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FANTASTIC FILTERS IN QUASI-ORDERED RESIDUATED SYSTEMS

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Abstract. The notion of quasi-ordered residuated systems was introduced by Bonzio and Chajda in 2018 as a generalization of both commutative residual lattices and hoop-algebras. With respect to the residuum, this system is a BE-algebra. After that, the concepts of ideals and filters in such systems as well as some types of filters in them were introduced by this author a little differently than it was done in the mentioned structures. In this paper, the concept of fantastic filters in a quasi-ordered residuated system is introduced. In addition, the relationships between this newly designed filter in the quasi-ordered residated system and other types of filters in it are analyzed.

1. Introduction

Logical algebras are the algebraic counterparts of the nonclassical logic and the algebraic foundation of reasoning mechanism in information sciences, computer sciences, theory of control, artificial intelligence, and other important fields. For example, BCK-algebra, BL-algebras, pseudo MTL-algebras, and noncommutative residuated lattice are algebraic counterparts of BCK Logic, Basic Logic, monoidal t-norm-based logic, and monoidal logic, respectively.

Filter theory plays a vital role not only in studying of algebraic structure, but also in nonclassical logic and computer science (see, for example [11, 26]).

The concept of residuated relational systems ordered under a quasi-order relation, or quasi-ordered residuated systems (briefly, QRS), was introduced in 2018 by S. Bonzio and I. Chajda [2] as a generalization of both commutative residual lattices and hoop-algebras. Previously, this idea was discussed in [6, 1]. The author introduced and developed the concepts of filters [13] and ideals [20] in this algebraic structure as well as several types of filters such as implicative, associated, comparative, weak implicative and normal filters ([16, 14, 18, 22, 23]) a little

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differently than it was done in the mentioned structures. In [22], it is shown that every comparative filter of a quasi-ordered residuated system $\mathfrak{A}$ is an implicative filter of $\mathfrak{A}$ and the reverse it need not be valid. The concept of a strong quasi-ordered residuated system was introduced and discussed in [17]. In such systems, comparative and implicative filters coincide. The specificity of strong QRSs is that they allow us to determine the least upper bound for each of their two elements. In this specific environment, in strong QRSs, the concepts of prime and irreducible filters as well as their interrelationships are analyzed ([19, 22, 21]).

In this paper, the notion of fantastic filters in a quasi-ordered residuated system is introduced, and then relations among some types of filters in this algebraic system previously introduced and fantastic filter are analyzed.

2. Preliminaries

In article [2], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

Definition 2.1 ([2], Definition 2.1). A *residuated relational system* is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and R is a binary relation on A and satisfying the following properties:

(1) $(A, \cdot, 1)$ is a commutative monoid;
(2) $(\forall x \in A)((x, 1) \in R)$;
(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.

We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

The basic properties for residuated relational systems are subsumed in the following:

Theorem 1 ([2], Proposition 2.1). *Let $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ be a residuated relational system. Then*

(4) $(\forall x, y \in A)(x \rightarrow y = 1 \Longrightarrow (x, y) \in R)$;
(5) $(\forall x \in A)((x, 1 \rightarrow 1) \in R)$;
(6) $(\forall x \in A)((1, x \rightarrow 1) \in R)$;
(7) $(\forall x, y, z \in A)(x \rightarrow y = 1 \Longrightarrow (z \cdot x, y) \in R)$;
(8) $(\forall x, y \in A)((x, y \rightarrow 1) \in R)$.

2.1. Concept of quasi-ordered residuated systems.

Recall that a *quasi-order relation* $'\preccurlyeq'$ on a set A is a binary relation which is reflexive and transitive.

Definition 2.2 ([2]). A *quasi-ordered residuated system* is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, where $\preccurlyeq$ is a quasi-order relation in the monoid $(A, \cdot)$.

Example 2.1. Let $A = \{1, a, b, c, d\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

and

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$\preccurlyeq :=$

$\{(1,1),(a,1),(b,1),(c,1),(d,1),(b,b),(a,a),(c,c),(d,d),(c,a),(d,a),(d,b),(d,c)\}$.

Example 2.2. For a commutative monoid A, let $\mathfrak{P}(A)$ denote the powerset of A ordered by set inclusion and '$\cdot$' the usual multiplication of subsets of A. Then $\langle \mathfrak{P}(A), \cdot, \to, A, \subseteq\rangle$ is a quasi-ordered residuated system in which the residuum are given by

$$(\forall X, Y \in \mathfrak{P}(A))(Y \to X := \{z \in A : Yz \subseteq X\}).$$

Example 2.3. Let $\mathbb{R}$ be the field of real numbers. Define a binary operations '$\cdot$' and '$\to$' on $A = [0,1] \subset \mathbb{R}$ by

$$(\forall x, y \in [0,1])(x \cdot y := \max\{0, x+y-1\}) \text{ and } x \to y := \min\{1, 1-x+y\}).$$

Then, A is a commutative monoid with the identity 1 and $\langle A, \cdot, \to, <, 1\rangle$ is a quasi-ordered residuated system.

Example 2.4. Let $A = \langle -\infty, 1] \subset \mathbb{R}$ (the real numbers field). If we define '$\cdot$' and '$\to$' as follows, $(\forall u, v \in A)(u \cdot v := \min\{u, v\})$ and $u \to v := 1$ if $u \leqslant v$ and $u \to v := v$ if $v < u$ for all $u, v \in A$, then $\mathfrak{A} := \langle A, \cdot, \to, 1, <\rangle$ is a quasi-ordered residuated system.

Example 2.5. Any commutative residuated lattice $\langle A, \cdot, \to, 0, 1, \sqcap, \sqcup, R\rangle$ where R is a lattice quasi-order is a quasi-ordered residuated system.

Example 2.6. Let $A = \{1, a, b, c, d, e\}$ and operations '$\cdot$' and '$\to$' defined on A be as follows:

$\cdot$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	b	a	c	d	e
b	b	b	a	c	d	e
c	c	b	c	c	d	e
d	d	d	d	d	e	e
e	e	e	e	e	e	d

and

$\to$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	c	1	1	1
b	1	c	1	1	1	1
c	1	c	c	1	1	1
d	1	d	d	d	1	e
e	1	e	e	e	d	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated system, where the relation '$\preccurlyeq$' is defined as follows

$\preccurlyeq := \{(1,1),\ (a,1)$,(b,1),(c,1),(d,1),(e,1),(a,a),(a,c),(a,d), $(a,e),\ (b,b),\ (b,c),\ (b,d),\ (b,e),\ (c,c),\ (c,d),\ (c,e),\ (d,d),\ (e,e)\}$.

It should be noted that the elements a and b are not comparable.

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 2.1 ([2], Proposition 3.1)**.** *Let A be a quasi-ordered residuated system. Then*
(9) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y))$;
(10) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (y \to z \preccurlyeq x \to z \wedge z \to x \preccurlyeq z \to y))$;
(11) $(\forall x, y \in A)(x \cdot y \preccurlyeq x \wedge x \cdot y \preccurlyeq y)$.

It is common knowledge that a quasi-order relation $\preccurlyeq$ generates an equivalence relation $\equiv_{\preccurlyeq} =: \preccurlyeq \cap \preccurlyeq^{-1}$. Due to (9) and (10), this relation is a congruence on $\mathfrak{A}$.

2.2. **Concept of filters.** In this subsection we give some notions that will be used in this article.

Definition 2.3. ([13], Definition 3.1) For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a *filter* of $\mathfrak{A}$ if it satisfies the conditions
(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$, and
(F3) $(\forall u, v \in A)((u \in F \wedge u \to v \in F) \implies v \in F)$.

It is shown ([13], Proposition 3.4 and Proposition 3.2), that if a non-empty subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the conditions:
(F0) $1 \in F$ and
(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$.

If $\mathfrak{F}(A)$ is the family of all filters in a QRS $\mathfrak{A}$, then $\mathfrak{F}(A)$ is a complete lattice ([13], Theorem 3.1).

Remark 2.1: In implicative algebras, the term 'implicative filter' is used instead of the term 'filter' we use (see, for example [3, 12]) because in the structure we study the concept of filter is determined more complexly than requirements (F3). It is obvious that our filter concept is also a filter in the sense of [3, 4, 12]. The term 'special implicative filter' is also used in the aforementioned sources if the implicative filter in the sense of [12] satisfies some additional condition.

Example 2.7. Let $A = \langle -\infty, 1] \subset \mathbb{R}$ (the real numbers field). If we define '$\cdot$' and '$\to$' as follows, $(\forall y, v \in A)(u \cdot v := min\{u, v\})$ and $u \to v := 1$ if $u \leqslant v$ and $u \to v := v$ if $v < u$ for all $u, v \in A$, then $\mathfrak{A} := \langle A, \cdot, \to, 1, \leqslant \rangle$ is a quasi-ordered residuated system. All filters in $\mathfrak{A}$ are in the form of $\langle x, 1]$, for $x \in \langle -\infty, 1]$.

Terms covering some of the requirements used herein to identify various types of filters in the observed algebraic structure are mostly taken from papers on UP-algebras. In some other algebraic systems, different terms are used to cover the concepts of implicative and comparative filters mentioned herein.

Definition 2.4. ([22], Definition 5) For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a *comparative filter* of $\mathfrak{A}$ if (F2) and the following condition
(CF) $(\forall u, v, z \in A)((u \to ((v \to z) \to v) \in F \wedge u \in F) \implies v \in F)$
are valid.

In the literature, this previous concept often appears under the name 'positive implicative filter'.

Example 2.8. Let $\mathfrak{A}$ be a quasi-ordered residuated system as in Example 3.2. Then the set $F := \{1, a, b\}$ is a comparative filter in $\mathfrak{A}$.

Notions and notations that are used but not previously determined in this paper can be found in ([1, 2, 13, 14, 15, 22]).

3. Fantastic filters

The concept of fantastic filters in a lattice implication algebra was introduced and analyzed in the paper [7] by Y. B. Jun. The notion of fantastic filters has become the subject of research by several authors (for example, [8, 5, 9, 10, 24, 25]) in a number of different circumstances. M. Haveshki et al. introduced the notion of fantastic filters in BL-algebras (see, [5], Definition 4.1). M. Kondo and W. A. Dudek also discussed the properties of fantastic filters in BL-algebras ([9]). M. Kondo analyzed this type of filter in residuated lattices ([10]). M. S. Rao analyzed the fuzzification of this type of filters in BE-algebras ([24], Definition 3.1). A. Soleimani Nasab and A. Borumand Saeid in [25] analyze fantastic filter in Hilbert algebras.

Of course, the condition that a subset F of one of the previously mentioned algebraic systems must meet in order for it to be a fantasric filter is always the same, but the logical circumstances in which this requirement appears are different. Here, we introduce the concept of fantastic filters in quasi-ordered residuated systems.

Definition 3.1. A nonempty subset F of a QRS $\mathfrak{A}$ is called a *fantastic filter* in $\mathfrak{A}$ if besides condition (F2) it satisfies the following condition:

(FF) $(\forall u, v, z \in A)((z \to (v \to u) \in F \wedge z \in F) \Longrightarrow ((u \to v) \to v) \to u \in F)$.

In what follows, we need the following lemma:

Lemma 1. *Let F be a nonempty subset of a quasi-ordered residuated system $\mathfrak{A}$ that satisfies the condition (F2). Then*

(12) $(\forall u \in A)(u \in F \iff 1 \to u \in F)$.

Proof. Let $u \in A$ be an element such that $u \in F$. From $u \cdot 1 = u \preccurlyeq u$ follows $u \preccurlyeq 1 \to u$ according to (3). Then, $1 \to u \in F$ due to (F2). Conversely, let $1 \to u \in F$ be holds. From $1 \to u \preccurlyeq 1 \to u$, we get $1 \to u \preccurlyeq u$ in accordance with (3). From here and from $1 \to u \in F$ it follows $u \in F$ according to (F2). ☐

Theorem 2. *Every fantastic filter F in a quasi-ordered resduated system $\mathfrak{A}$ is a filter in $\mathfrak{A}$.*

Proof. Let F be a fantastic filter in a quasi-ordered residuated system $\mathfrak{A}$. This means that F satisfies the conditions (F2) and (FF). Let us prove that it satisfies the condition (F3). Let $u, v \in A$ be such that $u \in F$ and $u \to v \in F$. On the other hand, from $u \to v \preccurlyeq u \to v$, we have $(u \to v) \cdot u \preccurlyeq v$ and $(u \to v) \cdot u \preccurlyeq v \preccurlyeq 1 \to v$.

From here it follows $u \to v \preccurlyeq u \to (1 \to v)$ according to (3). Now, $u \to (1 \to v) \in F$ by (F2). From here and from $u \in F$ follows $((v \to 1) \to 1) \to v \in F$ according to (FF). Thus $v \in F$ due (12) since $(v \to 1) \to 1 \equiv_{\preccurlyeq} 1$. □

The reverse of the previous theorem does not have to be valid, as the following example shows:

Example 3.1. Let $A = \{1, a, b\}$ and operations '$\cdot$' and '$\to$' defined on A be as follows:

$\cdot$	1	a	b
1	1	a	b
a	a	a	a
b	b	a	b

and

$\to$	1	a	b
1	1	a	b
a	1	1	1
b	1	a	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated system where the relation '$\preccurlyeq$' is defined as follows $\preccurlyeq := \{(1,1), (a,1), (b,1), (b,b), (a,b)\}$. Subsets $F =: \{1\}$ and $G =: \{1, b\}$ are filters in $\mathfrak{A}$ but F is not a fantastic filter in $\mathfrak{A}$ since $1 \to (a \to b) = 1 \to 1 = 1 \in F$ and $1 \in F$ but $((b \to a) \to a) \to b = b \notin F$. Filter G is a fantastic filter in $\mathfrak{A}$.

Theorem 3. *A filter F in a quasi-ordered residuated system $\mathfrak{A}$ is a fantastic filter in $\mathfrak{A}$ if and only if it satisfies:*
(FF2) $(\forall u, v \in a)(v \to u \in F \implies ((u \to v) \to v) \to u \in F)$.

Proof. Assume that F is a fantastic filter of a quasi-ordered residuated system $\mathfrak{A}$. Let $u, v \in A$ be such that $v \to u \in F$. Then $1 \to (v \to u) \in F$ by (12). Since $1 \in F$ holds, it follows that $((u \to v) \to v) \to u \in F$ according to (FF). So the implication $(F2) \wedge (FF) \implies (FF2)$ is a valid formula.

Conversely, let F be a filter of a quasi-ordered resodiated system $\mathfrak{A}$ satisfying the condition (FF2). Let us prove (FF). Let us take arbitrary elements $u, v, z \in A$ such that $z \to (v \to u) \in F$ and $z \in F$. Then $v \to u \in F$ by (F2). From here, according to (FF2), we get $((u \to v) \to v) \to u \in F$. Thus, the implication $(F2) \wedge (FF2) \implies (FF)$ is proved. □

The previous theorem enables obtaining a sufficient condition for a filter in a quasi-ordered residuated system A to be a fantastic filter in A.

Corollary 3.1. *If a filter F in a quasi-ordered residuated system $\mathfrak{A}$ satisfies the following condition*
(FF3) $(\forall x, y, u \in A)((x \to u \in F \wedge y \to u \in F) \implies ((x \to y) \to y) \to u \in F)$,
then F is a fantastic filter in $\mathfrak{A}$.

Proof. If we put $u = x$ in the formula (FF3), we get that $y \to x \in F$ implies $((x \to y) \to y) \to x \in F$, which means that F is a fantastic filter in $\mathfrak{A}$ because it satisfies the condition (FF2). □

Recall ([15], Theorem 3.2) that a filter F in a quasi-ordered residuated system $\mathfrak{A}$ is a comparative filter in $\mathfrak{A}$ if it satisfies the condition
(F4) $(\forall y, z \in A)((v \to z) \to v \in F \implies v \in F)$.

In what follows, in addition to the previous one, we will use the following valid formulas ([2], Proposition 3.1):

(13) $(\forall x, y, z \in A)(x \to y \preccurlyeq (y \to z) \to (x \to z))$,

(14) $(\forall x, y, z \in A)(y \to z \preccurlyeq (x \to y) \to (x \to z))$ and

(15) $(\forall x, y, z \in A)(x \to (y \to z) \equiv_{\preccurlyeq} y \to (x \to z))$.

Theorem 4. *If F is a comparative filter in a quasi-ordered residuated system $\mathfrak{A}$, then F is a fantastic filter in $\mathfrak{A}$.*

Proof. Let F be a comparative filter in a quasi-ordered residuated system $\mathfrak{A}$. This means that, among other things, it satisfies the conditions (F2) and (F3). We will prove that (FF2) holds. Let $u, v \in A$ such that $v \to u \in F$. We start from the valid formula $u \cdot ((u \to v) \to v) \preccurlyeq u$ in accordance with (11). From here, according to (3), we have $u \preccurlyeq ((u \to v) \to v) \to u$. If we apply (13) to this inequality, we get

(16) $(((u \to v) \to v) \to u) \to v \preccurlyeq u \to v$.

On the other hand, according to (14), we have

$$v \to u \preccurlyeq ((u \to v) \to v) \to ((u \to v) \to u)$$
$$\equiv_{\preccurlyeq} (u \to v) \to (((u \to v) \to v) \to u) \qquad \text{by (15).}$$

Now, applying (10) to inequality (16), acting on the right with $(((u \to v) \to v) \to u)$, we obtain

$$(u \to v) \to (((u \to v) \to v) \to u)$$
$$\preccurlyeq ((((u \to v) \to v) \to u) \to v) \to (((u \to v) \to v) \to u).$$

This inequality with the previous one gives

$$v \to u \preccurlyeq ((((u \to v) \to v) \to u) \to v) \to (((u \to v) \to v) \to u).$$

From here and from $v \to u \in F$ it follows

$$((((u \to v) \to v) \to u) \to v) \to (((u \to v) \to v) \to u) \in F,$$

according to (F2). From here, by (F4), it follows $((u \to v) \to v) \to u \in F$. This shows that F satisfies the condition (FF2). So, F is a fantastic filter in $\mathfrak{A}$. □

Example 3.2. Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	a	a
c	c	a	a	b

and

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	1
c	1	b	c	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1 \rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows $a \preccurlyeq b \preccurlyeq c \preccurlyeq 1$. The subset $F =: \{1\}$ is a fantastic filter in $\mathfrak{A}$ but it is not a comparative filter in $\mathfrak{A}$.

4. Conclusion and final comments

In this paper, we have introduced the notion of fantastic filters in quasi-ordered residuated systems. In addition, an equivalent condition for determining this concept was found. Also, the connections between the fantastic filter, on the one hand, and the filter (comparative filter), on the other hand, were established.

Since, in the general case, the relation R on a quasi-ordered residuated system $\mathfrak{A}$, determined by the filter F in $\mathfrak{A}$ in the following way

$$(\forall x, y \in A)((x, y) \in R \iff (x \to y \in F \wedge y \to x \in F)),$$

need not be a congruence on $\mathfrak{A}$, the factor-set A/R does not have to be a quasi-ordered residuated system. Therefore, the question about the character of this factor-set in the case when F is a fantastic filter in $\mathfrak{A}$, remains unanswered.

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