Математички Билтен
46 (LXXII) 1 (2022), 51–65

UDC: 510.64:512.533.5
DOI: 10.37560/matbil22461051m

FUZZY IDEALS IN $(n+k,n)$-SEMIGROUPS

VALENTINA MIOVSKA[1], DELCHO LESHKOVSKI[2],
AND VESNA CELAKOSKA-JORDANOVA[1]

Abstract. In this paper we introduce the concepts of fuzzy subset of vector valued groupoid (semigroup), as well as fuzzy subgroupoid, fuzzy subsemigroup, fuzzy i-ideal (ideal) and bi-ideal of vector valued groupoid (semigroup), investigate their properties and present suitable examples. Prime and fuzzy prime, semiprime and fuzzy semiprime subsets of vector valued groupoids are defined and their properties are investigated. We characterize the Green's relations $\mathcal{J}_i$ on a vector valued semigroup $\mathbf{S}$ in terms of fuzzy subsets. Green's relations $\mathcal{J}_i^F$ on $\mathbf{S}$ are suitably defined and it is shown that they coincide with the relation $\mathcal{J}_i$ on $\mathbf{S}$.

1. Introduction

The notion of fuzzy sets, introduced by Zadeh in [15], is a fundamental mathematical concept that deals with uncertainty. It has many applications in a wide range of mathematical areas, as well as in engineering and economics. According to [15], a fuzzy subset μ of a nonempty set S is a function of S into the closed interval $[0,1]$, i.e. $\mu : S \to [0,1]$. For all $x \in S$, $\mu(x)$ is called the grade of membership of x. If $\mu(x) = 1$, then we say that x is fully included in S, and, if $\mu(x) = 0$, then we say that x is not included in S. If the set S bears some structure, one may distinguish some fuzzy subsets of S in terms of that additional structure. Rosenfeld in [12] inspired the fuzzification of algebraic structures and introduced the notion of fuzzy (left, right) ideal in a groupoid and the notion of fuzzy subgroup of a group.

Motivated by the study of fuzzy ideals in semigroups ([7, 8, 9, 10, 3, 4, 5, 6]), we extend these notions on vector valued semigroups, introduced in [13] and investigated in [1], [2], [11] and [14]. In Section 2 we present the notion of fuzzy subset of vector valued groupoid (semigroup), as well as fuzzy subgroupoid, fuzzy subsemigroup, fuzzy i-ideal (ideal) of vector valued groupoid (semigroup) and suitable examples are presented. In Section 3, prime and fuzzy prime, semiprime and fuzzy semiprime subsets of vector valued groupoids are defined and their

2010 *Mathematics Subject Classification.* 08A72, 20N99; 03E72, 20M12.
Key words and phrases. Vector valued groupoid, vector valued semigroup, fuzzy subset, fuzzy ideal, Green's relations.

properties are investigated. Properties of fuzzy subsets of vector valued semigroups are considered in Section 4 as well as the notion of vector valued bi-ideal and fuzzy bi-ideal on a vector valued semigroup and some of its properties are investigated. Section 5 deals with a characterization of Green's relations $\mathcal{J}_i$ on a vector valued semigroup in terms of fuzzy subsets.

2. Preliminaries

Let S be a nonempty set. By S^n we denote the n-th Cartesian product of S, where n is a positive integer. Throughout the paper, we will use the following simplified notation. The elements of S^n, i.e. the sequences $(x_1, x_2, \ldots, x_n)$ will be denoted by $x_1x_2\ldots x_n$ or x_1^n. The symbol x_i^j will denote the sequence $x_ix_{i+1}\ldots x_j$ of elements of S when $i \leq j$ and the empty symbol when $i > j$. If $x_{i+1} = x_{i+2} = \cdots = x_{i+r} = x$, then the sequence x_{i+1}^{i+r} is denoted by $\overset{r}{x}$. Under this convention the sequence $x_1 \ldots x_i \underbrace{x \ldots x}_{r} x_{i+r+1} \ldots x_n$ will be denoted by $x_1^i \overset{r}{x} x_{i+r+1}^n$.

We define an (m, n)-operation on S as a mapping from the m-th Cartesian product S^m into S^n, where m, n are two positive integers, by

$$[\;] : (x_1, \ldots, x_m) \mapsto [x_1, \ldots, x_m].$$

The S with the (m, n)-operation forms a structure $\mathbf{S} = (S, [\;])$ called a *vector valued groupoid*, i.e. an (m, n)-*groupoid*.

Let $m - n = k \geq 1$. An $(n + k, n)$-groupoid $\mathbf{S} = (S, [\;])$ is called an $(n + k, n)$-*semigroup* or *vector valued semigroup* if

$$[x_1^i[x_{i+1}^{i+n+k}]x_{i+n+k+1}^{n+2k}] = [x_1^j[x_{j+1}^{j+n+k}]x_{j+n+k+1}^{n+2k}],$$

for all $x_1, \ldots, x_{n+2k} \in S$ and $i, j \in \{0, \ldots, k\}$.

An $(n + k, n)$-*subgroupoid* ($(n + k, n)$-*subsemigroup*) $\mathbf{P}$ is a (nonempty) subset of $\mathbf{S}$ that is an $(n+k, n)$-groupoid ($(n+k, n)$-semigroup) regarding the $(n+k, n)$-operation defined on $\mathbf{S}$.

A nonempty subset J of S is said to be:

(1) an $(n + k, n) - i$-*ideal* of $\mathbf{S}$ for some $i \in \{0, \ldots, k\}$ if

$$(\forall x_1^k \in S^k, a_1^n \in J^n) \;\; [x_1^i a_1^n x_{i+1}^k] \in J^n;$$

(2) an $(n + k, n)$-*ideal* of $\mathbf{S}$ if it is an $(n + k, n) - i$-ideal (i-ideal) of $\mathbf{S}$ for all $i \in \{0, \ldots, k\}$.

The concept of ideals of $(n + k, n)$-semigroups is introduced in [14]. Note that, in [14], $(n + k, n)$-ideal of $\mathbf{S}$ is a subset T of S^n. In our work we consider the set $T = A^n$, where $A \subseteq S$. Instead of A^n is an i-ideal od $\mathbf{S}$ we will say that A is an i-ideal od $\mathbf{S}$.

Fuzzy subsets and fuzzy i-ideals (ideals) in $(n + k, n)$-groupoids ($(n + k, n)$-semigroups) have not yet been defined and studied. Here, we introduce the notion of fuzzy subset of $(n + k, n)$-groupoid ($(n + k, n)$-semigroup) and inspired by [12] and [9], fuzzy subsemigroups and fuzzy ideals, as well.

A *fuzzy subset* μ of $(n + k, n)$-groupoid ($(n + k, n)$-semigroup) $\mathbf{S}$ is a function from the Cartesian product S^n into $[0, 1]$. The set of all fuzzy subsets of $\mathbf{S}$ is denoted by $F(S)$. Analogously as for binary groupoids (semigroups), a *grade of membership* of a sequence is a value $\mu(x_1^n) \in [0,1]$ that is assigned to each sequence $x_1^n \in S^n$. If $\mu(x_1^n) = 1$, then we say that the sequence x_1^n is *fully included* in S^n. A (partial) ordering on $F(S)$ can be defined in the following way:

$$\mu \subseteq \alpha \text{ if and only if } \mu(x_1^n) \leq \alpha(x_1^n).$$

A *fuzzy subgroupoid* (*fuzzy subsemigroup*) μ of $(n+k,n)$-groupoid ($(n+k,n)$-semigroup) $\mathbf{S}$ is a fuzzy subset of S such that

$$\mu([x_1^{n+k}]) \geq \min\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,\dots,k\}\}.$$

We denote by f_A the characteristic function of a subset A of S, which is defined as the mapping of S into [0, 1] by

$$f_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A. \end{cases}$$

Note that we will use the characteristic function on A^n and instead of f_{A^n} we will simply write f_A.

A fuzzy subset μ of $(n+k,n)$-groupoid ($(n+k,n)$-semigroup) $\mathbf{S}$ is said to be a *fuzzy i-ideal* of $\mathbf{S}$, $i \in \{0,1,\dots,k\}$, if

$$\mu([x_1^{n+k}]) \geq \mu(x_{i+1}^{i+n}),$$

for all $x_1^{n+k} \in S^{n+k}$. A fuzzy subset μ of $\mathbf{S}$ is called a *fuzzy ideal* of $\mathbf{S}$ if it is a fuzzy i-ideal for all $i \in \{0,1,\dots,k\}$. It is clear that μ is a fuzzy ideal of $\mathbf{S}$ if for all $x_1^{n+k} \in S^{n+k}$

$$\mu([x_1^{n+k}]) \geq \max\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\dots,k\}\}.$$

It is also clear that if μ is a fuzzy i-ideal (ideal) of $\mathbf{S}$, then it is a fuzzy subgroupoid (subsemigroup) of $\mathbf{S}$.

Example 2.1. Let $\mathbf{S} = (\mathbb{N}, [\ ])$ be a $(3,2)$-groupoid with $[xyz] = (x+y, y+z)$. Note that $\mathbf{S}$ is not a $(3,2)$-semigroup, since $[[111]1] = [221] = (4,3) \neq (3,4) = [122] = [1[111]]$. Let $\mu : \mathbb{N}^2 \to [0,1]$ be defined by:

$$\mu(x,y) = \begin{cases} \frac{1}{3}, \text{ if } x+y \text{ is even number} \\ \\ 0, \quad \text{otherwise.} \end{cases}$$

Clearly, μ is a fuzzy subset of $\mathbf{S}$.

Let $x,y,z \in \mathbb{N}$. If $\mu([xyz]) = \frac{1}{3}$ then $\mu([xyz]) = \frac{1}{3} \geq \min\{\mu(x,y), \mu(y,z)\}$. If $\mu([xyz]) = \mu(x+y, y+z) = 0$, then $x+2y+z$ is an odd number and so $x+z$ is an odd number.

1. If x, y are even numbers and z is an odd number, then $\min\{\mu(x,y), \mu(y,z)\} = \min\{\frac{1}{3}, 0\} = 0$ and so, $\mu([xyz]) = 0 = \min\{\mu(x,y), \mu(y,z)\}$.

2. If x is an even number and y, z are odd, then $\min\{\mu(x,y), \mu(y,z)\} = \min\{0, \frac{1}{3}\} = 0$ and so, $\mu([xyz]) = 0 = \min\{\mu(x,y), \mu(y,z)\}$.

3. If x is an odd number, y, z are even, then $\min\{\mu(x,y), \mu(y,z)\} = \min\{0, \frac{1}{3}\} = 0$ and so $\mu([xyz]) = 0 = \min\{\mu(x,y), \mu(y,z)\}$.

4. If x, y are odd numbers and z is even, then $\min\{\mu(x,y),\mu(y,z)\} = \min\{\frac{1}{3},0\} = 0$ and so $\mu([xyz]) = 0 = \min\{\mu(x,y),\mu(y,z)\}$.

Hence, $\mu([xyz]) \geq \min\{\mu(x,y),\mu(y,z)\}$, for all $x,y,z \in \mathbb{N}$, μ is a fuzzy subgroupoid of $\mathbf{S}$.

Since $\mu([223]) = \mu(4,5) = 0 < \frac{1}{3} = \mu(2,2)$ and $\mu([124]) = \mu(3,6) = 0 < \frac{1}{3} = \mu(2,4)$, μ is not a fuzzy $0-$ideal and fuzzy $1-$ideal of $\mathbf{S}$, and so μ is not a fuzzy ideal of $\mathbf{S}$.

Example 2.2. Let $\mathbf{S} = (\mathbb{N},[\,])$ be a $(3,2)$-groupoid with $[xyz] = (x,y+z)$. Since $[[xyz]w] = [x\ y+z\ w] = (x,y+z+w) = [x\ y\ z+w] = [x[yzw]]$, $\mathbf{S}$ is a $(3,2)$-semigroup. Let $\mu : \mathbb{N}^2 \to [0,1]$ be defined by:

$$\mu(x,y) = \begin{cases} \frac{1}{4}, \text{ if } x \text{ is even number} \\ 0, \quad \text{otherwise.} \end{cases}$$

Clearly, μ is a fuzzy subset of $\mathbf{S}$.

Let $x,y,z \in \mathbb{N}$. If $\mu([xyz]) = \frac{1}{4}$ then $\mu([xyz]) = \frac{1}{4} \geq \min\{\mu(x,y),\mu(y,z)\}$. If $\mu([xyz]) = \mu(x,y+z) = 0$, then x is an odd number. Then $\mu(x,y) = 0$ and

$$\min\{\mu(x,y),\mu(y,z)\} = \min\{0,\mu(y,z)\} = 0$$

and so,

$$\mu([xyz]) = 0 = \min\{\mu(x,y),\mu(y,z)\}.$$

Hence, $\mu([xyz]) \geq \min\{\mu(x,y),\mu(y,z)\}$, for all $x,y,z \in \mathbb{N}$, so μ is a fuzzy subsemigroup of $\mathbf{S}$.

Let $x,y,z \in \mathbb{N}$. If $\mu([xyz]) = \frac{1}{4}$ then x is an even number and so $\mu([xyz]) = \frac{1}{4} = \mu(x,y)$. If $\mu([xyz]) = 0$, then x is an odd number and so $\mu([xyz]) = 0 = \mu(x,y)$. Hence, μ is a fuzzy $0-$ideal of $\mathbf{S}$. Since $\mu([124]) = \mu(1,6) = 0 < \frac{1}{4} = \mu(2,4)$, μ is not a fuzzy $1-$ideal of $\mathbf{S}$, and so μ is not a fuzzy ideal of $\mathbf{S}$.

Example 2.3. Let $\mathbf{S} = (\mathbb{R},[\,])$ be a $(5,2)-$groupoid with $[x_1^5] = (1,2)$.
Let $\mu : \mathbb{R}^2 \to [0,1]$ be defined by:

$$\mu(x,y) = \begin{cases} 1, \text{ if } (x,y) = (1,2) \\ \frac{1}{2}, \quad \text{otherwise.} \end{cases}$$

Then, $\mu([x_1^5]) = \mu(1,2) = 1 \geq \max\{\mu(x_1^2),\mu(x_2^3),\mu(x_3^4),\mu(x_4^5)\}$, for all $x_1^5 \in \mathbb{R}^5$. So, μ is a fuzzy ideal of $\mathbf{S}$.

3. Fuzzy prime and semiprime subsets of $(n+k,n)$-groupoids

In this section let $\mathbf{S} = (S,[\,])$ be an $(n+k,n)$-groupoid. A subset A of $\mathbf{S}$ is called $(n+k,n)$-*prime* if the following implication holds:

$$x_1^{n+k} \in S^{n+k}, [x_1^{n+k}] \in A^n \text{ implies } x_{i+1}^{i+n} \in A^n, \text{ for some } i \in \{0,1,\ldots,k\}.$$

A fuzzy subset μ on **S** is said to be *prime* in **S**, if

$$\mu([x_1^{n+k}]) \leq \max\{\mu(x_{i+1}^{i+n}) | i \in \{0,1,\ldots,k\}\}.$$

It is clear that a fuzzy subset μ on **S** is a *prime fuzzy ideal* of **S** if and only if

$$\mu([x_1^{n+k}]) = \max\{\mu(x_{i+1}^{i+n}) | i \in \{0,1,\ldots,k\}\}.$$

Example 3.1. An $(n+k,n)-$groupoid $\mathbf{S} = (S, [\ ])$ is said to be an *n-zero* $(n+k,n)$-*groupoid* or *left zero* $(n+k,n)$-*groupoid* if $[x_1^{n+k}] = x_1^n$ for all $x_1^{n+k} \in S^{n+k}$. Left zero $(n+k,n)$-groupoid is an example of an $(n+k,n)$-semigroup. Let μ be a fuzzy subset of **S**. Then, $\mu([x_1^{n+k}]) = \mu(x_1^n) \leq \max\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\}$, for all $x_1^{n+k} \in S^{n+k}$. The fuzzy subset μ is prime.

Example 3.2. Let's consider the Example 2.2. Let $x,y,z \in \mathbb{N}$. If $\mu([xyz]) = 0$, then $\mu([xyz]) = 0 \leq \max\{\mu(x,y),\mu(y,z)\}$. If $\mu([xyz]) = \mu(x,y+z) = \frac{1}{4}$, then x is an even number. From here, $\mu(x,y) = \frac{1}{4}$ and $\max\{\mu(x,y),\mu(y,z)\} = \max\{\frac{1}{4},\mu(y,z)\} = \frac{1}{4}$ and so, $\mu([xyz]) = \frac{1}{4} = \max\{\mu(x,y),\mu(y,z)\}$.

Hence, $\mu([xyz]) \leq \max\{\mu(x,y),\mu(y,z)\}$, for all $x,y,z \in \mathbb{N}$, so the fuzzy subsemigroup μ is prime.

Proposition 3.1. *Let* **S** *be an* $(n+k,n)$*-groupoid,* $A \neq \emptyset$*,* $A \subseteq S$ *and let* f_A *be the characteristic function on* A^n*. Then* A *is a prime* $(n+k,n)$*-ideal of* **S** *if and only if* f_A *is a prime fuzzy ideal of* **S***.*

Proof. Let A be a prime $(n+k,n)$-ideal of **S** and let $x_1^{n+k} \in G^{n+k}$. If $[x_1^{n+k}] \notin A^n$, then for all $i \in \{0,1,\ldots,k\}$, $x_{i+1}^{i+n} \notin A^n$ and so $f_A(x_{i+1}^{i+n}) = 0$, for all $i \in \{0,1,\ldots,k\}$. Hence, $f_A([x_1^{n+k}]) = 0 = \max\{0,\ldots,0\}$. If $[x_1^{n+k}] \in A^n$, then $f_A([x_1^{n+k}]) = 1$. In that case, there exists a sequence x_{i+1}^{i+n} for some $i \in \{0,1,\ldots,k\}$ such that $x_{i+1}^{i+n} \in A^n$ and so $f_A(x_{i+1}^{i+n}) = 1$. Hence, $f_A([x_1^{n+k}]) = 1 = \max\{f_A(x_{j+1}^{j+n}) |\ j \in \{0,1,\ldots,k\}\}$. □

Let **S** be an $(n+k,n)$-groupoid, $x_1^n \in S^n$ and $\lambda \in [0,1]$. The mapping $(x_1^n)_\lambda : S^n \to [0,1]$

$$(x_1^n)_\lambda(y_1^n) = \begin{cases} \lambda, \text{ if } x_1^n = y_1^n \\ 0, \text{ if } x_1^n \neq y_1^n. \end{cases}$$

is called a *fuzzy point* of S^n.

The subset A of an $(n+k,n)$-groupoid **S** is called $(n+k,n)$-*semiprime* if

$$a \in S,\ [\overset{n+k}{a}] \in A^n \text{ implies } a \in A.$$

A fuzzy subset μ on **S** is said to be a *semiprime* subset of **S**, if for every $x \in S$ and each $\lambda \in [0,1]$ such that $([\overset{n+k}{x}])_\lambda \subseteq \mu$, we have $(\overset{n}{x})_\lambda \subseteq \mu$.

Proposition 3.2. *Let* **S** *be an* $(n+k,n)$*-groupoid and* μ *a fuzzy subset of* **S***. Then* μ *is semiprime if and only if* $\mu(\overset{n}{x}) \geq \mu([\overset{n+k}{x}])$ *for every* $x \in S$*.*

Proof. Let μ be semiprime of **S**. Let $\mu([\overset{n+k}{x}]) = \lambda$ and $y_1^n \in G^n$. If $y_1^n \neq [\overset{n+k}{x}]$, then $([\overset{n+k}{x}])_\lambda(y_1^n) = 0 \leq \mu(y_1^n)$. If $y_1^n = [\overset{n+k}{x}]$, then $([\overset{n+k}{x}])_\lambda(y_1^n) = \lambda = \mu([\overset{n+k}{x}]) =$

$\mu(y_1^n)$. So, $([\overset{n+k}{x}])_\lambda \subseteq \mu$ and because μ is semiprime, $(\overset{n}{x})_\lambda \subseteq \mu$. Therefore, $\mu(\overset{n}{x}) \geq (\overset{n}{x})_\lambda(\overset{n}{x}) = \lambda = \mu([\overset{n+k}{x}])$. Conversely, let $\mu(\overset{n}{x}) \geq \mu([\overset{n+k}{x}])$ for every $x \in G$. Let $\lambda \in [0,1]$ such that $([\overset{n+k}{x}])_\lambda \subseteq \mu$ and $y_1^n \in S^n$. If $y_1^n \neq \overset{n}{x}$, then $(\overset{n}{x})_\lambda(y_1^n) = 0 \leq \mu(y_1^n)$. If $y_1^n = \overset{n}{x}$, then $(\overset{n}{x})_\lambda(\overset{n}{x}) = \lambda = ([\overset{n+k}{x}])_\lambda([\overset{n+k}{x}]) \leq \mu([\overset{n+k}{x}]) \leq \mu(\overset{n}{x})$. Hence, $(\overset{n}{x})_\lambda \subseteq \mu$. □

Let **S** be an $(n+k,n)$-groupoid, $a \in S$. If μ is a fuzzy ideal of **S**, then $\mu([\overset{n+k}{a}]) \geq \max\{\mu(\overset{n}{a}), \dots, \mu(\overset{n}{a})\} = \mu(\overset{n}{a})$. Hence, if μ is a semiprime fuzzy ideal of **S**, then, by Prop. 3.2, we have $\mu(\overset{n}{a}) = \mu([\overset{n+k}{a}])$.

If μ is a prime fuzzy ideal of **S**, then $\mu([\overset{n+k}{a}]) = \max\{\mu(\overset{n}{a}), \dots, \mu(\overset{n}{a})\} = \mu(\overset{n}{a})$. By Prop. 3.2, μ is semiprime.

Proposition 3.3. *Let **S** be an $(n+k,n)$-groupoid and $A \neq \emptyset$, $A \subseteq S$ and let f_A be the characteristic function on A^n. The following are equivalent:*
*(i) A is an $(n+k,n)-$ semiprime subset of **S**.*
*(ii) f_A is a semiprime fuzzy subset of **S**.*

Proof. Let A is an $(n+k,n)-$ semiprime subset of **S** and let $x \in S$. If $[\overset{n+k}{x}] \in A^n$, then $x \in A$, i.e. $\overset{n}{x} \in A^n$ and so $f_A([\overset{n+k}{x}]) = 1$ and $f_A(\overset{n}{x}) = 1$. Hence, $f_A(\overset{n}{x}) \geq f_A([\overset{n+k}{x}])$. If $[\overset{n+k}{x}] \notin A^n$, then $f_A([\overset{n+k}{x}]) = 0 \leq f_A(\overset{n}{x})$. By Prop. 3.2 we have that f_A is a semiprime fuzzy subset of **S**. Conversely, let f_A be a semiprime fuzzy subset of **S** and let $a \in A$ such that $[\overset{n+k}{a}] \in A^n$. Then $f_A([\overset{n+k}{a}]) = 1$ and by Prop. 3.2 we have $f_A([\overset{n+k}{a}]) \leq f_A(\overset{n}{a})$. So, $f_A(\overset{n}{a}) = 1$, i.e. $a \in A$. Hence, A is an $(n+k,n)-$ semiprime subset of **S**. □

By Prop. 3.2 and 3.3, we obtain

Proposition 3.4. *Let **S** be an $(n+k,n)$-groupoid and $A \neq \emptyset$, $A \subseteq S$ and let f_A be the characteristic function on A^n. Then A is an $(n+k,n)-$ semiprime ideal of **S** if and only if f_A is a semiprime fuzzy ideal of **S**.*

4. Properties of fuzzy subsets of vector valued semigroups

Inspired by [6] we obtain the following results for vector valued semigroups. Let **S** be an $(n+k,n)$-semigroup and μ be a fuzzy subset of **S**. The mapping $\mu' : S^n \to [0,1]$ defined by $\mu'(x_1^n) = 1 - \mu(x_1^n)$, is a fuzzy subset of **S** called *the complement of* μ in **S**. It is obvious that for all $x_1^n, y_1^n \in S^n$ the following statements hold:

$\mu(x_1^n) \leq \mu(y_1^n) \Leftrightarrow \mu'(x_1^n) \geq \mu'(y_1^n)$;
$\mu(x_1^n) = \mu(y_1^n) \Leftrightarrow \mu'(x_1^n) = \mu'(y_1^n)$;
$(\mu')' = \mu$.

Proposition 4.1. *Let **S** be an $(n+k,n)$-semigroup and μ a fuzzy subset of **S**. Then $1 - \min\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\dots,k\}\} = \max\{\mu'(x_{i+1}^{i+n}) |\ i \in \{0,1,\dots,k\}\}$, for all $x_1^{n+k} \in S^{n+k}$.*

Proof. Let $x_1^{n+k} \in S^{n+k}$. Since $\mu(x_{i+1}^{i+n}) \in [0,1]$, there is $j \in \{0,1,\ldots,k\}$ such that $\mu(x_{j+1}^{j+n}) \le \mu(x_{i+1}^{i+n})$, for all $i \in \{0,1,\ldots,k\}$. Then

$$\mu(x_{j+1}^{j+n}) = \min\{\mu(x_{i+1}^{i+n}) | i \in \{0,1,\ldots,k\}\}, \text{ so}$$

$$1 - \min\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\} = 1 - \mu(x_{j+1}^{j+n}) = \mu'(x_{j+1}^{j+n}).$$

As $\mu(x_{j+1}^{j+n}) \le \mu(x_{i+1}^{i+n})$, for all $i \in \{0,1,\ldots,k\}$, we obtain that $1 - \mu(x_{j+1}^{j+n}) \ge 1 - \mu(x_{i+1}^{i+n})$, for all $i \in \{0,1,\ldots,k\}$, i.e. $\mu'(x_{j+1}^{j+n}) \ge \mu'(x_{i+1}^{i+n})$. Hence,

$$\mu'(x_{j+1}^{j+n}) = \max\{\mu'(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\} \text{ and}$$

$$1 - \min\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\} = \max\{\mu'(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\}. \qquad \square$$

Corollary 4.1. *Let $\boldsymbol{S}$ be an $(n+k,n)$-semigroup and μ a fuzzy subset of $\boldsymbol{S}$. Then $1 - \min\{\mu'(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\} = \max\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\}$, for all $x_1^{n+k} \in S^{n+k}$.*

Proof. By Prop. 4.1 we obtain that

$$1 - \min\{\mu'(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\} = \max\{(\mu')'(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\} = \\ = \max\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\}. \qquad \square$$

As a consequence of Prop. 4.1 and Cor. 4.1, one obtains the following

Corollary 4.2. *Let $\boldsymbol{S}$ be an $(n+k,n)$-semigroup and μ be a fuzzy subset of $\boldsymbol{S}$. Then for all $x_1^{n+k} \in S^{n+k}$ the following statements hold:*
a) $1 - \max\{\mu'(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\} = \min\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\}$
b) $1 - \max\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\} = \min\{\mu'(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\}$.

Corollary 4.3. *Let $\boldsymbol{S}$ be an $(n+k,n)$-semigroup and μ be a fuzzy subset of $\boldsymbol{S}$. The following statements are equivalent:*
i) $\mu([x_1^{n+k}]) = \min\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\}$, $x_1^{n+k} \in S^{n+k}$;
ii) $\mu'([x_1^{n+k}]) = \max\{\mu'(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\}$, $x_1^{n+k} \in S^{n+k}$.

Proof. Let $x_1^{n+k} \in S^{n+k}$ and let *i)* holds. Then

$$\mu'([x_1^{n+k}]) = 1 - \mu([x_1^{n+k}]) = 1 - \min\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\} = \\ = \max\{\mu'(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\} \text{ So, } ii) \text{ is true.}$$

Conversely, let *ii)* holds. Then

$$\mu([x_1^{n+k}]) = 1 - \mu'([x_1^{n+k}]) = 1 - \max\{\mu'(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\} = \\ = \min\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\} \text{ So, } i) \text{ is true.} \qquad \square$$

A fuzzy subset μ of $(n+k,n)-$semigroup **S** is called a *fuzzy filter of* **S** if

$$\mu([x_1^{n+k}]) = \min\{\mu(x_{i+1}^{i+n}) |\ i \in \{0,1,\ldots,k\}\}$$

for all $x_1^{n+k} \in S^{n+k}$.

Proposition 4.2. *Let $\boldsymbol{S}$ be an $(n+k,n)-$semigroup. A fuzzy subset μ of $\boldsymbol{S}$ is a fuzzy filter of $\boldsymbol{S}$ if and only if the fuzzy subset μ' of $\boldsymbol{S}$ is a prime fuzzy ideal of $\boldsymbol{S}$.*

Proof. Let μ be a fuzzy filter of **S**. Then

$$\mu([x_1^{n+k}]) = \min\{\mu(x_{i+1}^{i+n}) | i \in \{0,1,\dots,k\}\}, \text{ i.e.}$$
$$\mu'([x_1^{n+k}]) = \max\{\mu'(x_{i+1}^{i+n}) | i \in \{0,1,\dots,k\}\}.$$

Clearly, $\mu'([x_1^{n+k}]) \geq \max\{\mu'(x_{i+1}^{i+n}) | i \in \{0,1,\dots,k\}\}$, so μ' is a fuzzy ideal of **S**. Also, by $\mu'([x_1^{n+k}]) \leq \max\{\mu'(x_{i+1}^{i+n}) | i \in \{0,1,\dots,k\}\}$, we conclude that μ' is prime. In a similar way one can show the opposite direction of the proposition. □

Lemma 1. *Let $\emptyset \neq A \subseteq \mathbb{R}$ and there is* $\inf A$ *in* $\mathbb{R}$. *Let* $b \in \mathbb{R}$. *Then*

$$b - \inf A = \sup\{b - a | a \in A\}.$$

Proposition 4.3. *Let **S** be an $(n+k,n)$-semigroup and μ be a fuzzy subset of **S**. Let $A \neq \emptyset$, $A \subseteq S$. Then* $1 - \inf\{\mu(a_1^n) | a_1^n \in A^n\} = \sup\{\mu'(a_1^n) | a_1^n \in A^n\}$.

Proof. Since $\mu(A^n) \subseteq [0,1]$ and $\mu(A^n) \neq \emptyset$, there is an $\inf \mu(A^n)$. By Lemma 1, one obtains $1 - \inf(\mu(A^n)) = \sup\{1 - x | x \in \mu(A^n)\} = \sup\{1 - \mu(a_1^n) | a_1^n \in A^n\} = \sup\{\mu'(a_1^n) | a_1^n \in A^n\}$. □

Corollary 4.4. *Let **S** be an $(n+k,n)$-semigroup and μ be a fuzzy subset of **S**. Let $A \neq \emptyset$, $A \subseteq S$. Then* $1 - \inf\{\mu'(a_1^n) | a_1^n \in A^n\} = \sup\{\mu(a_1^n) | a_1^n \in A^n\}$.

Proof. It is obvious as a consequence of Prop. 4.3. □

The following proposition gives a characterization of $(n+k,n)$-subsemigroup, $(n+k,n)-i$-ideal and $(n+k,n)$-ideal of **S** in terms fuzzy subsemigroup, fuzzy i-ideal and fuzzy ideal of **S**, respectively.

Proposition 4.4. *Let $A \neq \emptyset$, $A \subseteq S$ and let f_A be the characteristic function on A^n. Then*

*(i) A is an $(n+k,n)$-subsemigroup of **S** if and only if f_A is a fuzzy subsemigroup of **S**.*

*(ii) A is an $(n+k,n)-i$-ideal of **S** if and only if f_A is a fuzzy i-ideal of **S**, for $i \in \{0,1,\dots,k\}$.*

*(iii) A is an $(n+k,n)$-ideal of **S** if and only if f_A is a fuzzy ideal of **S**.*

Proof. (i) Let A be an $(n+k,n)$-subsemigroup of **S** and let $x_1^{n+k} \in S^{n+k}$. If $[x_1^{n+k}] \notin A^n$, then $f_A([x_1^{n+k}]) = 0$ and there is $j \in \{0,\dots,n+k\}$ such that $x_j \notin A$. In that case there exists a sequence x_{i+1}^{i+n} such that $x_{i+\alpha} = x_j$, for some $\alpha \in \{1,\dots,n\}$, $i \in \{0,\dots,k\}$. Hence, $f_A(x_{i+1}^{i+n}) = 0$ and $f_A([x_1^{n+k}]) = 0 \geq 0 = \min\{f_A(x_{i+1}^{i+n}) | \, i \in \{0,\dots,k\}\}$. If $[x_1^{n+k}] \in A^n$, then $f_A([x_1^{n+k}]) = 1 \geq \min\{f_A(x_{i+1}^{i+n}) | \, i \in \{0,\dots,k\}\}$.

Conversely, let f_A is a fuzzy subsemigroup of **S** and let $x_1^{n+k} \in A^{n+k}$. Then $f_A([x_1^{n+k}]) \geq \min\{f_A(x_{i+1}^{i+n}) | \, i \in \{0,\dots,k\}\} = \min\{1,\dots,1\} = 1$. Therefore $f_A([x_1^{n+k}]) = 1$ and hence $[x_1^{n+k}] \in A^n$.

(ii) Let A is an $(n+k,n)-i$-ideal of **S** and let $x_1^{n+k} \in S^{n+k}$. If $[x_1^{n+k}] \notin A^n$, then $x_{i+1}^{i+n} \notin A^n$, and therefore $f_A([x_1^{n+k}]) = 0 \geq 0 = f_A(x_{i+1}^{i+n})$. If $[x_1^{n+k}] \in A^n$, then $f_A([x_1^{n+k}]) = 1 \geq f_A(x_{i+1}^{i+n})$.

Conversely, let f_A be a fuzzy i-ideal of **S** for some $i \in \{0,\dots,k\}$ and let $x_1^k \in S^k$,

$a_1^n \in A^n$. Then $f_A([x_1^i a_1^n x_{i+1}^k]) \geq f_A(a_1^n) = 1$. Therefore $f_A([x_1^i a_1^n x_{i+1}^k]) = 1$ and hence $[x_1^i a_1^n x_{i+1}^k] \in A^n$.

(iii) As a direct consequence of the definitions and (ii) the proof of the claim is obvious. □

Now we will introduce the notion of fuzzy bi-ideal of a vector valued semigroup and investigate some of its properties.

An $(n+k,n)$-subsemigroup A of **S** is said to be an $(n+k,n)$-*bi-ideal* on **S** if $[\overset{k}{(a_1^n x_1^l)}\, b_1^n] \in A^n$, for all $a_1^n, b_1^n \in A^n$, $x_1^l \in S^l$, $l \in \mathbb{N}_0$. A fuzzy subset μ of **S** is a *fuzzy bi-ideal* on **S** if

$$\mu([\overset{k}{(x_1^n y_1^l)}\, z_1^n]) \geq \min\{\mu(x_1^n), \mu(z_1^n)\},$$

for all $x_1^n, z_1^n \in S^n$, $x_1^l \in S^l$, $l \in \mathbb{N}_0$.

Proposition 4.5. *Let A be an $(n+k,n)$-subsemigroup of **S** and let f_A be the characteristic function on A^n. Then A is an $(n+k,n)$-bi-ideal of **S** if and only if f_A is a fuzzy bi-ideal of **S**.*

Proof. Let A be an $(n+k,n)$-bi-ideal of **S** and let $x_1^n, z_1^n \in S^n$, $x_1^l \in S^l$, $l \in \mathbb{N}_0$. If $[\overset{k}{(x_1^n y_1^l)}\, z_1^n] \notin A^n$, then $f_A([\overset{k}{(x_1^n y_1^l)}\, z_1^n]) = 0$ and there $x_1^n \notin A^n$ or $z_1^n \notin A^n$. Hence, $f_A([\overset{k}{(x_1^n y_1^l)}\, z_1^n]) = 0 \geq 0 = \min\{f_A(x_1^n), f_A(z_1^n)\}$. If $[\overset{k}{(x_1^n y_1^l)}\, z_1^n] \in A^n$, then $f_A([\overset{k}{(x_1^n y_1^l)}\, z_1^n]) = 1 \geq \min\{f_A(x_1^n), f_A(z_1^n)\}$.

Conversely, let f_A be a fuzzy bi-ideal of **S** and let $x_1^n, z_1^n \in A^n$, $x_1^l \in S^l$, $l \in \mathbb{N}_0$. Then $f_A([\overset{k}{(x_1^n y_1^l)}\, z_1^n]) \geq \min\{f_A(x_1^n), f_A(z_1^n)\} = \min\{1,1\} = 1$. Therefore $f_A([\overset{k}{(x_1^n y_1^l)}\, z_1^n]) = 1$ and hence $[\overset{k}{(x_1^n y_1^l)}\, z_1^n] \in A^n$. □

Proposition 4.6. *Every fuzzy i-ideal (fuzzy ideal) of **S** is a fuzzy bi-ideal of **S**.*

Proof. Let μ be a fuzzy i-ideal of **S** for some $i \in \{0,\ldots,k\}$ and let $a_1^n, b_1^n \in A^n$, $x_1^l \in S^l$, $l \in \mathbb{N}_0$. Then

$$\mu([\overset{k}{(a_1^n x_1^l)}\, b_1^n]) = \mu([y_1^{n+(n+l)k}]) = \mu([y_1^i[y_{i+1}^{n+(n+l)k-k+i}]y_{n+(n+l)k-k+i+1}^{n+(n+l)k}]) \geq$$
$$\geq \mu([y_{i+1}^{n+(n+l)k-k+i}]) \geq \ldots \geq \mu([y_{(n+l)i+1}^{(n+l)i+n}]) = \mu(a_1^n) \geq \min\{\mu(a_1^n), \mu(b_1^n)\}.$$

Thus, μ is a fuzzy bi-ideal of **S**.

If μ is a fuzzy ideal of **S**, then μ is a fuzzy i-ideal for all $i \in \{0,\ldots,k\}$ and therefore μ is a fuzzy bi-ideal of **S**. □

Corollary 4.5. *Every $(n+k,n)-i$-ideal ($(n+k,n)$-ideal) of **S** is an $(n+k,n)$-bi-ideal of **S**.*

Proof. The proof is obvious as a consequence of the previous two propositions. □

$(k+1)$-*product* of fuzzy subsets $\mu_1, \ldots, \mu_{k+1}$ of **S**, denoted by $((\mu_1, \ldots, \mu_{k+1}))$ or $((\mu_1^{k+1}))$, is defined by:

$$((\mu_1^{k+1}))(x_1^n) = \begin{cases} \sup\limits_{x_1^n = [y_1^{n+k}]} \min\{\mu_1(y_1^n), \mu_2(y_2^{n+1}), \ldots, \mu_{k+1}(y_{k+1}^{n+k})\}, & \text{if } x_1^n = [y_1^{n+k}] \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 4.7. *A fuzzy subset μ of an $(n+k,n)-$semigroup **S** is a fuzzy subsemigroup of **S** if and only if $((\overset{k+1}{\mu})) \subseteq \mu$.*

Proof. Let μ be a fuzzy subsemigroup of **S** and let $x_1^n \in S^n$. If $((\overset{k+1}{\mu}))(x_1^n) = 0$, then $((\overset{k+1}{\mu}))(x_1^n) = 0 \leq \mu(x_1^n)$. Let $x_1^n = [y_1^{n+k}]$. Then

$$((\overset{k+1}{\mu}))(x_1^n) = \sup_{x_1^n = [y_1^{n+k}]} \min\{\mu(y_1^n), \mu(y_2^{n+1}), \ldots, \mu(y_{k+1}^{n+k})\}$$

and

$$\mu(x_1^n) = \mu([y_1^{n+k}]) \geq \min\{\mu(y_1^n), \mu(y_2^{n+1}), \ldots, \mu(y_{k+1}^{n+k})\}.$$

Hence, $((\overset{k+1}{\mu}))(x_1^n) \leq \mu(x_1^n)$.
Conversely, let $((\overset{k+1}{\mu})) \subseteq \mu$ and let $x_1^{n+k} \in S^{n+k}$. Then

$$\mu([x_1^{n+k}]) \geq ((\overset{k+1}{\mu}))([x_1^{n+k}]) = \sup_{[x_1^{n+k}] = [y_1^{n+k}]} \min\{\mu(y_1^n), \ldots, \mu(y_{k+1}^{n+k})\} \geq$$

$$\geq \min\{\mu(x_1^n), \ldots, \mu(x_{k+1}^{n+k})\}.$$

□

Proposition 4.8. *A fuzzy subset μ of an $(n+k,n)-$semigroup **S** is a fuzzy $i-$ideal of **S**, for $i \in \{0, 1, \ldots, k\}$, if and only if*

$$((\overset{i}{\varepsilon}\mu\overset{k-i}{\varepsilon})) \subseteq \mu,$$

where $\varepsilon : S^n \to [0,1], \varepsilon(x_1^n) = 1$, for all $x_1^n \in S^n$.

Proof. Let μ be a fuzzy $i-$ideal of **S** and let $x_1^n \in S^n$. If $((\overset{i}{\varepsilon}\mu\overset{k-i}{\varepsilon}))(x_1^n) = 0$, then $((\overset{i}{\varepsilon}\mu\overset{k-i}{\varepsilon}))(x_1^n) = 0 \leq \mu(x_1^n)$. Let $x_1^n = [y_1^{n+k}]$. Then
$((\overset{i}{\varepsilon}\mu\overset{k-i}{\varepsilon}))(x_1^n) = \sup\limits_{x_1^n = [y_1^{n+k}]} \min\{\varepsilon(y_1^n), \ldots, \varepsilon(y_i^{i+n-1}), \mu(y_{i+1}^{i+n}), \ldots, \varepsilon(y_{k+1}^{n+k})\} =$
$\sup\limits_{x_1^n = [y_1^{n+k}]} \min\{1, \mu(y_{i+1}^{i+n})\} = \sup\limits_{x_1^n = [y_1^{n+k}]} (\mu(y_{i+1}^{i+n}))$ and $\mu(x_1^n) = \mu([y_1^{n+k}]) \geq \mu(y_{i+1}^{i+n})$.
Hence, $((\overset{i}{\varepsilon}\mu\overset{k-i}{\varepsilon}))(x_1^n) \leq \mu(x_1^n)$.

Conversely, let $((\overset{i}{\varepsilon}\mu\overset{k-i}{\varepsilon})) \subseteq \mu$ and let $x_1^{n+k} \in S^{n+k}$. Then

$$\mu([x_1^{n+k}]) \geq ((\overset{i}{\varepsilon}\mu\overset{k-i}{\varepsilon}))([x_1^{n+k}]) =$$

$$\sup_{[x_1^{n+k}] = [y_1^{n+k}]} \min\{\varepsilon(y_1^n), \ldots, \varepsilon(y_i^{i+n-1}), \mu(y_{i+1}^{i+n}), \ldots, \varepsilon(y_{k+1}^{n+k})\} =$$

$$\sup_{[x_1^{n+k}]=[y_1^{n+k}]} \min\{1, \mu(y_{i+1}^{i+n})\} = \sup_{[x_1^{n+k}]=[y_1^{n+k}]} \{\mu(y_{i+1}^{i+n})\} \geq \mu(y_{i+1}^{i+n}).$$

□

Proposition 4.9. *A fuzzy subset μ of an $(n+k,n)-$semigroup $\boldsymbol{S}$ is a fuzzy ideal of $\boldsymbol{S}$ if and only if $((\overset{i}{\varepsilon}\mu\overset{k-i}{\varepsilon})) \subseteq \mu$ for all $i \in \{0,1,\ldots,k\}$.*

Proof. It is a direct consequence of the definition and the previous proposition. □

5. Green's relations in vector valued semigroups in terms of fuzzy subsets

The Green's equivalence relations on vector valued semigroups were considered in [14]. We will use the following notion and notation given in [14].

Let $a_1^n \in S^n$. Then the smallest $(n+k,n)-i$-ideal containing a_1^n, for $0 \leq i \leq n$, is

$$J_i(a_1^n) = \{a_1^n\} \cup \{[z_1^{li} a_1^n z_{li+1}^{lk}] \mid z_1^{lk} \in S^{lk}, l \in \mathbb{N}\}.$$

It is called the *principal* $(n+k,n) - i$*-ideal* generated by a_1^n.

The *principal* $(n+k,n)$*-ideal* $J(a_1^n)$ generated by a_1^n is defined in a similar way, for every $i \in \{0,1,\ldots,n\}$.

The notion of fuzzy i-ideal in a vector valued semigroup defined in Section 2 leads naturally to consideration of the Green's equivalence relations on such ideals.

In this section we will assume that $\mathbf{S} = (S,[\ ])$ is an $(n+k,n)$-semigroup and that μ is a fuzzy subset of $\mathbf{S}$. We denote the set of all fuzzy i-ideals of $\mathbf{S}$ that contain μ by J_i^μ, i.e.

$$J_i^\mu = \{\alpha \mid \alpha \text{ is a fuzzy } i-\text{ideal of } \mathbf{S},\ \mu \subseteq \alpha\}.$$

The fuzzy subset ρ of $F(S)$ is a *fuzzy i-ideal generated by* μ if and only if

$$\rho \in J_i^\mu \tag{5.1}$$

$$(\forall \alpha \in J_i^\mu)\ \ \rho \subseteq \alpha. \tag{5.2}$$

Proposition 5.1. *Let $\boldsymbol{S}$ be an $(n+k,n)$-semigroup and $\mu \in F(S)$. The mapping $\bigwedge_{\alpha \in J_i^\mu} \alpha : S^n \to [0,1]$ defined by:*

$$(\bigwedge_{\alpha \in J_i^\mu} \alpha)(x_1^n) = \inf\{\alpha(x_1^n) | \alpha \in J_i^\mu\},$$

for all $x_1^n \in S^n$ is a fuzzy ideal generated by μ.

Proof. Obviously, the mapping $\bigwedge_{\alpha \in J_i^\mu} \alpha$ is well defined. Let $x_1^{n+k} \in S^{n+k}$. Then

$$(\bigwedge_{\alpha \in J_i^\mu} \alpha)([x_1^{n+k}]) = \inf\{\alpha([x_1^{n+k}]) | \alpha \in J_i^\mu\} \geq$$

$$\geq \inf\{\alpha(x_{i+1}^{i+n}) | \alpha \in J_i^\mu\} = (\bigwedge_{\alpha \in J_i^\mu} \alpha)(x_{i+1}^{i+n}),$$

so $\bigwedge\limits_{\alpha\in J_i^\mu}\alpha$ is a fuzzy i-ideal of $\mathbf{S}$.

Let $\alpha \in J_i^\mu$. Then $\mu \subseteq \alpha$, so $\mu(x_1^n) \leq \alpha(x_1^n)$. From here we obtain that $\mu(x_1^n) \leq \inf\{\alpha(x_1^n) | \alpha \in J_i^\mu\} = (\bigwedge\limits_{\alpha\in J_i^\mu}\alpha)(x_1^n)$, for all $x_1^n \in S^n$, so $\mu \subseteq \bigwedge\limits_{\alpha\in J_i^\mu}\alpha$ and therefore $\bigwedge\limits_{\alpha\in J_i^\mu}\alpha \in J_i^\mu$. Moreover, $(\bigwedge\limits_{\alpha\in J_i^\mu}\alpha)(x_1^n) = \inf\{\alpha(x_1^n)|\alpha \in J_i^\mu\}$, for all $x_1^n \in S^n$. Hence, $\bigwedge\limits_{\alpha\in J_i^\mu}\alpha \subseteq \alpha$, for all $\alpha \in J_i^\mu$. □

Proposition 5.2. *Let $\boldsymbol{S}$ be an $(n+k, n)$-semigroup and $\mu \in F(S)$. If β is a fuzzy i-ideal of $\boldsymbol{S}$ generated by μ, then $\bigwedge\limits_{\alpha\in J_i^\mu}\alpha = \beta$.*

Proof. Let β be a fuzzy i-ideal of $\mathbf{S}$ generated by μ. Then $\beta \in J_i^\mu$ and $\beta \subseteq \alpha$, for all $\alpha \in J_i^\mu$. Let $x_1^n \in S^n$. Then

$$(\bigwedge_{\alpha\in J_i^\mu}\alpha)(x_1^n) = \inf\{\alpha(x_1^n)|\alpha \in J_i^\mu\} \leq \beta(x_1^n),$$

since $\beta \in J_i^\mu$. Thus $\bigwedge\limits_{\alpha\in J_i^\mu}\alpha \subseteq \beta$. For the reverse inclusion, since $\beta \subseteq \alpha$, for all $\alpha \in J_i^\mu$, it follows that $\beta(x_1^n) \subseteq \alpha(x_1^n)$, for all x_1^n. Therefore

$$\beta(x_1^n) \leq \inf\{\alpha(x_1^n)|\alpha \in J_i^\mu\} = (\bigwedge_{\alpha\in J_i^\mu}\alpha)(x_1^n)$$

and so, $\beta \subseteq \bigwedge\limits_{\alpha\in J_i^\mu}\alpha$. □

In what follows, instead of $\bigwedge\limits_{\alpha\in J_i^\mu}\alpha$ we will use the notation $J_i(\mu)$ for the fuzzy ideal of $\mathbf{S}$ generated by μ.

However, first we will show that the set of all fuzzy i-ideals of $\mathbf{S}$ such that the sequence x_1^n is fully included in S^n coincides with the fuzzy i-ideal of $\mathbf{S}$ generated by the characteristic function $f_{\{x_1^n\}}$. For that purpose, we denote the set $\{\alpha|\ \alpha \text{ is a fuzzy } i-\text{ideal of } \mathbf{S},\ \alpha(x_1^n) = 1\}$ by J_{i,x_1^n}.

Theorem 1. $J_{i,x_1^n} = J_i^{f_{\{x_1^n\}}}$.

Proof. Let $\mu \in J_{i,x_1^n}$ and let $y_1^n \in Q^n$. If $y_1^n = x_1^n$, then $\mu(y_1^n) = \mu(x_1^n) = 1$ and $f_{\{x_1^n\}}(y_1^n) = f_{\{x_1^n\}}(x_1^n) = 1$, so $f_{\{x_1^n\}}(y_1^n) = \mu(y_1^n)$.

Let $y_1^n \neq x_1^n$. Then $f_{\{x_1^n\}}(y_1^n) = 0 \leq \mu(y_1^n)$. Thus, $f_{\{x_1^n\}}(y_1^n) \leq \mu(y_1^n)$, for all $y_1^n \in S^n$, so $f_{\{x_1^n\}} \subseteq \mu$. Since $\mu \in J_i^{f_{\{x_1^n\}}}$, it follows that $J_{i,x_1^n} \subseteq J_i^{f_{\{x_1^n\}}}$. For the reverse inclusion, let $\mu \in J_i^{f_{\{x_1^n\}}}$. Then μ is a fuzzy i-ideal of $\mathbf{S}$ and $f_{\{x_1^n\}} \subseteq \mu$. From here, $\mu(x_1^n) \geq f_{\{x_1^n\}}(x_1^n) = 1$, so $\mu(x_1^n) = 1$. Therefore, $\mu \in J_{i,x_1^n}$ and so $J_i^{f_{\{x_1^n\}}} \subseteq J_{i,x_1^n}$. □

Theorem 2. $J_i(f_{\{x_1^n\}}) = f_{J_i(x_1^n)}$.

Proof. Using Theorem 1 we obtain:

$$J_i(f_{\{x_1^n\}}) = \bigwedge_{\alpha\in J_i^{f_{\{x_1^n\}}}} \alpha = \bigwedge_{\alpha\in J_{i,x_1^n}} \alpha \subseteq \alpha, \tag{5.3}$$

for all $\alpha \in J_{i,x_1^n}$. Since $x_1^n \in J_i(x_1^n)$, then

$$f_{J_i(x_1^n)}(x_1^n) = 1. \tag{5.4}$$

Let $y_1^{n+k} \in S^{n+k}$. If $[y_1^{n+k}] \in J_i(x_1^n)$, then

$$f_{J_i(x_1^n)}([y_1^{n+k}]) = 1 \geq f_{J_i(x_1^n)}(y_{i+1}^{i+n}).$$

Let $[y_1^{n+k}] \notin J_i(x_1^n)$. Then $f_{J_i(x_1^n)}([y_1^{n+k}]) = 0$. Suppose that $f_{J_i(x_1^n)}(y_{i+1}^{i+n}) = 1$. Then $y_{i+1}^{i+n} \in J_i(x_1^n)$, so

$$[y_1^{n+k}] = [y_1^i[a_1^{li}x_1^n a_{li+1}^{lk}]y_{i+1}^k] = [y_1^i a_1^{li}\ x_1^n\ a_{li+1}^{lk} y_{i+1}^k] = [z_1^{(l+1)i} x_1^n z_{(l+1)i+1}^{(l+1)k}]$$

and thus $[y_1^{n+k}] \in J_i(x_1^n)$, that contradicts the supposition. So, $f_{J_i(x_1^n)}(y_{i+1}^{i+n}) = 0$ and $f_{J_i(x_1^n)}([y_1^{n+k}]) = 0 \geq 0 = f_{J_i(x_1^n)}(y_{i+1}^{i+n})$. Hence, $f_{J_i(x_1^n)}$ is a fuzzy i-ideal of $\mathbf{S}$ and using (5.4) we obtain that $f_{J_i(x_1^n)} \in J_{i,x_1^n}$. From (5.3) it follows that $J_i(f_{\{x_1^n\}}) \subseteq f_{J_i(x_1^n)}$.

For the reverse inclusion, we will show that $f_{J_i(x_1^n)} \subseteq \alpha$, for every $\alpha \in J_{i,x_1^n}$. Let $\alpha \in J_{i,x_1^n}$, i.e. α is a fuzzy i-ideal of $\mathbf{S}$ and $\alpha(x_1^n) = 1$. Let $y_1^n \in S^n$. If $y_1^n \notin J_i(x_1^n)$, then $f_{J_i(x_1^n)}(y_1^n) = 0 \leq \alpha(y_1^n)$. If $y_1^n \in J_i(x_1^n)$, then $y_1^n = [a_1^{li}x_1^n a_{li+1}^{lk}]$ and

$$\alpha(y_1^n) = \alpha([a_1^{li}x_1^n a_{li+1}^{lk}]) = \alpha([a_1^i[a_{i+1}^{li}x_1^n a_{li+1}^{(l-1)k}]a_{(l-1)k+1}^{lk}]) \geq$$
$$\geq \alpha([a_{i+1}^{li}x_1^n a_{li+1}^{(l-1)k}]) \geq \ldots \geq \alpha(x_1^n) = 1.$$

Thus $\alpha(y_1^n) = 1$, so $f_{J_i(x_1^n)}(y_1^n) = 1 \leq 1 = \alpha(y_1^n)$. Therefore, $f_{J_i(x_1^n)} \subseteq \alpha$, for every $\alpha \in J_{i,x_1^n}$ and

$$f_{J_i(x_1^n)} \subseteq \inf\{\alpha|\ \alpha \in J_{i,x_1^n}\} = \inf\{\alpha|\ \alpha \in J_i^{f_{\{x_1^n\}}}\} = J_i(f_{\{x_1^n\}}),$$

that completes the proof. □

Remark 5.1: We denote $J_i(f_{\{x_1^n\}})$ by $J_i^F(x_1^n)$, for every $x_1^n \in S^n$, to simplify the notation. $J_i^F(x_1^n)$ is called a *fuzzy i-ideal of* $\mathbf{S}$ *generated by* x_1^n. $J^F(x_1^n)$ is a *fuzzy ideal of* $\mathbf{S}$ *generated by* x_1^n if $J_i^F(x_1^n)$ is a fuzzy i-ideal of $\mathbf{S}$ generated by x_1^n for every $i \in \{0,\ldots,k\}$. By the previous notations and Theorem 1 we have

$$J_i^F(x_1^n) = J_i(f_{\{x_1^n\}}) = \bigwedge_{\alpha\in J_i^{f_{\{x_1^n\}}}} \alpha = \bigwedge_{\alpha\in J_{i,x_1^n}} \alpha.$$

Note that $J_i^F(x_1^n) \in J_{i,x_1^n}$. Namely, for $J_i^F(x_1^n) = \bigwedge_{\alpha\in J_{i,x_1^n}} \alpha$ we obtain that

$$(\bigwedge_{\alpha\in J_{i,x_1^n}} \alpha)(x_1^n) = \inf\{\alpha(x_1^n)|\ \alpha \in J_{i,x_1^n}\} = \inf\{1|\ \alpha \in J_{i,x_1^n}\} = 1,$$

$$\bigwedge_{\alpha\in J_{i,x_1^n}} \alpha([y_1^{n+k}]) = \inf\{\alpha([y_1^{n+k}])|\ \alpha \in J_{i,x_1^n}\} \geq$$

$$\geq \inf\{\alpha([y_{i+1}^{i+n}]) \mid \alpha \in J_{i,x_1^n}\} = \bigwedge_{\alpha \in J_{i,x_1^n}} \alpha([y_{i+1}^{i+n}]),$$

so $J_i^F(x_1^n)$ is a fuzzy i-ideal of $\mathbf{S}$ and $(J_i^F(x_1^n))(x_1^n) = 1$. Thus, x_1^n if fully included in $J_i^F(x_1^n)$. Moreover, $J_i^F(x_1^n) \subseteq \alpha$, for every $\alpha \in J_{i,x_1^n}$.

Green's equivalences $\mathcal{J}_i$ on $\mathbf{S}$, for each $i \in \{0, 1, \ldots, n\}$, are introduced in [14] and defined by:

$$a_1^n \; \mathcal{J}_i \; b_1^n \quad \Leftrightarrow \quad J_i(a_1^n) = J_i(b_1^n),$$

as well as Green's equivalence $\mathcal{J}$, for all $i \in \{0, 1, \ldots, n\}$, defined by:

$$a_1^n \; \mathcal{J} \; b_1^n \quad \Leftrightarrow \quad J(a_1^n) = J(b_1^n).$$

We will now consider Green's equivalences on fuzzy i-ideals of $\mathbf{S}$. For that purpose, define a relation $\mathcal{J}_i^F$ on S^n for each $i \in \{0, 1, \ldots, n\}$, by

$$a_1^n \; \mathcal{J}_i^F \; b_1^n \quad \Leftrightarrow \quad J_i^F(a_1^n) = J_i^F(b_1^n),$$

as well as a relation $\mathcal{J}^F$, by:

$$a_1^n \; \mathcal{J}^F \; b_1^n \quad \Leftrightarrow \quad J^F(a_1^n) = J^F(b_1^n).$$

It is clear that these relations are equivalences on $\mathbf{S}$. We will show that the Green's relation $\mathcal{J}_i^F$ on $\mathbf{S}$ coincides with the relation $\mathcal{J}_i$ on $\mathbf{S}$.

Theorem 3. $\mathcal{J}_i^F = \mathcal{J}_i$.

Proof. $a_1^n \; \mathcal{J}_i^F \; b_1^n \;\Leftrightarrow\; \mathcal{J}_i^F(a_1^n) = \mathcal{J}_i^F(b_1^n) \;\Leftrightarrow\; \mathcal{J}_i(f_{a_1^n}) = \mathcal{J}_i(f_{b_1^n}) \;\Leftrightarrow$
$\Leftrightarrow\; f_{J_i(a_1^n)} = f_{J_i(b_1^n)} \;\Leftrightarrow\; J_i(a_1^n) = J_i(b_1^n) \;\Leftrightarrow\; a_1^n \; \mathcal{J}_i \; b_1^n.$ □

Note that in a similar way one can show that $\mathcal{J}^F = \mathcal{J}$.

Acknowledgment

The third author is partially supported by Ss. Cyril and Methodius University's project "Mathematical models and applications" (Grant: NIP.UKIM.20-21.6).

References

[1] Ǵ. Čupona, *Vector valued semigroups*, Semigroup Forum, 26 (1983), 65–74.
[2] Ǵ. Čupona, N. Celakoski, S. Markovski, D. Dimovski, *Vector valued groupoids, semigroups and groups* in *Vector Valued Semigroups and Groups*, Macedonian Academy of Sciences and Arts, Skopje (1988), 1–78.
[3] N. Kehayopulu, X-Y. Xie, M. Tsingelis, *A characterization of prime and semiprime ideals of groupoids in terms of fuzzy subsets*, Soochow Journal of Mathematics, Vol. 27, No. 2 (2001), 139–144.
[4] N. Kehayopulu, M. Tsingelis, *Green's relations in ordered groupoids in terms of fuzzy subsets*, Soochow Journal of Mathematics, Vol. 33, No. 3 (2007), 383–397.
[5] N. Kehayopulu, M. Tsingelis, *Fuzzy Sets in Ordered Groupoids*, Semigroup Forum Vol. 65 (2002), 128–132.
[6] N. Kehayopulu, M. Tsingelis, *A note on fuzzy sets in semigroups*, Scientiae Mathematicae Vol. 2, no. 3 (1999), 411–413.

[7] N. Kuroki, *On fuzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets and Systems 5 (1981), 203–215.
[8] N. Kuroki, *On fuzzy semiprime ideals in semigroups*, Fuzzy Sets and Systems 8 (1982), 71–79.
[9] N. Kuroki, *On fuzzy semigroups*, Inform. Sci. 53 (1991), 203–236.
[10] R. G. McLean, H. Kummer, *Fuzzy ideals in semigroups*, Fuzzy Sets and Systems 48 (1992), 137–140.
[11] V. Miovska, D, Dimovski, *$(m+k, m)$-bands*, Mathematica Macedonica 4 (2006), 11–24.
[12] A. Rozenfeld, *Fuzzy Groups*, Journal of Mathematical Analysis and Applications, 35 (1971), 512–517.
[13] B. Trpenovski, Ǵ. Čupona, *$[m, n]$-groupoids*, Bull. Soc. Math. Phys. Macédoine, 21 (1970), 19–29 (in Macedonian).
[14] Y. Zhu, *On (n, m)-semigroups*, Semigroup Forum 84 (2012), 342–364.
[15] L. A. Zadeh, *Fuzzy sets*, Inform. Control 8 (1965), 338–353.

Ss. Cyril and Methodius University, Skopje
Faculty of Natural Sciences and Mathematics
Arhimedova 3, 1000 Skopje, North Macedonia
Email address: miovska@pmf.ukim.mk
Email address: vesnacj@pmf.ukim.mk, celakoska@gmail.com

International Balkan University, Skopje
Faculty of Engineering
Makedonsko-Kosovska Brigada bb 1000 Skopje, North Macedonia
Email address: d.leshkovski@ibu.edu.mk, dleskovski@gmail.com

Received 15.11.2022
Revised 13.12.2022
Accepted 14.12.2022