for all $\widetilde{\chi}_1, \widetilde{\chi}_2 \in \mathbb{I}$ and $\jmath \in [0,1]$.

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Remark 1.1: From [40], every nonnegative convex function is also an n-polynomial convex function. Moreover, we get the convex function by taking $n = 1$ in Definition 1.2.

Symmetry has a significant role in integral inequality models with convexity. Furthermore, for convex functions and their types, many basic inequalities are found, such as Hermite–Hadamard type [13, 16], Hermite–Hadamard–Fejér type [28], Ostrowski type [14, 7, 39], Simpson type [5, 41], Hardy type [19], Olsen type [15] and Opial type [27].

Let us recall some published papers about above inequalities using $\acute{q}$-calculus that inspired us.

The $\acute{q}$-analogue of the trapezium's inequality was discovered by Tariboon and Ntouyas [38] using the concepts of $\acute{q}$-calculus, also known as calculus without limits, on the finite intervals. See [18] for more information on how to get classical calculus by taking $\acute{q} \to 1^-$. An updated version of the $\acute{q}$-analogue of the inequality of the trapezium was discovered by Alp *et al.* [4]. Meanwhile, $\acute{q}$-analogues of trapezium-like inequalities involving first order $\acute{q}$-differentiable convex functions were deduced by Sudsutad *et al.* [37] and Noor *et al.* [33]. These analogues were created by Liu and Zhuang [21] using twice $\acute{q}$-differentiable convex functions. Budak *et al.* [8] are able to further develop certain $\acute{q}$-Hermite–Hadamard-type inequality. Ali *et al.* [2] presented some $\acute{q}$-Ostrowski-type inequalities for twice $\acute{q}$-differentiable functions. Convexity was used by Butt *et al.* [9] to generate some new $\acute{q}$-Simpson-Newton-like estimates in the frame of Mercer type inequalities. Aljinović *et al.* [3] established Ostrowski inequality for $\acute{q}$-calculus.

Wang *et al.* [42] developed new Ostrowski type inequalities via $\acute{q}$-fractional integrals involving s-convex functions. Inspired by this paper, we attempt to give some new $\acute{q}$-fractional integral inequalities of Newton's type.

The article is set up as follows: The purpose of the Section 2 is to review several earlier findings of fractional calculus and $\acute{q}$-calculus to provide the main interpretation of this paper. In Section 3, we look at proving a new Newton's type quantum fractional integral identity and demonstrate some integral inequalities via n-polynomial convex functions. From our main results, we will discuss in detail several special cases. In Section 4, we offer an example and application to special means of positive real numbers in order to show the efficiency of our theoretical results. The conclusion and future research will be given in Section 5.

2. Preliminaries

Let us denote, respectively, $\mathcal{L}[\widetilde{\chi}_1, \widetilde{\chi}_2]$ the set of all Lebesgue integrable functions on $[\widetilde{\chi}_1, \widetilde{\chi}_2]$ and $\mathcal{C}[\widetilde{\chi}_1, \widetilde{\chi}_2]$ the set of all differentiable continuous functions on $[\widetilde{\chi}_1, \widetilde{\chi}_2]$.

2.1. **Fractional Calculus.**

Definition 2.1. Let $\alpha > 0$, $0 \leq \widetilde{\chi}_1 < \widetilde{\chi}_2$ and $\hbar \in \mathcal{L}[\widetilde{\chi}_1, \widetilde{\chi}_2]$. Then the Riemann–Liouville fractional integral operators of order α are defined by

$$\mathrm{J}^{\alpha}_{\widetilde{\chi}_1^+}\hbar(\theta) = \frac{1}{\Gamma(\alpha)} \int_{\widetilde{\chi}_1}^{\theta} (\theta - \jmath)^{\alpha-1}\hbar(\jmath)d\jmath, \quad \widetilde{\chi}_1 < \theta \tag{2.1}$$

and

$$\mathrm{J}^{\alpha}_{\widetilde{\chi}_2^-}\hbar(\theta) = \frac{1}{\Gamma(\alpha)}\int_{\theta}^{\widetilde{\chi}_2}(\jmath-\theta)^{\alpha-1}\hbar(\jmath)d\jmath, \quad \theta < \widetilde{\chi}_2,$$

where $\Gamma(\cdot)$ is gamma function, defined by

$$\Gamma(\alpha) = \int_0^{\infty}\jmath^{\alpha-1}e^{-\jmath}d\jmath, \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$

For $\alpha = 1$, we get the classical Riemann integrals.

The following fractional version of Simpson's type inequalities for differentiable s-convex functions was given by Chen and Huang:

Theorem 1. [10] *Suppose that $\hbar : \mathbb{I} \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable function with $\hbar \in \mathcal{L}[\widetilde{\chi}_1, \widetilde{\chi}_2]$, where $0 \le \widetilde{\chi}_1 < \widetilde{\chi}_2$ and $\widetilde{\chi}_1, \widetilde{\chi}_2 \in \mathbb{I}^{\circ}$ (interior of $\mathbb{I}$). If $|\hbar'|$ is a s-convex function for some fixed $s \in (0,1]$, then for $\alpha > 0$, the following inequality holds true:*

$$\left|\frac{1}{6}\left[\hbar(\widetilde{\chi}_1) + 4\hbar\left(\frac{\widetilde{\chi}_1+\widetilde{\chi}_2}{2}\right) + \hbar(\widetilde{\chi}_2)\right] - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\widetilde{\chi}_2-\widetilde{\chi}_1)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\widetilde{\chi}_1^+}\hbar\left(\frac{\widetilde{\chi}_1+\widetilde{\chi}_2}{2}\right) + \mathrm{J}^{\alpha}_{\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+\widetilde{\chi}_2}{2}\right)\right]\right|$$
$$\le \frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{s+1}\left[|\hbar'(\widetilde{\chi}_1)| + |\hbar'(\widetilde{\chi}_2)|\right]\mathrm{I}(s,\alpha), \tag{2.2}$$

where

$$\mathrm{I}(s,\alpha) := \int_0^1\left|\frac{\jmath^{\alpha}}{2} - \frac{1}{3}\right|\left[(1-\jmath)^s + (1+\jmath)^s\right]d\jmath.$$

2.2. **Quantum Calculus.** Throughout the remaining paper, let us consider $0 < \acute{\mathsf{q}} < 1$ as a constant.

Definition 2.2. [18] *For $\hbar \in \mathcal{C}[\widetilde{\chi}_1, \widetilde{\chi}_2]$, the left $\acute{\mathsf{q}}$-derivative of $\hbar$ at $\theta \in [\widetilde{\chi}_1, \widetilde{\chi}_2]$ is given by*

$${}_{\widetilde{\chi}_1}\mathrm{D}_{\acute{\mathsf{q}}}\hbar(\theta) = \frac{\hbar(\theta) - \hbar\left(\acute{\mathsf{q}}\theta + (1-\acute{\mathsf{q}})\widetilde{\chi}_1\right)}{(1-\acute{\mathsf{q}})(\theta-\widetilde{\chi}_1)}, \quad \theta \ne \widetilde{\chi}_1. \tag{2.3}$$

The function $\hbar$ is said to be $\acute{\mathsf{q}}$-differentiable on $[\widetilde{\chi}_1, \widetilde{\chi}_2]$ if ${}_{\widetilde{\chi}_1}\mathrm{D}_{\acute{\mathsf{q}}}\hbar(\theta)$ exists for all $\theta \in [\widetilde{\chi}_1, \widetilde{\chi}_2]$. If we choose $\widetilde{\chi}_1 = 0$, then we will use the notation ${}_{\widetilde{\chi}_1}\mathrm{D}_{\acute{\mathsf{q}}}\hbar(\theta) = \mathrm{D}_{\acute{\mathsf{q}}}\hbar(\theta)$, which is the $\acute{\mathsf{q}}$-Jackson derivative. See [17, 4] *for more details.*

The $\acute{\mathsf{q}}$-integer is expressed as follows:

$$[n]_{\acute{\mathsf{q}}} := \frac{\acute{\mathsf{q}}^n - 1}{\acute{\mathsf{q}} - 1} = 1 + \acute{\mathsf{q}} + \acute{\mathsf{q}}^2 + \cdots + \acute{\mathsf{q}}^{n-1}, \quad n \in \mathbb{N}, \ \acute{\mathsf{q}} \in (0,1).$$

The following $\acute{\mathsf{q}}$-integral along with its properties can be studied in [4].

Definition 2.3. *Suppose that $\hbar \in \mathcal{C}[\widetilde{\chi}_1, \widetilde{\chi}_2]$. Then $\acute{\mathsf{q}}$-definite integral for $\theta \in [\widetilde{\chi}_1, \widetilde{\chi}_2]$ is defined as*

$$\int_{\widetilde{\chi}_1}^{\theta}\hbar(\jmath)\,{}_{\widetilde{\chi}_1}d_{\acute{\mathsf{q}}}\jmath = (1-\acute{\mathsf{q}})(\theta-\widetilde{\chi}_1)\sum_{r=0}^{\infty}\acute{\mathsf{q}}^r\hbar\left(\acute{\mathsf{q}}^r\theta + (1-\acute{\mathsf{q}}^r)\widetilde{\chi}_1\right). \tag{2.4}$$

Choosing $\widetilde{\chi}_1 = 0$ in (2.4), we have

$$\int_0^\theta \hbar(\jmath) d_{\acute{q}}\jmath = (1-\acute{q})\theta \sum_{r=0}^{\infty} \acute{q}^r \hbar\left(\acute{q}^r\theta\right),$$

which gives

$$\int_0^1 \jmath^{\alpha+s} d_{\acute{q}}\jmath = \frac{1}{[\alpha+s+1]_{\acute{q}}}, \qquad \int_0^1 \jmath^{\alpha}(1-\jmath)^s d_{\acute{q}}\jmath = \frac{\Gamma_{\acute{q}}(\alpha+1)\Gamma_{\acute{q}}(s+1)}{\Gamma_{\acute{q}}(\alpha+s+2)},$$

where $\acute{q}$-gamma function for $\theta > 0$ is defined by

$$\Gamma_{\acute{q}}(\theta) = \int_0^\infty \jmath^{\theta-1}\mathcal{E}_{\acute{q}}^{-\acute{q}\jmath} d_{\acute{q}}\jmath,$$
$$\Gamma_{\acute{q}}(\theta+1) = [\theta]_{\acute{q}}\Gamma_{\acute{q}}(\theta),$$

and $\acute{q}$-exponential function is given as

$$\mathcal{E}_{\acute{q}}^{\jmath} = \sum_{r=0}^{\infty} \acute{q}^{\frac{r(r-1)}{2}} \frac{\jmath^r}{[r]_{\acute{q}}!}.$$

The following $\acute{q}$-fractional integrals can be studied in [22].

Definition 2.4. *Let $\alpha > 0$, $0 \leq \widetilde{\chi}_1 < \widetilde{\chi}_2$ and $\hbar \in \mathcal{L}[\widetilde{\chi}_1, \widetilde{\chi}_2]$. Then the Riemann–Liouville $\acute{q}$-fractional integrals of order α are defined by*

$$\mathrm{J}^{\alpha}_{\acute{q},\widetilde{\chi}_1^+}\hbar(\theta) = \frac{1}{\Gamma_{\acute{q}}(\alpha)}\int_{\widetilde{\chi}_1}^{\theta} (\theta - \acute{q}\jmath)^{\alpha-1}\hbar(\jmath)\, {}_{\widetilde{\chi}_1}d_{\acute{q}}\jmath, \quad \widetilde{\chi}_1 < \theta \tag{2.5}$$

and

$$\mathrm{J}^{\alpha}_{\acute{q},\widetilde{\chi}_2^-}\hbar(\theta) = \frac{1}{\Gamma_{\acute{q}}(\alpha)}\int_{\acute{q}\theta}^{\widetilde{\chi}_2} (\jmath - \acute{q}\theta)^{\alpha-1}\hbar(\jmath) d_{\acute{q}}\jmath, \quad \theta < \widetilde{\chi}_2,$$

where $\Gamma_{\acute{q}}(\cdot)$ is $\acute{q}$-gamma function. For $\acute{q} \to 1^-$, we get the Riemann–Liouville fractional integral operators.

Theorem 2. *($\acute{q}$-integration by parts)* [38] *Let $\hbar_1, \hbar_2 \in \mathcal{C}[\widetilde{\chi}_1, \widetilde{\chi}_2]$. Then for all $\theta \in [\widetilde{\chi}_1, \widetilde{\chi}_2]$, we have*

$$\begin{aligned}\int_{\widetilde{\chi}_1}^{\theta} \hbar_1(\jmath)\, {}_{\widetilde{\chi}_1}\mathrm{D}_{\acute{q}}\hbar_2(\jmath)\, {}_{\widetilde{\chi}_1}d_{\acute{q}}\jmath &= \hbar_1(\theta)\hbar_2(\theta) - \hbar_1(\widetilde{\chi}_1)\hbar_2(\widetilde{\chi}_1)\\ &\quad - \int_{\widetilde{\chi}_1}^{\theta} \hbar_2\left(\acute{q}\jmath + (1-\acute{q})\widetilde{\chi}_1\right)\, {}_{\widetilde{\chi}_1}\mathrm{D}_{\acute{q}}\hbar_1(\jmath)\, {}_{\widetilde{\chi}_1}d_{\acute{q}}\jmath.\end{aligned} \tag{2.6}$$

Theorem 3. *($\acute{q}$-Hölder's inequality)* [35] *Let $\hbar_1$, $\hbar_2$ be two $\acute{q}$-integrable functions on $[\widetilde{\chi}_1, \widetilde{\chi}_2]$, such that p, $\acute{q}_* > 1$ and $\frac{1}{p} + \frac{1}{\acute{q}_*} = 1$. Then we have*

$$\int_{\widetilde{\chi}_1}^{\widetilde{\chi}_2} |\hbar_1(\jmath)\hbar_2(\jmath)|\, {}_{\widetilde{\chi}_1}d_{\acute{q}}\jmath \leq \left(\int_{\widetilde{\chi}_1}^{\widetilde{\chi}_2} |\hbar_1(\jmath)|^p\, {}_{\widetilde{\chi}_1}d_{\acute{q}}\jmath\right)^{\frac{1}{p}} \left(\int_{\widetilde{\chi}_1}^{\widetilde{\chi}_2} |\hbar_2(\jmath)|^{\acute{q}_*}\, {}_{\widetilde{\chi}_1}d_{\acute{q}}\jmath\right)^{\frac{1}{\acute{q}_*}}. \tag{2.7}$$

Theorem 4. *($\acute{\mathsf{q}}$-Power mean inequality)* [35] *Let $\hbar_1$, $\hbar_2$ be two $\acute{\mathsf{q}}$-integrable functions on $[\widetilde{\chi}_1, \widetilde{\chi}_2]$ such that $\acute{\mathsf{q}}_* \geq 1$. Then we have*

$$\int_{\widetilde{\chi}_1}^{\widetilde{\chi}_2} |\hbar_1(\jmath)\hbar_2(\jmath)| \; {}_{\widetilde{\chi}_1}d_{\acute{\mathsf{q}}}\jmath \leq \left(\int_{\widetilde{\chi}_1}^{\widetilde{\chi}_2} |\hbar_1(\jmath)| \; {}_{\widetilde{\chi}_1}d_{\acute{\mathsf{q}}}\jmath\right)^{1-\frac{1}{\acute{\mathsf{q}}_*}} \left(\int_{\widetilde{\chi}_1}^{\widetilde{\chi}_2} |\hbar_1(\jmath)| \, |\hbar_2(\jmath)|^{\acute{\mathsf{q}}_*} \; {}_{\widetilde{\chi}_1}d_{\acute{\mathsf{q}}}\jmath\right)^{\frac{1}{\acute{\mathsf{q}}_*}}. \tag{2.8}$$

3. Main Results

For the simplicities of notations, let

$$\delta(\theta, \alpha) := \int_0^1 |\jmath^\alpha - \theta| \, d\jmath, \quad \rho(\theta, p, \alpha) := \int_0^1 |\jmath^\alpha - \theta|^p \, d\jmath.$$

Let us recall the well-known beta and hypergeometric functions below:

$$\beta(x, y) := \int_0^1 \jmath^{x-1}(1-\jmath)^{y-1} d\jmath, \quad x,\, y > 0$$

and

$${}_2\mathcal{F}_1\left(\widetilde{\chi}_1, \widetilde{\chi}_2; \tau; z\right) := \frac{1}{\beta(\widetilde{\chi}_2, \tau - \widetilde{\chi}_2)} \int_0^1 \jmath^{\widetilde{\chi}_2 - 1}(1-\jmath)^{\tau - \widetilde{\chi}_2 - 1}(1 - z\jmath)^{-\widetilde{\chi}_1} d\jmath,$$

for $\Re(\tau) > \Re(\widetilde{\chi}_2) > 0$ and $|z| \leq 1$.
The following two lemmas are very useful in the sequel.

Lemma 1. *For $\alpha > 0$ and $0 \leq \theta \leq 1$, we have*

$$\delta(\theta, \alpha) := \begin{cases} \dfrac{1}{\alpha+1}, & \text{for } \theta = 0; \\ \dfrac{2\alpha\theta^{1+\frac{1}{\alpha}} + 1}{\alpha+1} - \theta, & \text{for } 0 < \theta < 1; \\ \dfrac{\alpha}{\alpha+1}, & \text{for } \theta = 1. \end{cases}$$

Proof. The proof is evident. □

Lemma 2. *For $\alpha > 0$, $p \geq 1$ and $0 \leq \theta \leq 1$, we have*

$$\rho(\theta, p, \alpha) := \begin{cases} \dfrac{1}{p\alpha+1}, & \text{for } \theta = 0; \\ \dfrac{\theta^{p+\frac{1}{\alpha}}}{\alpha}\beta\left(\dfrac{1}{\alpha}, p+1\right) + \dfrac{(1-\theta)^{p+1}}{\alpha(p+1)} {}_2\mathcal{F}_1\left(1 - \dfrac{1}{\alpha}, 1; p+2; 1-\theta\right), & \text{for } 0 < \theta < 1; \\ \dfrac{1}{\alpha}\beta\left(\dfrac{1}{\alpha}, p+1\right), & \text{for } \theta = 1. \end{cases}$$

Proof. The proof is a straightforward computations. We omit here their details. □

We are in position to prove a new lemma including Riemann–Liouville $\acute{\mathsf{q}}$-fractional integrals, in order to establish our main results.

Lemma 3. *Suppose that $\hbar : [\widetilde{\chi}_1, \widetilde{\chi}_2] \to \mathbb{R}$ be a $\acute{\mathtt{q}}$-differentiable function, where $0 < \acute{\mathtt{q}} < 1$ such that $0 \le \widetilde{\chi}_1 < \widetilde{\chi}_2$. If $\mathtt{D}_{\acute{\mathtt{q}}}\hbar \in \mathcal{L}[\widetilde{\chi}_1, \widetilde{\chi}_2]$ and $\alpha \in \mathbb{N}$, then we have*

$$\begin{aligned}
&\frac{1}{8}\left[\hbar(\widetilde{\chi}_1) + 3\hbar\left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right) + 3\hbar\left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3}\right) + \hbar(\widetilde{\chi}_2)\right] \\
&\quad + \frac{[\alpha]_{\acute{\mathtt{q}}}(1-\acute{\mathtt{q}})}{3\acute{\mathtt{q}}^{\alpha}}\left[\hbar\left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right) + \hbar\left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3}\right) + \hbar(\widetilde{\chi}_2)\right] \\
&- \frac{3^{\alpha-1}\Gamma_{\acute{\mathtt{q}}}(\alpha+1)}{\acute{\mathtt{q}}^{\alpha}(\widetilde{\chi}_2 - \widetilde{\chi}_1)^{\alpha}}\left[\mathtt{J}^{\alpha}_{\acute{\mathtt{q}},\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1) + \mathtt{J}^{\alpha}_{\acute{\mathtt{q}},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right) + \mathtt{J}^{\alpha}_{\acute{\mathtt{q}},\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3}\right)\right] \\
&= \frac{(\widetilde{\chi}_2 - \widetilde{\chi}_1)}{9}\Big\{\int_0^1\left(\jmath^{\alpha} - \frac{3}{8}\right)\mathtt{D}_{\acute{\mathtt{q}}}\hbar\left(\jmath\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} + (1-\jmath)\widetilde{\chi}_1\right)d_{\acute{\mathtt{q}}}\jmath \\
&\quad + \int_0^1\left(\jmath^{\alpha} - \frac{1}{2}\right)\mathtt{D}_{\acute{\mathtt{q}}}\hbar\left(\jmath\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} + (1-\jmath)\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right)d_{\acute{\mathtt{q}}}\jmath \\
&\quad + \int_0^1\left(\jmath^{\alpha} - \frac{5}{8}\right)\mathtt{D}_{\acute{\mathtt{q}}}\hbar\left(\jmath\widetilde{\chi}_2 + (1-\jmath)\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3}\right)d_{\acute{\mathtt{q}}}\jmath\Big\}. \qquad (3.1)
\end{aligned}$$

Proof. Let denote, respectively,

$$\mathbb{I}_1 := \int_0^1\left(\jmath^{\alpha} - \frac{3}{8}\right)\mathtt{D}_{\acute{\mathtt{q}}}\hbar\left(\jmath\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} + (1-\jmath)\widetilde{\chi}_1\right)d_{\acute{\mathtt{q}}}\jmath,$$

$$\mathbb{I}_2 := \int_0^1\left(\jmath^{\alpha} - \frac{1}{2}\right)\mathtt{D}_{\acute{\mathtt{q}}}\hbar\left(\jmath\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} + (1-\jmath)\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right)d_{\acute{\mathtt{q}}}\jmath$$

and

$$\mathbb{I}_3 := \int_0^1\left(\jmath^{\alpha} - \frac{5}{8}\right)\mathtt{D}_{\acute{\mathtt{q}}}\hbar\left(\jmath\widetilde{\chi}_2 + (1-\jmath)\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3}\right)d_{\acute{\mathtt{q}}}\jmath.$$

With the help of $\acute{\mathtt{q}}$-integration by parts, we have

$$\begin{aligned}
\mathbb{I}_1 &= \int_0^1 \jmath^{\alpha}\mathtt{D}_{\acute{\mathtt{q}}}\hbar\left(\jmath\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} + (1-\jmath)\widetilde{\chi}_1\right)d_{\acute{\mathtt{q}}}\jmath - \frac{3}{8}\int_0^1\mathtt{D}_{\acute{\mathtt{q}}}\hbar\left(\jmath\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} + (1-\jmath)\widetilde{\chi}_1\right)d_{\acute{\mathtt{q}}}\jmath \\
&= -\frac{9}{8(\widetilde{\chi}_2 - \widetilde{\chi}_1)}\left[\hbar\left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right) - \hbar(\widetilde{\chi}_1)\right] + 3\jmath^{\alpha}\frac{\hbar\left(\jmath\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3} + (1-\jmath)\widetilde{\chi}_1\right)}{\widetilde{\chi}_2 - \widetilde{\chi}_1}\Bigg|_0^1 \\
&\quad - \frac{3[\alpha]_{\acute{\mathtt{q}}}}{\widetilde{\chi}_2 - \widetilde{\chi}_1}\int_0^1 \jmath^{\alpha-1}\hbar\left(\acute{\mathtt{q}}\jmath\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} + (1-\acute{\mathtt{q}}\jmath)\widetilde{\chi}_1\right)d_{\acute{\mathtt{q}}}\jmath \\
&= -\frac{9}{8(\widetilde{\chi}_2 - \widetilde{\chi}_1)}\left[\hbar\left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right) - \hbar(\widetilde{\chi}_1)\right] + \frac{3}{\widetilde{\chi}_2 - \widetilde{\chi}_1}\hbar\left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right) + \\
&\quad + \frac{3[\alpha]_{\acute{\mathtt{q}}}(1-\acute{\mathtt{q}})}{\acute{\mathtt{q}}^{\alpha}(\widetilde{\chi}_2 - \widetilde{\chi}_1)}\hbar\left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right) - \frac{3^{\alpha+1}[\alpha]_{\acute{\mathtt{q}}}\Gamma_{\acute{\mathtt{q}}}(\alpha)}{\acute{\mathtt{q}}^{\alpha}(\widetilde{\chi}_2 - \widetilde{\chi}_1)^{\alpha+1}}\frac{1}{\Gamma_{\acute{\mathtt{q}}}(\alpha)}\int_{\acute{\mathtt{q}}\widetilde{\chi}_1}^{\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}}(\jmath - \acute{\mathtt{q}}\widetilde{\chi}_1)^{\alpha-1}\hbar(\jmath)d_{\acute{\mathtt{q}}}\jmath \\
&= \frac{15}{8(\widetilde{\chi}_2 - \widetilde{\chi}_1)}\hbar\left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right) + \frac{9}{8(\widetilde{\chi}_2 - \widetilde{\chi}_1)}\hbar(\widetilde{\chi}_1) + \frac{3[\alpha]_{\acute{\mathtt{q}}}(1-\acute{\mathtt{q}})}{\acute{\mathtt{q}}^{\alpha}(\widetilde{\chi}_2 - \widetilde{\chi}_1)}\hbar\left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right) \\
&\quad - \frac{3^{\alpha+1}\Gamma_{\acute{\mathtt{q}}}(\alpha+1)}{\acute{\mathtt{q}}^{\alpha}(\widetilde{\chi}_2 - \widetilde{\chi}_1)^{\alpha+1}}\mathtt{J}^{\alpha}_{\acute{\mathtt{q}},\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1). \qquad (3.2)
\end{aligned}$$

Similarly, we get

$$\mathbb{I}_2 = \int_0^1 \jmath^{\alpha} \mathtt{D}_{\acute{\mathtt{q}}} \hbar \left(\jmath \frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} + (1-\jmath) \frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} \right) d_{\acute{\mathtt{q}}} \jmath - \tag{3.3}$$

$$- \frac{1}{2} \int_0^1 \mathtt{D}_{\acute{\mathtt{q}}} \hbar \left(\jmath \frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} + (1-\jmath) \frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} \right) d_{\acute{\mathtt{q}}} \jmath$$

$$= \frac{3}{2(\widetilde{\chi}_2 - \widetilde{\chi}_1)} \hbar \left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} \right) + \frac{3}{2(\widetilde{\chi}_2 - \widetilde{\chi}_1)} \hbar \left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} \right) +$$

$$+ \frac{3[\alpha]_{\acute{\mathtt{q}}}(1-\acute{\mathtt{q}})}{\acute{\mathtt{q}}^{\alpha}(\widetilde{\chi}_2 - \widetilde{\chi}_1)} \hbar \left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} \right) - \frac{3^{\alpha+1}\Gamma_{\acute{\mathtt{q}}}(\alpha+1)}{\acute{\mathtt{q}}^{\alpha}(\widetilde{\chi}_2 - \widetilde{\chi}_1)^{\alpha+1}} \mathrm{J}^{\alpha}_{\acute{\mathtt{q}}, \left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3}\right)^-} \hbar \left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} \right) \tag{3.4}$$

and

$$\mathbb{I}_3 = \int_0^1 \jmath^{\alpha} \mathtt{D}_{\acute{\mathtt{q}}} \hbar \left(\jmath \widetilde{\chi}_2 + (1-\jmath) \frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} \right) d_{\acute{\mathtt{q}}} \jmath -$$

$$- \frac{5}{8} \int_0^1 \mathtt{D}_{\acute{\mathtt{q}}} \hbar \left(\jmath \widetilde{\chi}_2 + (1-\jmath) \frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} \right) d_{\acute{\mathtt{q}}} \jmath$$

$$= \frac{9}{8(\widetilde{\chi}_2 - \widetilde{\chi}_1)} \hbar(\widetilde{\chi}_2) + \frac{15}{8(\widetilde{\chi}_2 - \widetilde{\chi}_1)} \hbar \left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} \right) + \frac{3[\alpha]_{\acute{\mathtt{q}}}(1-\acute{\mathtt{q}})}{\acute{\mathtt{q}}^{\alpha}(\widetilde{\chi}_2 - \widetilde{\chi}_1)} \hbar(\widetilde{\chi}_2)$$

$$- \frac{3^{\alpha+1}\Gamma_{\acute{\mathtt{q}}}(\alpha+1)}{\acute{\mathtt{q}}^{\alpha}(\widetilde{\chi}_2 - \widetilde{\chi}_1)^{\alpha+1}} \mathrm{J}^{\alpha}_{\acute{\mathtt{q}}, \widetilde{\chi}_2^-} \hbar \left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} \right). \tag{3.5}$$

Thus, we obtain the required identity (3.1) by adding equalities (3.2)–(3.5) and multiplying the resultant one by $\frac{\widetilde{\chi}_2 - \widetilde{\chi}_1}{9}$. □

Remark 3.1: Considering $\acute{\mathtt{q}} \to 1^-$ in Lemma 3, we have the following fractional identity:

$$\frac{1}{8} \left[\hbar(\widetilde{\chi}_1) + 3\hbar \left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} \right) + 3\hbar \left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} \right) + \hbar(\widetilde{\chi}_2) \right]$$

$$- \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\widetilde{\chi}_2 - \widetilde{\chi}_1)^{\alpha}} \left[\mathrm{J}^{\alpha}_{\left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right)^-} \hbar(\widetilde{\chi}_1) + \mathrm{J}^{\alpha}_{\left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3}\right)^-} \hbar \left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} \right) + \mathrm{J}^{\alpha}_{\widetilde{\chi}_2^-} \hbar \left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} \right) \right]$$

$$= \frac{(\widetilde{\chi}_2 - \widetilde{\chi}_1)}{9} \Big\{ \int_0^1 \left(\jmath^{\alpha} - \frac{3}{8} \right) \hbar' \left(\jmath \frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} + (1-\jmath)\widetilde{\chi}_1 \right) d\jmath$$

$$+ \int_0^1 \left(\jmath^{\alpha} - \frac{1}{2} \right) \hbar' \left(\jmath \frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} + (1-\jmath) \frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} \right) d\jmath$$

$$+ \int_0^1 \left(\jmath^{\alpha} - \frac{5}{8} \right) \hbar' \left(\jmath \widetilde{\chi}_2 + (1-\jmath) \frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} \right) d\jmath \Big\}, \tag{3.6}$$

which was established in ([1], Lemma 3.1).

Remark 3.2: Choosing $\alpha = 1$ in Lemma 3, we get the following $\acute{\mathtt{q}}$-identity:

$$\frac{1}{8} \left[\hbar(\widetilde{\chi}_1) + 3\hbar \left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} \right) + 3\hbar \left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} \right) + \hbar(\widetilde{\chi}_2) \right]$$

$$+ \frac{(1-\acute{\mathtt{q}})}{3\acute{\mathtt{q}}} \left[\hbar \left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} \right) + \hbar \left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} \right) + \hbar(\widetilde{\chi}_2) \right]$$

$$-\frac{1}{\acute{\mathsf{q}}(\widetilde{\chi}_2-\widetilde{\chi}_1)}\left[\int_{\acute{\mathsf{q}}\widetilde{\chi}_1}^{\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}}\hbar(\jmath)d_{\acute{\mathsf{q}}}\jmath+\int_{\acute{\mathsf{q}}\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}}^{\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}}\hbar(\jmath)d_{\acute{\mathsf{q}}}\jmath+\int_{\acute{\mathsf{q}}\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}}^{\widetilde{\chi}_2}\hbar(\jmath)d_{\acute{\mathsf{q}}}\jmath\right]$$
$$=\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Big\{\int_0^1\left(\jmath-\frac{3}{8}\right)\mathtt{D}_{\acute{\mathsf{q}}}\hbar\left(\jmath\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}+(1-\jmath)\widetilde{\chi}_1\right)d_{\acute{\mathsf{q}}}\jmath$$
$$+\int_0^1\left(\jmath-\frac{1}{2}\right)\mathtt{D}_{\acute{\mathsf{q}}}\hbar\left(\jmath\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}+(1-\jmath)\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)d_{\acute{\mathsf{q}}}\jmath$$
$$+\int_0^1\left(\jmath-\frac{5}{8}\right)\mathtt{D}_{\acute{\mathsf{q}}}\hbar\left(\jmath\widetilde{\chi}_2+(1-\jmath)\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)d_{\acute{\mathsf{q}}}\jmath\Big\}. \tag{3.7}$$

By using Lemmas 1, 2 and 3, we established the following $\acute{\mathsf{q}}$-fractional integral inequalities.

Theorem 5. *Assume that $\hbar:[\widetilde{\chi}_1,\widetilde{\chi}_2]\to\mathbb{R}$ is a $\acute{\mathsf{q}}$-differentiable function with $0<\acute{\mathsf{q}}<1$, such that $0\le\widetilde{\chi}_1<\widetilde{\chi}_2$. Suppose that $\mathtt{D}_{\acute{\mathsf{q}}}\hbar\in\mathcal{L}[\widetilde{\chi}_1,\widetilde{\chi}_2]$ and $|\mathtt{D}_{\acute{\mathsf{q}}}\hbar|^{\acute{\mathsf{q}}_*}$ is n-polynomial convex function for all $n\in\mathbb{N}$, where p, $\acute{\mathsf{q}}_*>1$ and $\frac{1}{p}+\frac{1}{\acute{\mathsf{q}}_*}=1$. Then for $\alpha\in\mathbb{N}$, the following $\acute{\mathsf{q}}$-fractional integral inequality holds true:*

$$\Big|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$+\frac{[\alpha]_{\acute{\mathsf{q}}}(1-\acute{\mathsf{q}})}{3\acute{\mathsf{q}}^\alpha}\left[\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$-\frac{3^{\alpha-1}\Gamma_{\acute{\mathsf{q}}}(\alpha+1)}{\acute{\mathsf{q}}^\alpha(\widetilde{\chi}_2-\widetilde{\chi}_1)^\alpha}\left[\mathrm{J}^{\alpha}_{\acute{\mathsf{q}},\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1)+\mathrm{J}^{\alpha}_{\acute{\mathsf{q}},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\mathrm{J}^{\alpha}_{\acute{\mathsf{q}},\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Big|$$

$$\le\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\left(1-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\acute{\mathsf{q}}}}\right)^{\frac{1}{\acute{\mathsf{q}}_*}}$$
$$\times\Big\{\mathrm{A}^{\frac{1}{p}}_{\acute{\mathsf{q}}}(p,\alpha)\left[|\mathtt{D}_{\acute{\mathsf{q}}}\hbar(\widetilde{\chi}_1)|^{\acute{\mathsf{q}}_*}+\left|\mathtt{D}_{\acute{\mathsf{q}}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{\mathsf{q}}_*}\right]^{\frac{1}{\acute{\mathsf{q}}_*}}$$
$$+\mathrm{B}^{\frac{1}{p}}_{\acute{\mathsf{q}}}(p,\alpha)\left[\left|\mathtt{D}_{\acute{\mathsf{q}}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{\mathsf{q}}_*}+\left|\mathtt{D}_{\acute{\mathsf{q}}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{\mathsf{q}}_*}\right]^{\frac{1}{\acute{\mathsf{q}}_*}}$$
$$+\mathrm{C}^{\frac{1}{p}}_{\acute{\mathsf{q}}}(p,\alpha)\left[\left|\mathtt{D}_{\acute{\mathsf{q}}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{\mathsf{q}}_*}+\left|\mathtt{D}_{\acute{\mathsf{q}}}\hbar(\widetilde{\chi}_2)\right|^{\acute{\mathsf{q}}_*}\right]^{\frac{1}{\acute{\mathsf{q}}_*}}\Big\}, \tag{3.8}$$

where

$$\mathrm{A}_{\acute{\mathsf{q}}}(p,\alpha):=\int_0^1\left|\jmath^\alpha-\tfrac{3}{8}\right|^p d_{\acute{\mathsf{q}}}\jmath,$$
$$\mathrm{B}_{\acute{\mathsf{q}}}(p,\alpha):=\int_0^1\left|\jmath^\alpha-\tfrac{1}{2}\right|^p d_{\acute{\mathsf{q}}}\jmath,$$
$$\mathrm{C}_{\acute{\mathsf{q}}}(p,\alpha):=\int_0^1\left|\jmath^\alpha-\tfrac{5}{8}\right|^p d_{\acute{\mathsf{q}}}\jmath.$$

Proof. By using Lemma 3, ǫ́-Hölder's inequality, n-polynomial convexity of $|\mathtt{D}_{\acute{q}}\hbar|^{\acute{q}_*}$ and properties of modulus, we have

$$\begin{aligned}
&\Big|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]\\
&+\frac{[\alpha]_{\acute{q}}(1-\acute{q})}{3\acute{q}^{\alpha}}\left[\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]\\
&-\frac{3^{\alpha-1}\Gamma_{\acute{q}}(\alpha+1)}{\acute{q}^{\alpha}(\widetilde{\chi}_2-\widetilde{\chi}_1)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\acute{q},\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1)+\mathrm{J}^{\alpha}_{\acute{q},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\mathrm{J}^{\alpha}_{\acute{q},\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Big|
\end{aligned}$$

$$\begin{aligned}
&\le\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Big\{\int_0^1\left|\jmath^{\alpha}-\frac{3}{8}\right|\left|\mathtt{D}_{\acute{q}}\hbar\left(\jmath\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}+(1-\jmath)\widetilde{\chi}_1\right)\right|d_{\acute{q}}\jmath\\
&\quad+\int_0^1\left|\jmath^{\alpha}-\frac{1}{2}\right|\left|\mathtt{D}_{\acute{q}}\hbar\left(\jmath\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}+(1-\jmath)\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|d_{\acute{q}}\jmath\\
&\quad+\int_0^1\left|\jmath^{\alpha}-\frac{5}{8}\right|\left|\mathtt{D}_{\acute{q}}\hbar\left(\jmath\widetilde{\chi}_2+(1-\jmath)\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|d_{\acute{q}}\jmath\Big\}\\
&\le\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Big\{\left(\int_0^1\left|\jmath^{\alpha}-\frac{3}{8}\right|^p d_{\acute{q}}\jmath\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathtt{D}_{\acute{q}}\hbar\left(\jmath\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}+(1-\jmath)\widetilde{\chi}_1\right)\right|^{\acute{q}_*}d_{\acute{q}}\jmath\right)^{\frac{1}{\acute{q}_*}}\\
&\quad+\left(\int_0^1\left|\jmath^{\alpha}-\frac{1}{2}\right|^p d_{\acute{q}}\jmath\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathtt{D}_{\acute{q}}\hbar\left(\jmath\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}+(1-\jmath)\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}d_{\acute{q}}\jmath\right)^{\frac{1}{\acute{q}_*}}\\
&\quad+\left(\int_0^1\left|\jmath^{\alpha}-\frac{5}{8}\right|^p d_{\acute{q}}\jmath\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathtt{D}_{\acute{q}}\hbar\left(\jmath\widetilde{\chi}_2+(1-\jmath)\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}d_{\acute{q}}\jmath\right)^{\frac{1}{\acute{q}_*}}\Big\}
\end{aligned}$$

$$\begin{aligned}
&\le\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Big\{\mathrm{A}_{\acute{q}}^{\frac{1}{p}}(p,\alpha)\left[\int_0^1\frac{1}{n}\sum_{\ell=1}^{n}\left(1-(1-\jmath)^{\ell}\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}d_{\acute{q}}\jmath+\int_0^1\frac{1}{n}\sum_{\ell=1}^{n}\left(1-\jmath^{\ell}\right)|\mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_1)|^{\acute{q}_*}d_{\acute{q}}\jmath\right]^{\frac{1}{\acute{q}_*}}\\
&\quad+\mathrm{B}_{\acute{q}}^{\frac{1}{p}}(p,\alpha)\left[\int_0^1\frac{1}{n}\sum_{\ell=1}^{n}\left(1-(1-\jmath)^{\ell}\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}d_{\acute{q}}\jmath+\int_0^1\frac{1}{n}\sum_{\ell=1}^{n}\left(1-\jmath^{\ell}\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}d_{\acute{q}}\jmath\right]^{\frac{1}{\acute{q}_*}}\\
&\quad+\mathrm{C}_{\acute{q}}^{\frac{1}{p}}(p,\alpha)\left[\int_0^1\frac{1}{n}\sum_{\ell=1}^{n}\left(1-(1-\jmath)^{\ell}\right)\left|\mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_2)\right|^{\acute{q}_*}d_{\acute{q}}\jmath+\int_0^1\frac{1}{n}\sum_{\ell=1}^{n}\left(1-\jmath^{\ell}\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}d_{\acute{q}}\jmath\right]^{\frac{1}{\acute{q}_*}}\Big\}
\end{aligned}$$

$$\begin{aligned}
&=\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\left(1-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\acute{q}}}\right)^{\frac{1}{\acute{q}_*}}\\
&\quad\times\Big\{\mathrm{A}_{\acute{q}}^{\frac{1}{p}}(p,\alpha)\left[|\mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_1)|^{\acute{q}_*}+\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\\
&\quad+\mathrm{B}_{\acute{q}}^{\frac{1}{p}}(p,\alpha)\left[\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\\
&\quad+\mathrm{C}_{\acute{q}}^{\frac{1}{p}}(p,\alpha)\left[\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+|\mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_2)|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\Big\}.
\end{aligned}$$

This concludes the desired proof. □

Corollary 5.1. *Suppose* $\acute{q} \to 1^-$ *in Theorem 5. Then we have*

$$\Big|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\widetilde{\chi}_2-\widetilde{\chi}_1)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1)+\mathrm{J}^{\alpha}_{\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\mathrm{J}^{\alpha}_{\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Big|$$
$$\leq\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\left(1-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{\ell+1}\right)^{\frac{1}{\acute{q}_*}}$$
$$\times\left\{\rho^{\frac{1}{p}}\left(\frac{3}{8},p,\alpha\right)\left[|\hbar'(\widetilde{\chi}_1)|^{\acute{q}_*}+\left|\hbar'\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\right.$$
$$+\rho^{\frac{1}{p}}\left(\frac{1}{2},p,\alpha\right)\left[\left|\hbar'\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+\left|\hbar'\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}$$
$$\left.+\rho^{\frac{1}{p}}\left(\frac{5}{8},p,\alpha\right)\left[\left|\hbar'\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+|\hbar'(\widetilde{\chi}_2)|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\right\}. \tag{3.9}$$

Corollary 5.2. *Considering* $\alpha = 1$ *in Theorem 5, we get*

$$\Big|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$+\frac{(1-\acute{q})}{3\acute{q}}\left[\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$-\frac{1}{\acute{q}(\widetilde{\chi}_2-\widetilde{\chi}_1)}\left[\int_{\acute{q}\widetilde{\chi}_1}^{\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}}\hbar(\jmath)d_{\acute{q}}\jmath+\int_{\acute{q}\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}}^{\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}}\hbar(\jmath)d_{\acute{q}}\jmath+\int_{\acute{q}\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}}^{\widetilde{\chi}_2}\hbar(\jmath)d_{\acute{q}}\jmath\right]\Big|$$
$$\leq\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\left(1-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\acute{q}}}\right)^{\frac{1}{\acute{q}_*}}$$
$$\times\left\{\mathrm{A}_{\acute{q}}^{\frac{1}{p}}(p)\left[|\mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_1)|^{\acute{q}_*}+\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\right.$$
$$+\mathrm{B}_{\acute{q}}^{\frac{1}{p}}(p)\left[\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}$$
$$\left.+\mathrm{C}_{\acute{q}}^{\frac{1}{p}}(p)\left[\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+\left|\mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_2)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\right\}, \tag{3.10}$$

where

$$\mathrm{A}_{\acute{q}}(p):=\int_0^1\left|\jmath-\frac{3}{8}\right|^p d_{\acute{q}}\jmath,\qquad \mathrm{B}_{\acute{q}}(p):=\int_0^1\left|\jmath-\frac{1}{2}\right|^p d_{\acute{q}}\jmath,\qquad \mathrm{C}_{\acute{q}}(p):=\int_0^1\left|\jmath-\frac{5}{8}\right|^p d_{\acute{q}}\jmath.$$

Corollary 5.3. *Taking* $|\mathtt{D}_{\acute{q}}\hbar|\leq\mathcal{K}$ *in Theorem 5, we obtain*

$$\Big|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$

$$+\frac{[\alpha]_{\acute{q}}(1-\acute{q})}{3\acute{q}^{\alpha}}\left[\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$-\frac{3^{\alpha-1}\Gamma_{\acute{q}}(\alpha+1)}{\acute{q}^{\alpha}(\widetilde{\chi}_2-\widetilde{\chi}_1)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\acute{q},\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1)+\mathrm{J}^{\alpha}_{\acute{q},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\mathrm{J}^{\alpha}_{\acute{q},\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Bigg|$$
$$\leq\frac{2^{\frac{1}{\acute{q}_*}}\mathcal{K}(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\left(1-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\acute{q}}}\right)^{\frac{1}{\acute{q}_*}}\left\{\mathrm{A}^{\frac{1}{p}}_{\acute{q}}(p,\alpha)+\mathrm{B}^{\frac{1}{p}}_{\acute{q}}(p,\alpha)+\mathrm{C}^{\frac{1}{p}}_{\acute{q}}(p,\alpha)\right\}. \tag{3.11}$$

Corollary 5.4. *Considering $n=1$ in Theorem 5, we have*

$$\Bigg|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$+\frac{[\alpha]_{\acute{q}}(1-\acute{q})}{3\acute{q}^{\alpha}}\left[\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$-\frac{3^{\alpha-1}\Gamma_{\acute{q}}(\alpha+1)}{\acute{q}^{\alpha}(\widetilde{\chi}_2-\widetilde{\chi}_1)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\acute{q},\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1)+\mathrm{J}^{\alpha}_{\acute{q},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\mathrm{J}^{\alpha}_{\acute{q},\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Bigg|$$
$$\leq\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\left(\frac{\acute{q}}{\acute{q}+1}\right)^{\frac{1}{\acute{q}_*}}$$
$$\times\left\{\mathrm{A}^{\frac{1}{p}}_{\acute{q}}(p,\alpha)\left[|\mathrm{D}_{\acute{q}}\hbar(\widetilde{\chi}_1)|^{\acute{q}_*}+\left|\mathrm{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\right.$$
$$+\mathrm{B}^{\frac{1}{p}}_{\acute{q}}(p,\alpha)\left[\left|\mathrm{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+\left|\mathrm{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}$$
$$\left.+\mathrm{C}^{\frac{1}{p}}_{\acute{q}}(p,\alpha)\left[\left|\mathrm{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+\left|\mathrm{D}_{\acute{q}}\hbar(\widetilde{\chi}_2)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\right\}. \tag{3.12}$$

Corollary 5.5. *Taking $\acute{q}\to 1^-$ in Corollary 5.4, we get*

$$\Bigg|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\widetilde{\chi}_2-\widetilde{\chi}_1)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1)+\mathrm{J}^{\alpha}_{\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\mathrm{J}^{\alpha}_{\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Bigg|$$
$$\leq\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\left(\frac{1}{2}\right)^{\frac{1}{\acute{q}_*}}$$
$$\times\left\{\mathrm{A}^{\frac{1}{p}}(p,\alpha)\left[|\hbar'(\widetilde{\chi}_1)|^{\acute{q}_*}+\left|\hbar'\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\right.$$
$$+\mathrm{B}^{\frac{1}{p}}(p,\alpha)\left[\left|\hbar'\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+\left|\hbar'\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}$$

$$+ \mathrm{C}^{\frac{1}{p}}(p,\alpha)\left[\left|\hbar'\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{\mathtt{q}}_*}+\left|\hbar'(\widetilde{\chi}_2)\right|^{\acute{\mathtt{q}}_*}\right]^{\frac{1}{\acute{\mathtt{q}}_*}}\Bigg\}, \tag{3.13}$$

where

$$\mathrm{A}(p,\alpha) := \int_0^1 \left|\jmath^\alpha - \tfrac{3}{8}\right|^p d\jmath,$$
$$\mathrm{B}(p,\alpha) := \int_0^1 \left|\jmath^\alpha - \tfrac{1}{2}\right|^p d\jmath,$$
$$\mathrm{C}(p,\alpha) := \int_0^1 \left|\jmath^\alpha - \tfrac{5}{8}\right|^p d\jmath.$$

Theorem 6. *Assume that* $\hbar : [\widetilde{\chi}_1, \widetilde{\chi}_2] \to \mathbb{R}$ *is a* $\acute{\mathtt{q}}$*-differentiable function with* $0 < \acute{\mathtt{q}} < 1$*, such that* $0 \le \widetilde{\chi}_1 < \widetilde{\chi}_2$*. Suppose that* $\mathtt{D}_{\acute{\mathtt{q}}}\hbar \in \mathcal{L}[\widetilde{\chi}_1, \widetilde{\chi}_2]$ *and* $|\mathtt{D}_{\acute{\mathtt{q}}}\hbar|^{\acute{\mathtt{q}}_*}$ *is* n*-polynomial convex function for all* $n \in \mathbb{N}$*, where* $\acute{\mathtt{q}}_* \ge 1$*. Then for* $\alpha \in \mathbb{N}$*, the following* $\acute{\mathtt{q}}$*-fractional integral inequality holds true:*

$$\Bigg|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$+\frac{[\alpha]_{\acute{\mathtt{q}}}(1-\acute{\mathtt{q}})}{3\acute{\mathtt{q}}^\alpha}\left[\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$-\frac{3^{\alpha-1}\Gamma_{\acute{\mathtt{q}}}(\alpha+1)}{\acute{\mathtt{q}}^\alpha(\widetilde{\chi}_2-\widetilde{\chi}_1)^\alpha}\left[\mathrm{J}^\alpha_{\acute{\mathtt{q}},\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1)+\mathrm{J}^\alpha_{\acute{\mathtt{q}},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\mathrm{J}^\alpha_{\acute{\mathtt{q}},\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Bigg|$$
$$\le \frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Bigg\{\mathrm{A}_{\acute{\mathtt{q}}}^{1-\frac{1}{\acute{\mathtt{q}}_*}}(\alpha)\left[\left(\mathrm{A}_{\acute{\mathtt{q}}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^n \mathrm{E}_{\acute{\mathtt{q}}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{\mathtt{q}}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{\mathtt{q}}_*}\right.$$
$$\left.+\left(\mathrm{A}_{\acute{\mathtt{q}}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^n \mathrm{F}_{\acute{\mathtt{q}}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{\mathtt{q}}}\hbar(\widetilde{\chi}_1)\right|^{\acute{\mathtt{q}}_*}\right]^{\frac{1}{\acute{\mathtt{q}}_*}}$$
$$+\mathrm{B}_{\acute{\mathtt{q}}}^{1-\frac{1}{\acute{\mathtt{q}}_*}}(\alpha)\left[\left(\mathrm{B}_{\acute{\mathtt{q}}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^n \mathrm{G}_{\acute{\mathtt{q}}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{\mathtt{q}}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{\mathtt{q}}_*}\right.$$
$$\left.+\left(\mathrm{B}_{\acute{\mathtt{q}}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^n \mathrm{H}_{\acute{\mathtt{q}}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{\mathtt{q}}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{\mathtt{q}}_*}\right]^{\frac{1}{\acute{\mathtt{q}}_*}}$$
$$+\mathrm{C}_{\acute{\mathtt{q}}}^{1-\frac{1}{\acute{\mathtt{q}}_*}}(\alpha)\left[\left(\mathrm{C}_{\acute{\mathtt{q}}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^n \mathrm{M}_{\acute{\mathtt{q}}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{\mathtt{q}}}\hbar(\widetilde{\chi}_2)\right|^{\acute{\mathtt{q}}_*}\right.$$
$$\left.+\left(\mathrm{C}_{\acute{\mathtt{q}}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^n \mathrm{N}_{\acute{\mathtt{q}}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{\mathtt{q}}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{\mathtt{q}}_*}\right]^{\frac{1}{\acute{\mathtt{q}}_*}}\Bigg\}, \tag{3.14}$$

where

$$\mathrm{A}_{\acute{\mathtt{q}}}(\alpha) := \int_0^1 \left|\jmath^\alpha - \frac{3}{8}\right| d_{\acute{\mathtt{q}}}\jmath, \quad \mathrm{B}_{\acute{\mathtt{q}}}(\alpha) := \int_0^1 \left|\jmath^\alpha - \frac{1}{2}\right| d_{\acute{\mathtt{q}}}\jmath, \quad \mathrm{C}_{\acute{\mathtt{q}}}(\alpha) := \int_0^1 \left|\jmath^\alpha - \frac{5}{8}\right| d_{\acute{\mathtt{q}}}\jmath$$

and

$$\mathrm{E}_{\acute{\mathtt{q}}}(\alpha,\ell) := \int_0^1 (1-\jmath)^\ell \left|\jmath^\alpha - \frac{3}{8}\right| d_{\acute{\mathtt{q}}}\jmath, \quad \mathrm{F}_{\acute{\mathtt{q}}}(\alpha,\ell) := \int_0^1 \jmath^\ell \left|\jmath^\alpha - \frac{3}{8}\right| d_{\acute{\mathtt{q}}}\jmath,$$

$$\mathrm{G}_{\acute{q}}(\alpha,\ell):=\int_0^1(1-\jmath)^{\ell}\left|\jmath^{\alpha}-\frac{1}{2}\right|d_{\acute{q}}\jmath,\quad \mathrm{H}_{\acute{q}}(\alpha,\ell):=\int_0^1\jmath^{\ell}\left|\jmath^{\alpha}-\frac{1}{2}\right|d_{\acute{q}}\jmath,$$
$$\mathrm{M}_{\acute{q}}(\alpha,\ell):=\int_0^1(1-\jmath)^{\ell}\left|\jmath^{\alpha}-\frac{5}{8}\right|d_{\acute{q}}\jmath,\quad \mathrm{N}_{\acute{q}}(\alpha,\ell):=\int_0^1\jmath^{\ell}\left|\jmath^{\alpha}-\frac{5}{8}\right|d_{\acute{q}}\jmath.$$

Proof. By using Lemma 3, $\acute{q}$-Power mean inequality, n-polynomial convexity of $|\mathtt{D}_{\acute{q}}\hbar|^{\acute{q}_*}$ and properties of modulus, we have

$$\Big|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$+\frac{[\alpha]_{\acute{q}}(1-\acute{q})}{3\acute{q}^{\alpha}}\left[\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$-\frac{3^{\alpha-1}\Gamma_{\acute{q}}(\alpha+1)}{\acute{q}^{\alpha}(\widetilde{\chi}_2-\widetilde{\chi}_1)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\acute{q},\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1)+\mathrm{J}^{\alpha}_{\acute{q},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\mathrm{J}^{\alpha}_{\acute{q},\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Big|$$

$$\leq\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Big\{\int_0^1\left|\jmath^{\alpha}-\frac{3}{8}\right|\left|\mathtt{D}_{\acute{q}}\hbar\left(\jmath\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}+(1-\jmath)\widetilde{\chi}_1\right)\right|d_{\acute{q}}\jmath$$
$$+\int_0^1\left|\jmath^{\alpha}-\frac{1}{2}\right|\left|\mathtt{D}_{\acute{q}}\hbar\left(\jmath\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}+(1-\jmath)\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|d_{\acute{q}}\jmath$$
$$+\int_0^1\left|\jmath^{\alpha}-\frac{5}{8}\right|\left|\mathtt{D}_{\acute{q}}\hbar\left(\jmath\widetilde{\chi}_2+(1-\jmath)\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|d_{\acute{q}}\jmath\Big\}$$

$$\leq\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Big\{\left(\int_0^1\left|\jmath^{\alpha}-\frac{3}{8}\right|d_{\acute{q}}\jmath\right)^{1-\frac{1}{\acute{q}_*}}\left(\int_0^1\left|\jmath^{\alpha}-\frac{3}{8}\right|\left|\mathtt{D}_{\acute{q}}\hbar\left(\jmath\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}+(1-\jmath)\widetilde{\chi}_1\right)\right|^{\acute{q}_*}d_{\acute{q}}\jmath\right)^{\frac{1}{\acute{q}_*}}$$
$$+\left(\int_0^1\left|\jmath^{\alpha}-\frac{1}{2}\right|d_{\acute{q}}\jmath\right)^{1-\frac{1}{\acute{q}_*}}\left(\int_0^1\left|\jmath^{\alpha}-\frac{1}{2}\right|\left|\mathtt{D}_{\acute{q}}\hbar\left(\jmath\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}+(1-\jmath)\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}d_{\acute{q}}\jmath\right)^{\frac{1}{\acute{q}_*}}$$
$$+\left(\int_0^1\left|\jmath^{\alpha}-\frac{5}{8}\right|d_{\acute{q}}\jmath\right)^{1-\frac{1}{\acute{q}_*}}\left(\int_0^1\left|\jmath^{\alpha}-\frac{5}{8}\right|\left|\mathtt{D}_{\acute{q}}\hbar\left(\jmath\widetilde{\chi}_2+(1-\jmath)\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}d_{\acute{q}}\jmath\right)^{\frac{1}{\acute{q}_*}}\Big\}$$

$$\leq\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}$$
$$\times\Big\{\mathrm{A}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[\int_0^1\left|\jmath^{\alpha}-\frac{3}{8}\right|\left(\frac{1}{n}\sum_{\ell=1}^{n}\left(1-(1-\jmath)^{\ell}\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+\frac{1}{n}\sum_{\ell=1}^{n}\left(1-\jmath^{\ell}\right)|\mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_1)|^{\acute{q}_*}\right)d_{\acute{q}}\jmath\right]^{\frac{1}{\acute{q}_*}}$$
$$+\mathrm{B}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[\int_0^1\left|\jmath^{\alpha}-\frac{1}{2}\right|\left(\frac{1}{n}\sum_{\ell=1}^{n}\left(1-(1-\jmath)^{\ell}\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+\frac{1}{n}\sum_{\ell=1}^{n}\left(1-\jmath^{\ell}\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right)d_{\acute{q}}\jmath\right]^{\frac{1}{\acute{q}_*}}$$
$$+\mathrm{C}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[\int_0^1\left|\jmath^{\alpha}-\frac{5}{8}\right|\left(\frac{1}{n}\sum_{\ell=1}^{n}\left(1-(1-\jmath)^{\ell}\right)\left|\mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_2)\right|^{\acute{q}_*}+\frac{1}{n}\sum_{\ell=1}^{n}\left(1-\jmath^{\ell}\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right)d_{\acute{q}}\jmath\right]^{\frac{1}{\acute{q}_*}}\Big\}$$

$$=\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Big\{\mathrm{A}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[\left(\mathrm{A}_{\acute{q}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{E}_{\acute{q}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right.$$

$$+\left(\mathrm{A}_{\acute{q}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{F}_{\acute{q}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_1)\right|^{\acute{q}_*}\Bigg]^{\frac{1}{\acute{q}_*}}$$
$$+\mathrm{B}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[\left(\mathrm{B}_{\acute{q}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{G}_{\acute{q}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right.$$
$$\left.+\left(\mathrm{B}_{\acute{q}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{H}_{\acute{q}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}$$
$$+\mathrm{C}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[\left(\mathrm{C}_{\acute{q}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{M}_{\acute{q}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_2)\right|^{\acute{q}_*}\right.$$
$$\left.+\left(\mathrm{C}_{\acute{q}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{N}_{\acute{q}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\Bigg\}.$$

This completes the proof. □

Corollary 6.1. *Taking* $\acute{q}_*=1$ *in Theorem 6, we have*

$$\Bigg|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$+\frac{[\alpha]_{\acute{q}}(1-\acute{q})}{3\acute{q}^{\alpha}}\left[\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$-\frac{3^{\alpha-1}\Gamma_{\acute{q}}(\alpha+1)}{\acute{q}^{\alpha}(\widetilde{\chi}_2-\widetilde{\chi}_1)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\acute{q},\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1)+\mathrm{J}^{\alpha}_{\acute{q},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\mathrm{J}^{\alpha}_{\acute{q},\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Bigg|$$
$$\leq\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}$$
$$\times\Bigg\{\left(\mathrm{A}_{\acute{q}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{E}_{\acute{q}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|+\left(\mathrm{A}_{\acute{q}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{F}_{\acute{q}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_1)\right|$$
$$+\left(\mathrm{B}_{\acute{q}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{G}_{\acute{q}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|$$
$$+\left(\mathrm{B}_{\acute{q}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{H}_{\acute{q}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|$$
$$+\left(\mathrm{C}_{\acute{q}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{M}_{\acute{q}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_2)\right|+\left(\mathrm{C}_{\acute{q}}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{N}_{\acute{q}}(\alpha,\ell)\right)\left|\mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|\Bigg\}. \tag{3.15}$$

Corollary 6.2. *If we choose* $\acute{q}\to 1^-$ *in Theorem 6, we get*

$$\Bigg|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\widetilde{\chi}_2-\widetilde{\chi}_1)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1)+\mathrm{J}^{\alpha}_{\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\mathrm{J}^{\alpha}_{\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Bigg|$$

$$\leq \frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Big\{\mathrm{A}^{1-\frac{1}{q_*}}(\alpha)\Bigg[\Bigg(\mathrm{A}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{E}(\alpha,\ell)\Bigg)\left|\hbar'\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{q_*}$$
$$+\Bigg(\mathrm{A}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{F}(\alpha,\ell)\Bigg)|\hbar'(\widetilde{\chi}_1)|^{q_*}\Bigg]^{\frac{1}{q_*}}$$
$$+\mathrm{B}^{1-\frac{1}{q_*}}(\alpha)\Bigg[\Bigg(\mathrm{B}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{G}(\alpha,\ell)\Bigg)\left|\hbar'\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{q_*}$$
$$+\Bigg(\mathrm{B}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{H}(\alpha,\ell)\Bigg)\left|\hbar'\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{q_*}\Bigg]^{\frac{1}{q_*}}$$
$$+\mathrm{C}^{1-\frac{1}{q_*}}(\alpha)\Bigg[\Bigg(\mathrm{C}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{M}(\alpha,\ell)\Bigg)|\hbar'(\widetilde{\chi}_2)|^{q_*}$$
$$+\Bigg(\mathrm{C}(\alpha)-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{N}(\alpha,\ell)\Bigg)\left|\hbar'\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{q_*}\Bigg]^{\frac{1}{q_*}}\Big\}, \tag{3.16}$$

where

$$\mathrm{A}(\alpha):=\int_0^1\left|\jmath^\alpha-\frac{3}{8}\right|d\jmath=\frac{2\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{\frac{\alpha+1}{\alpha}}+\frac{1}{\alpha+1}-\frac{3}{8},$$
$$\mathrm{B}(\alpha):=\int_0^1\left|\jmath^\alpha-\frac{1}{2}\right|d\jmath=\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{\alpha+1}{\alpha}}+\frac{1}{\alpha+1}-\frac{1}{2},$$
$$\mathrm{C}(\alpha):=\int_0^1\left|\jmath^\alpha-\frac{5}{8}\right|d\jmath=\frac{2\alpha}{\alpha+1}\left(\frac{5}{8}\right)^{\frac{\alpha+1}{\alpha}}+\frac{1}{\alpha+1}-\frac{5}{8}$$

and

$$\mathrm{E}(\alpha,\ell):=\int_0^1(1-\jmath)^\ell\left|\jmath^\alpha-\frac{3}{8}\right|d\jmath,\quad \mathrm{F}(\alpha,\ell):=\int_0^1\jmath^\ell\left|\jmath^\alpha-\frac{3}{8}\right|d\jmath,$$
$$\mathrm{G}(\alpha,\ell):=\int_0^1(1-\jmath)^\ell\left|\jmath^\alpha-\frac{1}{2}\right|d\jmath,\quad \mathrm{H}(\alpha,\ell):=\int_0^1\jmath^\ell\left|\jmath^\alpha-\frac{1}{2}\right|d\jmath,$$
$$\mathrm{M}(\alpha,\ell):=\int_0^1(1-\jmath)^\ell\left|\jmath^\alpha-\frac{5}{8}\right|d\jmath,\quad \mathrm{N}(\alpha,\ell):=\int_0^1\jmath^\ell\left|\jmath^\alpha-\frac{5}{8}\right|d\jmath.$$

Corollary 6.3. *Taking $\alpha=1$ in Theorem 6, we obtain*

$$\Bigg|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$+\frac{(1-q)}{3q}\left[\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]$$
$$-\frac{1}{q(\widetilde{\chi}_2-\widetilde{\chi}_1)}\left[\int_{q\widetilde{\chi}_1}^{\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}}\hbar(\jmath)d_q\jmath+\int_{q\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}}^{\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}}\hbar(\jmath)d_q\jmath+\int_{q\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}}^{\widetilde{\chi}_2}\hbar(\jmath)d_q\jmath\right]\Bigg|$$
$$\leq\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Big\{\mathrm{A}_q^{1-\frac{1}{q_*}}\Bigg[\Bigg(\mathrm{A}_q-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{E}_q(\ell)\Bigg)\left|\mathrm{D}_q\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{q_*}$$
$$+\Bigg(\mathrm{A}_q-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{F}_q(\ell)\Bigg)|\mathrm{D}_q\hbar(\widetilde{\chi}_1)|^{q_*}\Bigg]^{\frac{1}{q_*}}$$

$$+ \mathrm{B}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}} \left[\left(\mathrm{B}_{\acute{q}} - \frac{1}{n}\sum_{\ell=1}^{n} \mathrm{G}_{\acute{q}}(\ell) \right) \left| \mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*} + \left(\mathrm{B}_{\acute{q}} - \frac{1}{n}\sum_{\ell=1}^{n} \mathrm{H}_{\acute{q}}(\ell) \right) \left| \mathtt{D}_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*} \right]^{\frac{1}{\acute{q}_*}}$$
$$+ \mathrm{C}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}} \left[\left(\mathrm{C}_{\acute{q}} - \frac{1}{n}\sum_{\ell=1}^{n} \mathrm{M}_{\acute{q}}(\ell) \right) \left| \mathtt{D}_{\acute{q}}\hbar(\widetilde{\chi}_2)\right|^{\acute{q}_*} + \left(\mathrm{C}_{\acute{q}} - \frac{1}{n}\sum_{\ell=1}^{n} \mathrm{N}_{\acute{q}}(\ell) \right) \left| \mathtt{D}_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*} \right]^{\frac{1}{\acute{q}_*}} \Bigg\}, \tag{3.17}$$

where

$$\mathrm{A}_{\acute{q}} := \int_0^1 \left|\jmath - \frac{3}{8}\right| d_{\acute{q}}\jmath, \quad \mathrm{B}_{\acute{q}} := \int_0^1 \left|\jmath - \frac{1}{2}\right| d_{\acute{q}}\jmath, \quad \mathrm{C}_{\acute{q}} := \int_0^1 \left|\jmath - \frac{5}{8}\right| d_{\acute{q}}\jmath$$

and

$$\mathrm{E}_{\acute{q}}(\ell) := \int_0^1 (1-\jmath)^{\ell} \left|\jmath - \frac{3}{8}\right| d_{\acute{q}}\jmath, \quad \mathrm{F}_{\acute{q}}(\ell) := \int_0^1 \jmath^{\ell} \left|\jmath - \frac{3}{8}\right| d_{\acute{q}}\jmath,$$
$$\mathrm{G}_{\acute{q}}(\ell) := \int_0^1 (1-\jmath)^{\ell} \left|\jmath - \frac{1}{2}\right| d_{\acute{q}}\jmath, \quad \mathrm{H}_{\acute{q}}(\ell) := \int_0^1 \jmath^{\ell} \left|\jmath - \frac{1}{2}\right| d_{\acute{q}}\jmath,$$
$$\mathrm{M}_{\acute{q}}(\ell) := \int_0^1 (1-\jmath)^{\ell} \left|\jmath - \frac{5}{8}\right| d_{\acute{q}}\jmath, \quad \mathrm{N}_{\acute{q}}(\ell) := \int_0^1 \jmath^{\ell} \left|\jmath - \frac{5}{8}\right| d_{\acute{q}}\jmath.$$

Corollary 6.4. *Choosing* $|\mathtt{D}_{\acute{q}}\hbar| \leq \mathcal{K}$ *in Theorem 6, we have*

$$\Bigg| \frac{1}{8}\left[\hbar(\widetilde{\chi}_1) + 3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right) + 3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right) + \hbar(\widetilde{\chi}_2)\right] + \frac{[\alpha]_{\acute{q}}(1-\acute{q})}{3\acute{q}^{\alpha}}\left[\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right) + \hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right) + \hbar(\widetilde{\chi}_2)\right]$$
$$- \frac{3^{\alpha-1}\Gamma_{\acute{q}}(\alpha+1)}{\acute{q}^{\alpha}(\widetilde{\chi}_2-\widetilde{\chi}_1)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\acute{q},\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1) + \mathrm{J}^{\alpha}_{\acute{q},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right) + \mathrm{J}^{\alpha}_{\acute{q},\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Bigg| \tag{3.18}$$

$$\leq \frac{\mathcal{K}(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Bigg\{ \mathrm{A}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[2\mathrm{A}_{\acute{q}}(\alpha) - \frac{1}{n}\sum_{\ell=1}^{n}\left(\mathrm{E}_{\acute{q}}(\alpha,\ell) + \mathrm{F}_{\acute{q}}(\alpha,\ell)\right)\right]^{\frac{1}{\acute{q}_*}}$$
$$+ \mathrm{B}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[2\mathrm{B}_{\acute{q}}(\alpha) - \frac{1}{n}\sum_{\ell=1}^{n}\left(\mathrm{G}_{\acute{q}}(\alpha,\ell) + \mathrm{H}_{\acute{q}}(\alpha,\ell)\right)\right]^{\frac{1}{\acute{q}_*}}$$
$$+ \mathrm{C}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[2\mathrm{C}_{\acute{q}}(\alpha) - \frac{1}{n}\sum_{\ell=1}^{n}\left(\mathrm{M}_{\acute{q}}(\alpha,\ell) + \mathrm{N}_{\acute{q}}(\alpha,\ell)\right)\right]^{\frac{1}{\acute{q}_*}} \Bigg\}. \tag{3.19}$$

Corollary 6.5. *Taking $n=1$ in Theorem 6, we get*

$$\begin{aligned}&\Bigg|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]\\&+\frac{[\alpha]_{\acute{q}}(1-\acute{q})}{3\acute{q}^{\alpha}}\left[\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]\\&-\frac{3^{\alpha-1}\Gamma_{\acute{q}}(\alpha+1)}{\acute{q}^{\alpha}(\widetilde{\chi}_2-\widetilde{\chi}_1)^{\alpha}}\left[J^{\alpha}_{\acute{q},\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1)+J^{\alpha}_{\acute{q},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+J^{\alpha}_{\acute{q},\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Bigg|\\&\le\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Bigg\{A_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[\left(A_{\acute{q}}(\alpha)-E_{\acute{q}}(\alpha)\right)\left|D_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+\left(A_{\acute{q}}(\alpha)-F_{\acute{q}}(\alpha)\right)\left|D_{\acute{q}}\hbar(\widetilde{\chi}_1)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\\&+B_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[\left(B_{\acute{q}}(\alpha)-G_{\acute{q}}(\alpha)\right)\left|D_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+\left(B_{\acute{q}}(\alpha)-H_{\acute{q}}(\alpha)\right)\left|D_{\acute{q}}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\\&+C_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[\left(C_{\acute{q}}(\alpha)-M_{\acute{q}}(\alpha)\right)\left|D_{\acute{q}}\hbar(\widetilde{\chi}_2)\right|^{\acute{q}_*}+\left(C_{\acute{q}}(\alpha)-N_{\acute{q}}(\alpha)\right)\left|D_{\acute{q}}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\Bigg\},\end{aligned}\tag{3.20}$$

where

$$\begin{aligned}E_{\acute{q}}(\alpha)&:=\int_0^1(1-\jmath)\left|\jmath^{\alpha}-\frac{3}{8}\right|d_{\acute{q}}\jmath,\quad F_{\acute{q}}(\alpha):=\int_0^1\jmath\left|\jmath^{\alpha}-\frac{3}{8}\right|d_{\acute{q}}\jmath,\\G_{\acute{q}}(\alpha)&:=\int_0^1(1-\jmath)\left|\jmath^{\alpha}-\frac{1}{2}\right|d_{\acute{q}}\jmath,\quad H_{\acute{q}}(\alpha):=\int_0^1\jmath\left|\jmath^{\alpha}-\frac{1}{2}\right|d_{\acute{q}}\jmath,\\M_{\acute{q}}(\alpha)&:=\int_0^1(1-\jmath)\left|\jmath^{\alpha}-\frac{5}{8}\right|d_{\acute{q}}\jmath,\quad N_{\acute{q}}(\alpha):=\int_0^1\jmath\left|\jmath^{\alpha}-\frac{5}{8}\right|d_{\acute{q}}\jmath.\end{aligned}$$

Corollary 6.6. *Choosing $\acute{q}\to 1^-$ in Corollary 6.5, we obtain*

$$\begin{aligned}&\Bigg|\frac{1}{8}\left[\hbar(\widetilde{\chi}_1)+3\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+3\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)+\hbar(\widetilde{\chi}_2)\right]\\&-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\widetilde{\chi}_2-\widetilde{\chi}_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^-}\hbar(\widetilde{\chi}_1)+J^{\alpha}_{\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^-}\hbar\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)+J^{\alpha}_{\widetilde{\chi}_2^-}\hbar\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right]\Bigg|\end{aligned}\tag{3.21}$$

$$\begin{aligned}&\le\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\Bigg\{A^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[F(\alpha)\left|\hbar'\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+(A(\alpha)-F(\alpha))\left|\hbar'(\widetilde{\chi}_1)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\\&+B^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[H(\alpha)\left|\hbar'\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}+(B(\alpha)-H(\alpha))\left|\hbar'\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\\&+C^{1-\frac{1}{\acute{q}_*}}(\alpha)\left[N(\alpha)\left|\hbar'(\widetilde{\chi}_2)\right|^{\acute{q}_*}+(C(\alpha)-N(\alpha))\left|\hbar'\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)\right|^{\acute{q}_*}\right]^{\frac{1}{\acute{q}_*}}\Bigg\},\end{aligned}\tag{3.22}$$

where

$$F(\alpha):=\int_0^1\jmath\left|\jmath^{\alpha}-\frac{3}{8}\right|d\jmath=\frac{\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{\frac{\alpha+2}{\alpha}}+\frac{1}{\alpha+2}-\frac{3}{16},$$

$$\mathrm{H}(\alpha) := \int_0^1 \jmath \left| \jmath^\alpha - \frac{1}{2} \right| d\jmath = \frac{\alpha}{\alpha+2} \left(\frac{1}{2}\right)^{\frac{\alpha+2}{\alpha}} + \frac{1}{\alpha+2} - \frac{1}{4},$$
$$\mathrm{N}(\alpha) := \int_0^1 \jmath \left| \jmath^\alpha - \frac{5}{8} \right| d\jmath = \frac{\alpha}{\alpha+2} \left(\frac{5}{8}\right)^{\frac{\alpha+2}{\alpha}} + \frac{1}{\alpha+2} - \frac{5}{16}.$$

4. Application to Special Means

Let $\hbar(\jmath) = \frac{\jmath^{\frac{\nu+1}{\acute{q}_*}+1}}{\left[\frac{\nu+1}{\acute{q}_*}+1\right]_{\acute{q}}}$, where $\nu \in \mathbb{N}$, $\acute{q}_* \geq 1$ and $\acute{q} \in (0,1)$. After simple calculations, we have $\left|\mathtt{D}_{\acute{q}}\hbar(\jmath)\right|^{\acute{q}_*} = \jmath^{\nu+1}$, which shows that $|\mathtt{D}_{\acute{q}}\hbar(\jmath)|^{\acute{q}_*}$ is convex function for all $\jmath > 0$ and $r \in \mathbb{N}$. Then, from Remark 1.1, $|\mathtt{D}_{\acute{q}}\hbar(\jmath)|^{\acute{q}_*}$ is n-polynomial convex function.

We consider the following arithmetic mean for real numbers $\widetilde{\chi}_1$ and $\widetilde{\chi}_2$ such that $0 \leq \widetilde{\chi}_1 < \widetilde{\chi}_2$:
$$\mathcal{A}(\widetilde{\chi}_1, \widetilde{\chi}_2) = \frac{\widetilde{\chi}_1 + \widetilde{\chi}_2}{2}.$$
For the simplicity of notations, let
$$\Delta_{\acute{q}}^{(1)}(\nu, \acute{q}_*; \widetilde{\chi}_1, \widetilde{\chi}_2) := (1-\acute{q})\left[(2-3\acute{q})\widetilde{\chi}_1 + \widetilde{\chi}_2\right] \sum_{n=0}^{\infty} \acute{q}^n \left(\acute{q}^n \frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3} + (1-\acute{q}^n)\acute{q}\widetilde{\chi}_1\right)^{\frac{\nu+1}{\acute{q}_*}+1},$$
$$\Delta_{\acute{q}}^{(2)}(\nu, \acute{q}_*; \widetilde{\chi}_1, \widetilde{\chi}_2) :=$$
$$:= (1-\acute{q})\left[(1-2\acute{q})\widetilde{\chi}_1 + (2-\acute{q})\widetilde{\chi}_2\right] \sum_{n=0}^{\infty} \acute{q}^n \left(\acute{q}^n \frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3} + (1-\acute{q}^n)\acute{q}\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right)^{\frac{\nu+1}{\acute{q}_*}+1},$$
$$\Delta_{\acute{q}}^{(3)}(\nu, \acute{q}_*; \widetilde{\chi}_1, \widetilde{\chi}_2) := (1-\acute{q})\left[(3-2\acute{q})\widetilde{\chi}_2 - \acute{q}\widetilde{\chi}_1\right] \sum_{n=0}^{\infty} \acute{q}^n \left(\acute{q}^n \widetilde{\chi}_2 + (1-\acute{q}^n)\acute{q}\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3}\right)^{\frac{\nu+1}{\acute{q}_*}+1},$$
where $r \in \mathbb{N}$, $\acute{q}_* \geq 1$, and $\acute{q} \in (0,1)$.

By applying Corollaries 5.2 and 6.3, we deduce the following $\acute{q}$-inequalities:

Proposition 4.1. *Let n, $\nu \in \mathbb{N}$, $\acute{q} \in (0,1)$ and $\widetilde{\chi}_1$, $\widetilde{\chi}_2 \in \mathbb{R}$, where $0 \leq \widetilde{\chi}_1 < \widetilde{\chi}_2$. Then for p, $\acute{q}_* > 1$ and $\frac{1}{p} + \frac{1}{\acute{q}_*} = 1$, we have*
$$\Bigg| \frac{1}{4} \left[\mathcal{A}\left(\widetilde{\chi}_1^{\frac{\nu+1}{\acute{q}_*}+1}, \widetilde{\chi}_2^{\frac{\nu+1}{\acute{q}_*}+1}\right) + 3\mathcal{A}\left(\left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right)^{\frac{\nu+1}{\acute{q}_*}+1}, \left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3}\right)^{\frac{\nu+1}{\acute{q}_*}+1}\right)\right]$$
$$+ \frac{(1-\acute{q})}{3\acute{q}} \left[2\mathcal{A}\left(\left(\frac{2\widetilde{\chi}_1 + \widetilde{\chi}_2}{3}\right)^{\frac{\nu+1}{\acute{q}_*}+1}, \left(\frac{\widetilde{\chi}_1 + 2\widetilde{\chi}_2}{3}\right)^{\frac{\nu+1}{\acute{q}_*}+1}\right) + \widetilde{\chi}_2^{\frac{\nu+1}{\acute{q}_*}+1}\right] -$$
$$- \frac{1}{3\acute{q}(\widetilde{\chi}_2 - \widetilde{\chi}_1)} \sum_{\mathrm{i}=1}^{3} \Delta_{\acute{q}}^{(\mathrm{i})}(\nu, \acute{q}_*; \widetilde{\chi}_1, \widetilde{\chi}_2) \Bigg|$$

$$\leq \frac{2^{\frac{1}{\acute{q}_*}}(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\left[\frac{\nu+1}{\acute{q}_*}+1\right]_{\acute{q}}\left(1-\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{[\ell+1]_{\acute{q}}}\right)^{\frac{1}{\acute{q}_*}}$$
$$\times\left\{\mathrm{A}_{\acute{q}}^{\frac{1}{p}}(p)\mathcal{A}^{\frac{1}{\acute{q}_*}}\left(\widetilde{\chi}_1^{\nu+1},\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^{\nu+1}\right)+\right.$$
$$+\mathrm{B}_{\acute{q}}^{\frac{1}{p}}(p)\mathcal{A}^{\frac{1}{\acute{q}_*}}\left(\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^{\nu+1},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^{\nu+1}\right)$$
$$\left.+\mathrm{C}_{\acute{q}}^{\frac{1}{p}}(p)\mathcal{A}^{\frac{1}{\acute{q}_*}}\left(\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^{\nu+1},\widetilde{\chi}_2^{\nu+1}\right)\right\}. \tag{4.1}$$

Proof. By applying Corollary 5.2 and Remark 1.1 with $\hbar(\jmath)=\frac{\jmath^{\frac{\nu+1}{\acute{q}_*}+1}}{\left[\frac{\nu+1}{\acute{q}_*}+1\right]_{\acute{q}}}$ for all $\jmath\in[\widetilde{\chi}_1,\widetilde{\chi}_2]$, we can get the desired result (4.1). □

Proposition 4.2. *Let $n,\nu\in\mathbb{N}$, $\acute{q}\in(0,1)$ and $\widetilde{\chi}_1,\widetilde{\chi}_2\in\mathbb{R}$, where $0\leq\widetilde{\chi}_1<\widetilde{\chi}_2$. Then for $\acute{q}_*\geq 1$, we have*

$$\left|\frac{1}{4}\left[\mathcal{A}\left(\widetilde{\chi}_1^{\frac{\nu+1}{\acute{q}_*}+1},\widetilde{\chi}_2^{\frac{\nu+1}{\acute{q}_*}+1}\right)+3\mathcal{A}\left(\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^{\frac{\nu+1}{\acute{q}_*}+1},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^{\frac{\nu+1}{\acute{q}_*}+1}\right)\right]\right.$$
$$+\frac{(1-\acute{q})}{3\acute{q}}\left[2\mathcal{A}\left(\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^{\frac{\nu+1}{\acute{q}_*}+1},\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^{\frac{\nu+1}{\acute{q}_*}+1}\right)+\widetilde{\chi}_2^{\frac{\nu+1}{\acute{q}_*}+1}\right]-$$
$$\left.-\frac{1}{3\acute{q}(\widetilde{\chi}_2-\widetilde{\chi}_1)}\sum_{\mathrm{i}=1}^{3}\Delta_{\acute{q}}^{(\mathrm{i})}(\nu,\acute{q}_*;\widetilde{\chi}_1,\widetilde{\chi}_2)\right|$$
$$\leq\frac{(\widetilde{\chi}_2-\widetilde{\chi}_1)}{9}\left[\frac{\nu+1}{\acute{q}_*}+1\right]_{\acute{q}}$$
$$\times\left\{\mathrm{A}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}\left[\left(\mathrm{A}_{\acute{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{E}_{\acute{q}}(\ell)\right)\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^{\nu+1}+\left(\mathrm{A}_{\acute{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{F}_{\acute{q}}(\ell)\right)\widetilde{\chi}_1^{\nu+1}\right]^{\frac{1}{\acute{q}_*}}\right.$$
$$+\mathrm{B}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}\left[\left(\mathrm{B}_{\acute{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{G}_{\acute{q}}(\ell)\right)\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^{\nu+1}+\left(\mathrm{B}_{\acute{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{H}_{\acute{q}}(\ell)\right)\left(\frac{2\widetilde{\chi}_1+\widetilde{\chi}_2}{3}\right)^{\nu+1}\right]^{\frac{1}{\acute{q}_*}}$$
$$\left.+\mathrm{C}_{\acute{q}}^{1-\frac{1}{\acute{q}_*}}\left[\left(\mathrm{C}_{\acute{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{M}_{\acute{q}}(\ell)\right)\widetilde{\chi}_2^{\nu+1}+\left(\mathrm{C}_{\acute{q}}-\frac{1}{n}\sum_{\ell=1}^{n}\mathrm{N}_{\acute{q}}(\ell)\right)\left(\frac{\widetilde{\chi}_1+2\widetilde{\chi}_2}{3}\right)^{\nu+1}\right]^{\frac{1}{\acute{q}_*}}\right\}. \tag{4.2}$$

Proof. By using Corollary 6.3 and Remark 1.1 with $\hbar(\jmath)=\frac{\jmath^{\frac{\nu+1}{\acute{q}_*}+1}}{\left[\frac{\nu+1}{\acute{q}_*}+1\right]_{\acute{q}}}$ for all $\jmath\in[\widetilde{\chi}_1,\widetilde{\chi}_2]$, we can obtain the desired result (4.2). □

5. Conclusions

First we consider a new Newton's type quantum fractional integral identity. By using this, we have established some Newton's type quantum fractional integral inequalities using n-polynomial convex functions, and several special cases are discussed in detail. In order to illustrate the efficiency of our main results, an example and application on special means of positive real numbers are provided. Interested readers can apply our results in mathematical inequalities and special functions, and can find new bounds related to convexity class of functions such that n-polynomial p-convex function, n-polynomial exponentially s-convex functions, n-polynomial harmonically convex function, and so on. Furthermore, they can use q́-deformed real numbers [30] to extend our results. We hope that this novel idea that mixed together fractional calculus and q́-calculus opens many avenues for interested researchers working in these fields and they can discover further approximations for different kinds of convexity.

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