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A NOTE REGARDING SOME FUZZY SSPO MAPPINGS

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Abstract. In this paper we will introduce the concept of fuzzy SSPO irresolute continuous mappings, fuzzy SSPO irresolute open (closed) mappings as well as the concept of SSPO homeomorphism. We will also study some of their properties and their relation with other forms of fuzzy continuity.

1. Introduction

Fuzzy sets were introduced by Zadeh in [19], while the concept of Fuzzy topological space was initially introduced by Chang in [4]. Following the introduction of fuzzy semi open sets [1], fuzzy preopen sets [3], fuzzy strongly semi open sets [3], fuzzy strongly preopen sets[11], many other forms of weaker fuzzy continuity have been studied by other authors in [7], [5], [10], [8], [14]. Following the introduction of fuzzy strongly semi pre-open sets in [13] we will introduce the new concept of fuzzy SSPO irresolute continuous mappings. We will also define and investigate the properties of fuzzy SSPO irresolute open (closed) mappings. The concept of semi-homeomorphism [6] was extended to fuzzy topological spaces by Yalvac in [18]. Other forms of fuzzy semi-homeomorphism were introduced by Krsteska in [12], Pankaj et al. in [15] and many other authors. We will introduce the concept of fuzzy SSPO homeomorphism and will investigate some of their properties.

2. Preliminaries

In this work we will use the notation (X, τ) or sometimes only by X to denote a fuzzy topological space, shortly *fts*, as defined by Chang in [4]. We will also denote by $intA$, clA and A^c the interior, closure and the complement of a fuzzy set A, respectively.

Definition 2.1. A fuzzy set A of a *fts* X is called:
(1) *Fuzzy semiopen* (*semiclosed*) if and only if $A \leq cl(intA)$ $(A \geq int(clA))$; [1];
(2) *Fuzzy preopen* (*preclosed*) if and only if $A \leq int(clA)$ $(A \geq cl(intA))$; [3];
(3) *Fuzzy strongly semiopen* (*strongly semiclosed*) if and only if $A \leq int(cl(intA))$

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$(A \geq cl(int(clA)))$; [3];
(4) *Fuzzy regular open* (*regular closed*) if and only if $A = int(clA)$ $(A = cl(intA))$; [1].

Definition 2.2. Let A be a fuzzy set of a *fts* X . Then:
$pintA = \vee\{B|B \leq A, B\ is\ a\ fuzzy\ preopen\ set\}$ is called the *fuzzy preinterior* of set A; [17];
$pclA = \wedge\{B|B \geq A, B\ is\ a\ fuzzy\ preclosed\ set\}$ is called the *fuzzy preclosure* of set A; [17].

Definition 2.3. A fuzzy set A of a *fts* X is called:
(a) *Fuzzy strongly preopen* (*strongly preclosed*) if and only if
$A \leq int(pclA)$ $(A \geq cl(pintA))$; [11];
(b) *Fuzzy strongly semi pre-open* (*strongly semi pre-closed*) if and only if
$A \leq int(pclA) \vee pcl(intA)$ ($A \geq cl(pintA) \wedge pint(clA)$); [13].

Given any fuzzy topological space (X, τ), the family of all fuzzy semiopen (semiclosed) sets is denoted $FSO(\tau)$ $(FSC(\tau))$; the family of all fuzzy preopen (preclosed) sets is denoted $FPO(\tau)$ $(FPC(\tau)$; the family of all fuzzy strongly semiopen (strongly semiclosed) sets is denoted $FSSO(\tau)$ $(FSSC(\tau))$; the family of all fuzzy regular open (regular closed) sets is denoted $FRO(\tau)$ $(FRC(\tau))$; the family of all fuzzy strongly preopen (strongly preclosed) sets is denoted $FSPO(\tau)$ $(FSPC(\tau))$; the family of all fuzzy strongly semi pre-open (strongly semi pre-closed) sets is denoted $FSSPO(\tau)$ $(FSSPC(\tau))$.

Definition 2.4. ([13]) If A is a fuzzy set of a *fts* X, then
(1) The union of all fuzzy strongly semi-pre-open sets contained in a set A is called a *fuzzy strong semi-pre-interior* of set A and is denoted by $sspintA$.
(2) The intersection of all fuzzy strongly semi-pre-closed sets containing set A is called the *strong semi-pre-closure* and is denoted by $sspclA$.

Definition 2.5. Let $f : (X, \tau) \to (Y, \delta)$ be a mapping from a fts (X, τ) to a fts (Y, δ). The mapping is called:
1) *Fuzzy continuous* if $f^{-1}(B)$ is a fuzzy open set of X, for each $B \in \delta$; [4];
2) *Fuzzy precontinuous* if $f^{-1}(B)$ is a fuzzy preopen set of X, for each $B \in \delta$; [17];
3) *Fuzzy strong precontinuous* if $f^{-1}(B)$ is a fuzzy strongly preopen set of X, for each $B \in \delta$; [11];
4) *Fuzzy strong semi pre-continuous* if $f^{-1}(B)$ is a fuzzy strongly semi pre-open set of X, for each $B \in \delta$; [13];
5) *Fuzzy preirresolute continuous* if $f^{-1}(B) \in FPO(\tau)$ for each $B \in FPO(\delta)$; [9];
6) *Fuzzy SP-irresolute continuous* if $f^{-1}(B) \in FSPO(\tau)$ for each $B \in FSPO(\delta)$; [12];
7) *Fuzzy R-continuous* if $f^{-1}(B) \in FRO(\tau)$ for each $B \in FRO(\delta)$; [2];
8) *Fuzzy homeomorphism* if it is a bijective mapping and the mapping f and its inverse are fuzzy continuous; [4].

Definition 2.6. ([16]) A *fuzzy point* x_α of a *fts* X is a fuzzy set defined as:

$$x_\alpha(z) = \begin{cases} \alpha & if\ z = x \\ 0 & if\ otherwise \end{cases}$$

for $0 < \alpha \leq 1$.

Definition 2.7. ([16]) A fuzzy set U of a *fts* X is a called the *neighborhood* of a fuzzy point x_α if there is a fuzzy open set O of X, such that $x_\alpha \leq O \leq U$.

Lemma 1. ([20]) *If A is a fuzzy set of a fts X, then*
(a) $pclA^c = (pintA)^c$;
(b) $pintA^c = (pclA)^c$.

Lemma 2. ([13]) *If A is a fuzzy set of a fts$X$, then*
(a) $sspclA^c = (sspintA)^c$;
(b) $sspintA^c = (sspclA)^c$.

Lemma 3. ([11]) *Let $f : X \to Y$ be a mapping. For fuzzy sets A and B of X and Y respectively, the following statements hold:*
(a) $ff^{-1}(B) \leq B$;
(b) $f^{-1}f(A) \geq A$;
(c) $f(A^c) \geq (f(A))^c$;
(d) $f^{-1}(B^c) = (f^{-1}(B)))^c$;
(e) if f is injective, then $f^{-1}f(A) = A$;
(f) if f is surjective, then $ff^{-1}(B) = B$;
(g) if f is bijective, then $f(A^c) = (f(A))^c$.

Lemma 4. *Let A be a fuzzy set of a fts (X, τ). If $A \neq 0_X$ and $A \in FSSPO(\tau)$ then $intA \neq 0_X$.*

Proof. Let us suppose that $intA = 0_X$ then $pcl(intA) = 0_X$ and since $A \geq cl(intA) = 0_X$ it is obvious that $A \in FPC(\tau)$ and subsequently $pclA = A$. Since $A \in FSSPO(\tau)$, we have

$$A \leq int(pclA) \vee pcl(intA) = intA = 0_X$$

which contradicts the fact that $A \neq 0_X$. □

3. Fuzzy SSPO-irresolute continuous mappings

Definition 3.1. A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ from a *fts* (X, τ_1) to a *fts* (Y, τ_2) is called a *fuzzy SSPO-irresolute continuous* if $f^{-1}(B) \in FSSPO(\tau_1)$ for each $B \in FSSPO(\tau_2)$.

It is obvious that any fuzzy SSPO-irresolute continuous mapping is also a fuzzy strongly semi pre-continuous mapping, while the converse is not true in general. Let us illustrate this with an example.

Example 3.1. Let $X = \{a, b, c\}$ be a set and let us define fuzzy topological spaces $\tau_1 = \{0, A, 1\}$ where $A = \{(a, 0.25); (b, 0.30); (c, 0.45)\}$ and the fuzzy topological space $\tau_2 = \{0, B, 1\}$ such that $B = \{(a, 0.55); (b, 0.63); (c, 0.55)\}$. The mapping $f = id : (X, \tau_1) \to (X, \tau_2)$ is a fuzzy strongly semi-precontinuous mapping which is not a fuzzy SSPO-irresolute continuous mapping. If we consider a set $V = \{(a, 0.8); (b, 0.8); (c, 0.9)\}$, it is obvious that it satisfies the condition:

$$V \leq int(pclV) \vee pcl(intV), \text{ that is } V \in FSSPO(\tau_2).$$

On the other side, if we consider $f^{-1}(V)$ and the fts (X, τ_1), we will conclude that $f^{-1}(V) \notin FSSPO(\tau_1)$.

Example 3.2. Let $X = \{a, b, c\}$ be a set and let us define fuzzy topological spaces $\delta_1 = \{0, A, B, A \vee B, A \wedge B, 1\}$ where $A = \{(a, 0.35); (b, 0.60); (c, 0.75)\}$, $B = \{(a, 0.75); (b, 0.80); (c, 0.45)\}$. Let us also define the fuzzy topological space $\delta_2 = \{0, D, 1\}$ such that $D = \{(a, 0.76); (b, 0.80); (c, 0.75)\}$. The mapping $h = id : (X, \delta_1) \to (X, \delta_2)$ is a fuzzy SSPO-irresolute continuous mapping which is not a fuzzy continuous mapping.

Example 3.3. Let $X = \{a, b, c\}$ be a set and let us define fuzzy topological spaces $\tau_1 = \{0, A, B, A \vee B, A \wedge B, 1\}$ where $A = \{(a, 0.25); (b, 0.30); (c, 0.55)\}$,$B = \{(a, 0.55); (b, 0.63); (c, 0.40)\}$. Let us also define the fuzzy topological space $\tau_2 = \{0, C, 1\}$ such that $C = \{(a, 0.55); (b, 0.63); (c, 0.55)\}$. The mapping $g = id : (X, \tau_1) \to (X, \tau_2)$ is not a fuzzy SSPO-irresolute continuous mapping but it is a fuzzy continuous mapping.
If we consider a set $G = \{(a, 0.76); (b, 0.85); (c, 0.89)\}$ which satisfies the condition:

$$G \leq int(pclG) \vee pcl(intG), \text{ that is } G \in FSSPO(\tau_2).$$

On the other side, if we consider $f^{-1}(G)$ and the fts (X, τ_1), we will conclude that $f^{-1}(G) \notin FSSPO(\tau_1)$.

Theorem 1. *Let $f : (X, \tau) \to (Y, \delta)$ be a mapping from a fuzzy topological space (X, τ) into a fuzzy topological space (Y, δ) then the following statements are equivalent:*
(a) f is a fuzzy SSPO-irresolute continuous mapping;
(b) $f^{-1}(B)$ is a $FSSPC(\tau)$, for each $B \in FSSPC(\delta)$;
(c) $f(sspclA) \leq sspclf(A)$ for each fuzzy set A of X;
(d) $sspclf^{-1}(B) \leq f^{-1}(sspclB)$, for each fuzzy set B of Y;
(e) $f^{-1}(sspintB) \leq sspint(f^{-1}(B))$, for each fuzzy set B of Y.

Proof. (a) $\Rightarrow$ (b) For each $B \in FSSPC(\delta)$, its compliment B^c is a $FSSPO$ set and subsequently $f^{-1}(B^c) \in FSSPO(\tau)$. Now, from the fact that $f^{-1}(B) = f^{-1}(Y \setminus B^c) = f^{-1}(Y) \setminus f^{-1}(B^c) = X \setminus f^{-1}(B^c)$, meaning that $f^{-1}(B)$ is a $FSSPC(\tau)$.
(b) $\Rightarrow$ (c) For each fuzzy set A of X the set $sspclf(A) \in FSSPC(\delta)$, that is $f^{-1}(sspclf(A)) \in FSSPC(\tau)$. From the fact that $A \leq f^{-1}(f(A))$ it is obvious that

$$sspclA \leq sspcl(f^{-1}(f(A)) \leq sspcl(f^{-1}(sspclf(A)) = f^{-1}(sspclf(A)).$$

From the last relation it is easy to see that $f(sspclA) \leq sspclf(A)$.
(c) $\Rightarrow$ (d) According to the assumption, given any fuzzy set B of Y,

$$f(sspclf^{-1}(B)) \leq sspcl(f(f^{-1}(B))) \leq sspclB.$$

Thus, $sspclf^{-1}(B) \leq f^{-1}(f(sspclf^{-1}(B))) \leq f^{-1}(sspclB)$.
(d) $\Rightarrow$ (e) It can be proved by using Lemma 2 and Lemma 3.
(e) $\Rightarrow$ (a) Let $B \in FSSPO(\delta)$. Then $sspintB = B$ and from the assumption we have that $f^{-1}(B) = f^{-1}(sspintB) \leq sspint(f^{-1}(B)) \leq f^{-1}(B)$. From the last relation we can deduct that $f^{-1}(B) = sspint(f^{-1}(B)$, thus $f^{-1}(B) \in FSSPO(\tau)$. In other words, the function f is fuzzy SSPO-irresolute continuous mapping. □

Theorem 2. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a bijective mapping from a fts X to a fts Y. The mapping f is fuzzy SSPO-irresolute continuous if and only if $sspintf(A) \leq f(sspintA)$ for each fuzzy set A of X.*

Proof. Let us suppose that the mapping f is a fuzzy SSPO-irresolute continuous. It is obvious that $f^{-1}(sspintf(A))$ is a $FSSPO$ set of X, and according to Theorem 1. we have:

$$f^{-1}(sspintf(A)) \leq sspint(f^{-1}(f(A)) = sspintA,$$

and similarly

$$f(f^{-1}(sspintf(A))) = sspintf(A) \leq f(sspintA).$$

Conversely, if we consider a set $B \in FSSPO(\tau_2)$ and the fact that $sspintB = B$, in accordance with the assumption we have:

$$f(sspintf^{-1}(B) \geq sspintf(f^{-1}(B)) = sspintB = B \implies$$
$$sspintf^{-1}(B) \geq f^{-1}(B)$$

From the other side we have that $sspintf^{-1}(B) \leq f^{-1}(B)$, which gives the following $sspintf^{-1}(B) = f^{-1}(B)$, that is $f^{-1}(B) \in FSSPO(\tau_1)$, meaning that f is a SSPO-irresolute continuous mapping. □

Theorem 3. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts X to a fts Y. The following statements are equivalent:*
(a) f is a fuzzy SSPO-irresolute continuous mapping;
(b) $cl(pintf^{-1}(B)) \wedge pint(clf^{-1}(B)) \leq f^{-1}(sspclB)$ for each fuzzy set B of Y;
(c) $f^{-1}(sspintB) \leq int(pclf^{-1}(B)) \vee pcl(intf^{-1}(B))$ for each fuzzy set B of Y;
(d) $f(cl(pintA) \wedge pint(clA) \leq sspclf(A)$,for each fuzzy set A of X.

Proof. $(a) \Rightarrow (b)$ If we suppose that f is a fuzzy SSPO-irresolute continuous mapping, then for each fuzzy set B of Y we have that $f^{-1}(sspclB) \in FSSPC(\tau_1)$ which means that:

$$f^{-1}(sspclB) \geq cl(pintf^{-1}(sspclB)) \wedge pint(clf^{-1}(sspclB)) \geq$$
$$cl(pintf^{-1}(B)) \wedge pint(clf^{-1}(B)).$$

$(b) \Rightarrow (c)$ According to assumption, for every fuzzy set B of Y we have:

$$(f^{-1}(sspclB^c)^c \leq (cl(pintf^{-1}(B^c)) \wedge pint(clf^{-1}(B^c))^c$$

which gives us the wanted result $f^{-1}(sspintB) \leq int(pclf^{-1}(B)) \vee pcl(intf^{-1}(B))$.

$(c) \Rightarrow (d)$ Given any fuzzy set A of X, if we put $f(A) = B$, we have:

$$(int(pclA^c)) \vee pcl(intA^c))^c \leq (int(pclf^{-1}(B^c)) \vee pcl(intf^{-1}(B^c)))^c \leq (f^{-1}(sspint(B^c))^c$$

Which gives the following

$$f(cl(pintA) \wedge pint(clA)) \leq f(f^{-1}(sspclB)) \leq sspclB$$

Now, if we put $B = f(A)$ we get the wanted result.

$(d) \Rightarrow (a)$ Given any set $B \in FSSPC(\tau_2)$, according to the assumption we have: $f(cl(pintf^{-1}(B) \wedge pint(clf^{-1}(B)) \leq sspclf(f^{-1}(B) \leq sspclB = B$. From the last relation we get that

$$f^{-1}(B) \geq (cl(pintf^{-1}(B) \wedge pint(clf^{-1}(B)),$$

which means that $f^{-1}(B) \in FSSPC(\tau_1)$, thus f is a fuzzy SSPO-irresolute continuous mapping. □

Theorem 4. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts X to a fts Y. If the mapping f is fuzzy SSPO-irresolute continuous then*

$$f^{-1}(B) \leq sspint(f^{-1}(int(pclB) \vee pcl(intB))),$$

for each fuzzy set $B \in FSSPO(\tau_2)$.

Proof. Let $B \in FSSPO(\tau_2)$, then $f^{-1}(B) \leq f^{-1}(int(pclB) \vee pcl(intB))$ and accordingly:

$$f^{-1}(B) \leq sspint(f^{-1}(int(pclB) \vee pcl(intB)))$$

□

Theorem 5. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts X to a fts Y. The following statements are equivalent:*
a) f is a fuzzy SSPO-irresolute continuous mapping;
b) For any fuzzy point x_α of X and any fuzzy set $B \in FSSPO(\tau_2)$ which contains $f(x_\alpha)$ there exists $A \in FSSPO(\tau_1)$ containing x_α, and $A \leq f^{-1}(B)$;
c) For any fuzzy point x_α of X and any fuzzy set $B \in FSSPO(\tau_2)$ which contains $f(x_\alpha)$ there exists $A \in FSSPO(\tau_1)$ containing x_α, and $f(A) \leq B$.

Proof. $(a) \Rightarrow (b)$ If f is a fuzzy SSPO-irresolute continuous mapping, then for any fuzzy point x_α of X and any fuzzy set $B \in FSSPO(\tau_2)$ which contains $f(x_\alpha)$, there exists $f^{-1}(B) \in FSSPO(\tau_1)$ containing x_α and $sspintf^{-1}(B) = f^{-1}(B)$. The conditions are fulfilled if $A = sspintf^{-1}(B) \leq f^{-1}(B)$.

$(b) \Rightarrow (c)$ It follows from the fact that $f(A) \leq f(f^{-1}(B)) \leq B$.

$(c) \Rightarrow (a)$ If we consider any fuzzy set $B \in FSSPO(\tau_2)$ and let $x_\alpha \in f^{-1}(B)$, then there exist $A \in FSSPO(\tau_1)$ containing x_α, that is:

$$\forall x_\alpha \in f^{-1}(B) \implies x_\alpha \in A = sspintA \leq sspintf^{-1}(B)$$

In other words, $f^{-1}(B) \in FSSPO(\tau_1)$ and f is a fuzzy SSPO-irresolute continuous mapping. □

Theorem 6. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping between fts X to a fts Y and let $g : (Y, \tau_2) \to (Z, \tau_3)$ be a mapping between fts Y to a fts Z. Then:*
1) If both f, g are fuzzy SSPO-irresolute continuous, then gf is also fuzzy SSPO-irresolute continuous.
2) If the mapping f is fuzzy SSPO-irresolute continuous and g is fuzzy strongly semi pre-continuous mapping then gf is also fuzzy strongly semi pre-continuous mapping.
3) If the mapping f is fuzzy SSPO-irresolute continuous and g is fuzzy continuous mapping then gf is also fuzzy strongly semi pre-continuous mapping.
4) If the mapping gf is fuzzy SSPO-irresolute continuous and g is fuzzy strongly semi pre-open (pre-closed) and injective mapping, then f is a fuzzy strongly semi pre-continuous mapping.
5) If the mapping gf is fuzzy strongly semi pre-open (pre-closed) and the mapping g is fuzzy SSPO-irresolute continuous and injective, then f is a fuzzy strongly semi pre-open (pre-closed) mapping.

Proof. 1) Since $(gf)^{-1}(B) = f^{-1}(g^{-1}(B))$, and according to the given assumptions, for any $B \in FSSPO(\tau_3)$ it follows that $f^{-1}(g^{-1}(B)) \in FSSPO(\tau_1)$, that is gf is a fuzzy SSPO-irresolute continuous mapping.
Similarly to the first case we can show 2) and 3).
4) Based on the assumption that g is a fuzzy strongly semi pre-open (pre-closed) and injective mapping, and since for any injective mapping and any fuzzy set$A$ of Y it follows that $g^{-1}(g(A)) = A$, it is obvious that for any fuzzy open (closed) set $B \in Y$ we have:

$$f^{-1}(B) = f^{-1}(g^{-1}(g(B))) = (gf)^{-1}(g(B)).$$

Now according to the conditions given by the theorem it is obvious that f is a fuzzy strongly semi pre-continuous mapping.
5) Similarly to 4).

□

Corollary 6.1. *If X, X_1, X_2 are fuzzy topological spaces and let $p_i : X_1 \times X_2 \to X_i$, $(i = 1, 2)$, be projections of $X_1 \times X_2$ onto X_i. If $f : X \to X_1 \times X_2$ is a fuzzy SSPO-irresolute continuous mapping then the compositions $p_i f$ are fuzzy strongly semi pre-continuous.*

Proof. It follows from the fact that projections $p_i : X_1 \times X_2 \to X_i$, $(i = 1, 2)$, are fuzzy continuous mappings and from Theorem 6. □

4. Fuzzy SSPO-irresolute open and fuzzy SSPO-irresolute closed mappings

Definition 4.1. A mapping $f : (X, \tau_1) \to (Y, \tau_2)$, from a *fts* X to a *fts* Y is called

(a) *fuzzy SSPO-irresolute open* if $f(A) \in FSSPO(\tau_2)$ for each $A \in FSSPO(\tau_1)$;
(b) *fuzzy SSPO-irresolute closed* if $f(A) \in FSSPC(\tau_2)$ for each $A \in FSSPC(\tau_1)$.

It is worth to mention that the concept of fuzzy SSPO-irresolute open (closed) mapping is a generalization of the concept of fuzzy strongly semi pre-open (pre-closed) mapping. Any fuzzy SSPO-irresolute open (closed) mapping is also a fuzzy strongly semi pre-open (pre-closed) mapping. The concept of fuzzy SSPO-irresolute open (closed) mapping is also independent from the concept of fuzzy open (closed) mappings.

We can show this with the following examples.

Example 4.1. If we consider fuzzy topological spaces given in the Example 3.3 and the mapping $f = id : (X, \tau_2) \to (X, \tau_1)$, it is easy to verify that the mapping f is a fuzzy strongly semi pre-open but it is not a fuzzy SSPO-irresolute open mapping. It is important to mention that this mapping is also a fuzzy open mapping.

Example 4.2. If we consider fuzzy topological spaces given in the Example 3.2 and the mapping $f = id : (X, \delta_2) \to (X, \delta_1)$, it is easy to verify that the mapping f is a fuzzy SSPO-irresolute open mapping which is not a fuzzy open mapping.

Theorem 7. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts X to a fts Y. The following statements are equivalent:*
(a) f is a fuzzy SSPO-irresolute open mapping;
(b) $f(sspintA) \le sspintf(A)$ for each fuzzy set A of X;
(c) $sspintf^{-1}(B) \le f^{-1}(sspintB)$ for each fuzzy set B of Y;
(d) $f^{-1}(sspclB) \le sspclf^{-1}(B)$, for each fuzzy set B of Y.

Proof. (1) $\Rightarrow$ (2) Given any fuzzy set A of X then

$$f(sspintA) = sspintf(sspintA) \le sspintf(A)$$

(2) $\Rightarrow$ (3) Let B be a fuzzy set of Y, then

$$f(sspintf^{-1}(B)) = sspintf(sspintA) \le sspintf(A).$$

The last relation then yields the result.
(3) $\Rightarrow$ (4) It follows by using Lemma 2 and Lemma 3.
(4) $\Rightarrow$ (1) Let $A \in FSSPO(\tau_1)$. It is obvious that $sspintA = A$. From the assumption we have that

$$A = sspintA \le sspintf^{-1}(f(A)) \le (sspcl(f^{-1}(f(A)^c))^c \le (f^{-1}(sspint(f(A))$$

Therefore, $f(A) \le sspintf(A)$ which means that f is a fuzzy SSPO-irresolute open mapping. □

Theorem 8. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts X to a fts Y. The mapping f is a fuzzy SSPO-irresolute closed if and only if the following condition is met $sspclf(A) \le f(sspclA)$, for each fuzzy set A of X.*

Proof. Given any fuzzy set A of X, $f(sspclA) \in FSSPC(\tau_2)$ and also $f(A) \leq f(sspclA) \implies sspclf(A) \leq f(sspclA)$.
Conversely, if $A \in FSSPC(\tau_1)$, then $f(A) = f(sspclA) \geq sspclf(A)$, which gives us the following $f(A) = sspclf(A)$, hence the mapping f is a fuzzy SSPO-irresolute closed. □

Theorem 9. *Let $f : X \to Y$ be a mapping from a fts X to a fts Y. The mapping f is a fuzzy SSPO-irresolute open if and only if $f(sspintA) \leq int(pcl(f(A)) \vee pcl(int(f(A))$, for each fuzzy set A of X.*

Proof. Let f be a fuzzy SSPO-irresolute open mapping and let A be any fuzzy set of X. Since $sspintA$ is a fuzzy strongly semi pre-open set, then $f(sspintA)$ is a fuzzy strongly semi-pre-open set of Y, that is

$$f(sspintA) \leq int(pcl(f(A)) \vee pcl(int(f(A)).$$

Conversely, if A is a fuzzy strongly semi pre-open set of X, then $A = sspintA$ and $f(A) = f(sspintA) \leq int(pcl(f(A)) \vee pcl(int(f(A))$ which means that $f(A)$ is an $FSSPO$ set and therefore the mapping f is a fuzzy SSPO-irresolute open mapping. □

Similarly to the above theorem we can formulate and prove the following theorem.

Theorem 10. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts X to a fts Y. The mapping f is a fuzzy SSPO-irresolute closed if and only if $cl(pintf(A) \wedge pint(clf(A)) \leq f(sspclA)$, for each fuzzy set A of X.*

Proof. In similar manner to Theorem 9. □

Theorem 11. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts X to a fts Y.*
a) If $f(int(pclA) \vee pcl(intA)) \leq int(pcl(f(A)) \vee pcl(int(f(A)$, for any set $A \in FSSPO(\tau_1)$, then the mapping f is a fuzzy SSPO-irresolute open.
b) If $f(cl(pint) \wedge pint(clA)) \geq cl(pint(f(A)) \wedge pint(cl(f(A)$, for any set $A \in FSSPC(\tau_1)$, then the mapping f is a fuzzy SSPO-irresolute closed.

Proof. a) If $A \in FSSPO(\tau_1)$ then $A \leq int(pclA) \vee pcl(intA)$ and subsequently $f(A) \leq f(int(pclA) \vee pcl(intA))$. Now, according to the assumption we get:

$$f(A) \leq f(int(pclA) \vee pcl(intA)) \leq int(pcl(f(A)) \vee pcl(int(f(A).$$

The last means that $f(A)$ is a $FSSPO$ set and accordingly f is a fuzzy SSPO-irresolute open mapping.
b) Similarly to a). □

Theorem 12. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a bijective mapping from a fts X to a fts Y. The following holds:*
1) Mapping f is a fuzzy SSPO-irresolute open if and only if it is a fuzzy SSPO-irresolute closed mapping.
2) Mapping f is a fuzzy SSPO-irresolute open (closed) if and only if f^{-1} is a fuzzy SSPO-irresolute continuous mapping.

Proof. 1) Let f be a fuzzy SSPO-irresolute open mapping and let $A \in FSSPO(\tau_1)$. It is obvious that $f(A) \in FSSPO(\tau_2)$.

From the other side, for any $B \in FSSPC(\tau_1)$, $f(B^c) \in FSSPO(\tau_2)$. Since f is bijective, then $f(B^c) = f(B)^c \in FSSPO(\tau_2) \implies f(B) \in FSSPC(\tau_2)$ which confirms the mentioned.
2) The proof follows from the fact that the relation $(f^{-1})^{-1}(C) = f(C)$ stands for any bijective mapping and for any fuzzy set C of X. □

Corollary 12.1. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a bijective mapping from a fts X to a fts Y. The mapping f is a fuzzy SSPO-irresolute open if and only if $sspclf(A) \leq f(sspclA)$, for each fuzzy set A of X.*

Corollary 12.2. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a bijective mapping from a fts X to a fts Y. The following statements are equivalent:*
i) The mapping f is a fuzzy SSPO-irresolute closed;
ii) $f(sspintA) \leq sspintf(A)$, for each fuzzy set A of X;
iii) $sspintf^{-1}(B) \leq f^{-1}(sspintB)$, for each fuzzy set B of Y;
iv) $f^{-1}(sspclB) \leq sspclf^{-1}(B)$, for each fuzzy set B of Y.

Theorem 13. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts X to a fts Y. The mapping f is a fuzzy SSPO-irresolute open if and only if for each fuzzy set B of Y and each set $A \in FSSPC(\tau_1)$ such that $f^{-1}(B) \leq A$ there exists a set $C \in FSSPC(\tau_2)$ such that $B \leq C$ and $f^{-1}(C) \leq A$.*

Proof. Let B be any fuzzy set of an $ftsY$ and let $A \in FSSPC(\tau_1)$ be such that $f^{-1}(B) \leq A$. Then $(f^{-1}(B))^c \geq A^c$ and $f(A^c) \leq f(f^{-1}(B^c)) \leq B^c$. Since the mapping f is a fuzzy SSPO-irresolute open, then $f(A^c)$ is a $FSSPO$ set which means that $f(A^c) \leq sspintB^c$. From the last we will get $f^{-1}(f(A^c)) \leq f^{-1}(sspintB^c)$ and hence $A \geq f^{-1}(sspclB)$. If we take $C = sspclB$ then it is obvious that $C \in FSSPC(\tau_2)$, $B \leq C$ and $f^{-1}(C) \leq A$.

Conversely, let as suppose that $V \in FSSPO(\tau_1)$. We have to show that $f(V) \in FSSPO(\tau_2)$. If we start from the fact that $f^{-1}(f(V)) \geq V \implies f^{-1}(f(V)^c) \leq V^c$ and then if we substitute $f(V)^c = B$, a fuzzy set of Y, and $V^c = A$, $A \in FSSPC(\tau_1)$, then from the assumption of the theorem, there is a set $C \in FSSPC(\tau_2)$ such that $B = f(V)^c \leq C$ and $f^{-1}(C) \leq A = V^c$.

From $B = f(V)^c \leq C$ we conclude that $sspcl(f(V)^c) \leq sspclC = C$ and subsequently $C^c \leq sspintf(V)$. From $f^{-1}(C) \leq A = V^c$ we obtain $f^{-1}(C^c) \geq V$ and $f(V) \leq C^c \leq sspintf(V)$.

It is also obvious that $sspintf(V) \leq f(V)$, so from the last two expressions we will get that $f(V) = sspintf(V)$, which means that $f(V) \in FSSPO(\tau_2)$. In other words the mapping f is a fuzzy SSPO-irresolute open. □

Theorem 14. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts X to a fts Y. The mapping f is a fuzzy SSPO-irresolute closed mapping if and only if for each fuzzy set B of Y and each fuzzy set $A \in FSSPO(\tau_1)$ such that $f^{-1}(B) \leq A$ there exists a fuzzy set $C \in FSSPO(\tau_2)$ such that $B \leq C$ and $f^{-1}(C) \leq A$.*

Proof. Similar to the proof of the Theorem 13. □

Theorem 15. *Let $f : X \to Y$ be a mapping from a fts X to a fts Y.The mapping is a fuzzy SSPO-irresolute continuous and SSPO-irresolute closed if and only if for each fuzzy set $U \in X$, $f(sspclU) = sspclf(U)$.*

Proof. If we recall Theorem 1, we have that $f(sspclU) \leq sspclf(U)$ for any fuzzy set $U \in X$. Since the given mapping is also a fuzzy SSPO-irresolute close, from Theorem 8 we have that $sspclf(U) \leq f(sspclU)$ for each fuzzy set $U \in X$. From the last two relations we can conclude that $f(sspclU) = sspclf(U)$, for every $U \in X$. Both theorems imply double implication, therefore the theorem is proved. □

Theorem 16. *Let $f : X \to Y$ be a mapping from a fts X to a fts Y. The mapping f is a fuzzy SSPO-irresolute continuous and SSPO-irresolute open if and only if for each fuzzy set $V \in Y$, $f^{-1}(sspclV) = sspclf^{-1}(V)$.*

Proof. Similarly to the Theorem 15 and by using Theorem 1 as well as Theorem 7. □

Theorem 17. *Let $f : X \to Y$ be a mapping from a fts X to a fts Y.The mapping f is a fuzzy SSPO-irresolute continuous and SSPO-irresolute open if and only if for each fuzzy set $V \in Y$, $f^{-1}(sspintV) = sspintf^{-1}(V)$.*

Proof. It follows from Theorem 1 and Theorem 7. □

Theorem 18. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping between fts X to a fts Y and let $g : (Y, \tau_2) \to (Z, \tau_3)$ be a mapping between fts Y to a fts Z. The following is true:*
1) If both f, g are fuzzy SSPO-irresolute open (closed), then gf is also fuzzy SSPO-irresolute open (closed).
2) If the mapping f is fuzzy strongly semi pre-open (pre-closed) and g is fuzzy SSPO-irresolute open (closed) mapping then gf is also fuzzy strongly semi pre-open (pre-closed) mapping.
3) If the mapping gf is fuzzy SSPO-irresolute continuous and the mapping g is fuzzy SSPO-irresolute open (closed) and injective, then f is fuzzy SSPO-irresolute continuous mapping.
4) If the mapping gf is fuzzy SSPO-irresolute open (closed) and the mapping g is fuzzy SSPO-irresolute continuous and injective, then f is a fuzzy SSPO-irresolute open (closed) mapping.
5) If the mapping gf is fuzzy SSPO-irresolute continuous and the mapping f is fuzzy SSPO-irresolute open (closed) and surjective, then g is a fuzzy SSPO-irresolute continuous mapping.
6) If the mapping gf is fuzzy strong semi pre-continuous and the mapping f is fuzzy SSPO-irresolute open (closed) and surjective, then g is a fuzzy strong semi pre-continuous mapping.
7) If the mapping gf is fuzzy SSPO-irresolute open (closed) continuous and the mapping f is fuzzy SSPO-irresolute continuous and surjective, then g is a fuzzy SSPO-irresolute open (closed) mapping.
8) If the mapping gf is fuzzy SSPO-irresolute open (closed) and the mapping f

is fuzzy strong semi pre-continuous and surjective, then g is a fuzzy strongly semi pre-open (pre-closed) mapping.

Proof. Let us prove statement 6). For any $U \in \tau_3$, $g^{-1}(U) = ff^{-1}(g^{-1}(U)) = f((gf)^{-1}(U))$. Since gf is fuzzy strong semi pre-continuous then $(gf)^{-1}(U) \in FSSPO(\tau_1)$, and subsequently $f((gf)^{-1}(U)) \in FSSPO(\tau_2)$. Therefore, g is a fuzzy strong semi pre-continuous mapping.

Other statements are proven similarly to Theorem 6 and by using Lemma 3. □

5. Fuzzy SSPO homeomorphism

Definition 5.1. A bijective mapping $f : (X, \tau) \to (Y, \delta)$ from a *fts* (X, τ) to a *fts* (Y, δ) is called a *fuzzy SSPO homeomorphism* if f and f^{-1} are both fuzzy SSPO-irresolute continuous.

We can show that in regards to the fuzzy SSPO homeomorphism we can formulate and prove the following theorem.

Theorem 19. *Let $f : (X, \tau) \to (Y, \delta)$ be a bijective mapping from a fts (X, τ) to a fts (Y, δ). The following statements are equivalent:*
i) The mapping f is a fuzzy SSPO homeomorphism;
j) The mapping f^{-1} is a fuzzy SSPO homeomorphism;
k) The mapping f and f^{-1} are both fuzzy SSPO irresolute open (closed);
l) The mapping f is a fuzzy SSPO-irresolute continuous and fuzzy SSPO irresolute open (closed);
m) $f(sspclA) = sspclf(A)$, for each fuzzy set A of X;
n) $f(sspintA) = sspintf(A)$, for each fuzzy set A of X;
o) $f^{-1}(sspintB) = sspintf^{-1}(B)$, for each fuzzy set B of Y;
p) $sspclf^{-1}(B) = f^{-1}(sspclB)$, for each fuzzy set B of Y.

Proof. $(i) \Rightarrow (j)$ Since the bijective mapping f is a fuzzy SSPO-homeomorphism, it follows from Definition 5.1. that the mapping f^{-1} is also a fuzzy SSPO-homeomorphism. It is obvious that $(f^{-1})^{-1} = f$.
$(j) \Rightarrow (k)$ Since both f and f^{-1} are fuzzy SSPO-irresolute continuous, it then follows by using conclusions from the Theorem 12.
$(k) \Rightarrow (l)$ It also follows from the Theorem 12.
$(l) \Rightarrow (m)$ Follows from Theorem 12 and Theorem 15.
$(m) \Rightarrow (n)$ It follows from Lemma 2 and Lemma 3
$(n) \Rightarrow (0)$ Let $B \in Y$ be any the fuzzy set, then according to the assumption we have

$$f(sspintf^{-1}(B)) = sspintf(f^{-1}(B)) = sspintB,$$

which then leads to

$$f^{-1}(f(sspintf^{-1}(B))) = f^{-1}(sspintB) \implies sspintf^{-1}(B) = f^{-1}(sspintB).$$

$(o) \Rightarrow (p)$ It follows from Lemma 2 and Lemma 3
$(p) \Rightarrow (i)$ Follows from Theorem 16 and Theorem 12. □

6. Conclusion

In this work we have introduced the concept of fuzzy SSPO irresolute mappings, and we have investigated some of their properties as well as their connections with other forms of fuzzy continuity.

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