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HYPER QUASI-ORDERED RESIDUATED SYSTEMS

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Abstract. The concept of quasi-ordered residuated systems as a generalization of both quasi-ordered residuated lattices and hoop-algebras was developed in 2018 by Bonzio and Chajda. In this paper, we apply the hyper structure theory to quasi-ordered residuated systems and we introduce the notion of hyper quasi-ordered residuated system which is a generalization of quasi-ordered residuated systems on the one hand and a generalization of both hyper quasi-ordered residuated lattices and hyper hoop-algebra, on the other hand, and we investigate some of their related properties. In addition to the previous one, the concept of deductive systems and concept of filters in this algebraic structure are presented and analyzed as well as the connections between them.

1. Introduction

Hyper structure theory was introduced in 1934 in [10], when F. Marty at the 8th congress of scandinavian mathematicians, gave the definition of hypergroup and illustrated some applications and showed its utility in the study of groups, algebraic functions, and rational fractions. Till now, the hyper structures have been studied from the theoretical point of view for their applications to many subjects of pure and applied mathematics. Some fields of applications of the mentioned structures are lattices, graphs, coding, ordered sets, median algebra, automata, and cryptography (see, for example [9]). Many researchers have worked on this area. For example, R. A. Borzooei et al. in [4, 5, 6] introduced and studied hyper hoop algebras. In addition to the previous one, the concept of hyper residuated lattices was studied for instance, in [2, 3, 17].

Quasi-ordered residuated system is a commutative residuated integral monoids ordered under a quasi-order, introduced by S. Bonzio and I. Chajda in [1]. In the last few years, the theory of quasi-ordered residuated systems was enriched with more results on ideals and filters in them (for example, see [11, 12, 14, 15]). This algebraic structure is a generalization of both residuated lattices and hoop-algebras.

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In this paper we construct and introduce the notion of hyper quasi-ordered residuated system which is a generalization of quasi-ordered residuated systems, on the one hand, and a generalization of hyper hoop-algebras and hyper residuated lattices, on the other. Then we study some properties of this structure. We also introduce the notion of (strong) deductive system and the notion of (strong) filter on hyper quasi-ordered residuated systems, and give several properties of them. Section 2 enables a potential reader to feel comfortable reading the material presented in Section 3, which is the main part of this work. Section 3 has two parts: In subsection 3.1 the concept of hyper quasi-ordered residuated systems is introduced and analyzed. In subsection 3.2 the concept of deductive systems and the concept of filters in a hyper quasi-ordered residuated system are introduced and analyzed. In this paper, the concepts of strong deductive systems and strong folters in a hyper quasi-ordered residuated system appear, which are not found in either hyper residuated lattices or hyper hoop-algebras. The paper also discusses the minimality (maximality) of these substructures in hyper quasi-ordered residuated systems.

2. Preliminaries

In this section, the necessary notions and notations and some of their interrelationships, mostly taken from papers [1, 11, 12, 16], are listed in the order to enable a reader to comfortably follow the presentation in this report. It should be pointed out here that the notations for logical conjunction, logical implication and others have a literal meaning. Thus, for example, the label $H \vDash Q$ has the meaning that the consequent Q can be demonstrated from the hypothesis H. The notation $=:$ in the formula $A =: B$ serves to indicate that A in it is the abbreviation for the formula B.

2.1. **Concept of quasi-ordered residuated systems.** In article [1], Bonzio and Chajda introduced and analyzed the concept of residual relational systems.

Definition 2.1 ([1], Definition 2.1)**.** *A* residuated relational system *is a structure* $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, R\rangle$, *where* $\langle A, \cdot, \rightarrow, 1\rangle$ *is an algebra of type* $\langle 2, 2, 0\rangle$ *and* R *is a binary relation on* A *and satisfying the following properties:*

(1) $(A, \cdot, 1)$ *is a commutative monoid,*
(2) $(\forall x \in A)((x, 1) \in R)$,
(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.

We will refer to the operation $\cdot$ *as (commutative) multiplication, to* $\rightarrow$ *as its residuum and to condition (3) as residuation.*

Recall that a *quasi-order relation* $'\preccurlyeq'$ on a set A is a binary relation which is reflexive and transitive.

Definition 2.2 ([1])**.** *A* quasi-ordered residuated system *is a residuated relational system* $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, *where* $\preccurlyeq$ *is a quasi-order relation in the monoid* $(A, \cdot)$

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 2.1 ([1], Proposition 3.1). *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then*

(4) *The operation '$\cdot$' preserves the pre-order in both positions;*

$$(\forall x, y, z \in A)(x \preccurlyeq y \implies (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y));$$

(5) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (y \to z \preccurlyeq x \to z \wedge z \to x \preccurlyeq z \to y))$;
(6) $(\forall y, z \in A)(x \cdot (y \to z) \preccurlyeq y \to x \cdot z)$;
(7) $(\forall x, y, z \in A)(x \cdot y \to z \preccurlyeq x \to (y \to z))$;
(8) $(\forall x, y, z \in A)(x \to (y \to z) \preccurlyeq x \cdot y \to z)$;
(9) $(\forall x, y, z \in A)(x \to (y \to z) \preccurlyeq y \to (x \to z))$;
(10) $(\forall x, yz \in A)((x \to y) \cdot (y \to z) \preccurlyeq x \to z)$;
(11) $(\forall x, y \in A)((x \cdot y \preccurlyeq x) \wedge (x \cdot y \preccurlyeq y))$;
(12) $(\forall x, y, z \in A)(x \to y \preccurlyeq (y \to z) \to (x \to z))$;
(13) $(\forall x, y, z \in A)(y \to z \preccurlyeq (x \to y) \to (x \to z))$.

It is generally known that a quasi-order relation $\preccurlyeq$ on a set A generates a equivalence relation $\equiv_{\preccurlyeq} := \preccurlyeq \cap \preccurlyeq^{-1}$ on A. Due to properties (4) and (5), this equality relation is compatible with the operations in A. Thus, $\equiv_{\preccurlyeq}$ is a congruence on A.

In the light of the previous note, it is easy to see that the following applies:
(7) and (8) give:
(14) $(\forall x, y, z \in A)(x \cdot y \to z \equiv_{\preccurlyeq} x \to (y \to z))$.
Due to the universality of formula (9) (or, due to the commutativity of the multiplication from (14)), we have:
(15) $(\forall x, y, z \in A)(x \to (y \to z) \equiv_{\preccurlyeq} y \to (x \to z))$.
Also, from (11) and (2), it follows
(16) $(\forall x \in A)(x \to x \equiv_{\preccurlyeq} 1)$
In the general case,
(17) $(\forall x, y \in A)(x \preccurlyeq y \iff x \to y \equiv_{\preccurlyeq} 1)$
is valid, which is obtained by referring to (11) and (2).

From the previous analysis it can be concluded that a quasi-ordered residuated system is a generalization of a hoop-algebra (in the sense of [7]) because the following formula

$$(\forall x, y \in A)(x \cdot (x \to y) \equiv_{\preccurlyeq} y \cdot (y \to x))$$

does not have to be a valid formula in the observed algebraic structure in the general case. Since the axioms by which the hoop algebra is determined are mutually independent, there must be a model that satisfies the conditions (1), (2) and (3) but it does not satisfy the mentioned condition.

A quasi-ordered residuated system $\mathfrak{A}$ is said to be a strong quasi-ordered residuated system ([13], Definition 6) if additionally the following

(18) $(\forall x, y \in A)((x \to y) \to y \equiv_{\preccurlyeq} (y \to x) \to x)$

is valid. If we recall that a hoop is a Weisberg hoop if condition (18) is added to the axioms that determine the concept of hoops, then we can conclude that a strong quasi-ordered residuated system is a generalization of Weisberg hoops. It is well known that the underlying ordering of a Weisberg hoop is a lattice ordering, and that the join is term-definable by $a \sqcup b =: (a \to b) \to b$. Since any hoop satisfies the equation ([8]) $(a \to b) \to (b \to a) = b \to a$, any Weisberg hoop satisfies the pre-linearity condition $(a \to b) \sqcup (b \to a) = 1$. However, a strong quasi-ordered residuated system, in the general case, does not have to satisfy the pre-linearity condition.

2.2. **Concept of filters.** The concept of filters in a quasi-ordered residuated system was introduced in the article [11]. This concept is somewhat different from the filter concept in both hoop-algebras and residuated lattices.

Definition 2.3 ([11], Definition 3.1)**.** *For a subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a* filter *of $\mathfrak{A}$ if it satisfies conditions*
(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$, *and*
(F3) $(\forall u, v \in A)((u \in F \wedge u \to v \in F) \implies v \in F)$.

Let it note that the empty subset of A satisfies the conditions (F2) and (F3). Therefore, $\emptyset$ is a filter in $\mathfrak{A}$. It is shown ([11], Proposition 3.4 and Proposition 3.2), that if a non-empty subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the following conditions
(F0) $1 \in F$ and
(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$.
Also, it can be seen without difficulty that

$$((F3) \wedge F \neq \emptyset) \implies (F2)$$

is valid. Indeed, if (F3) holds, then the formula $u \in F \wedge u \preccurlyeq v$, can be transformed into the formula $u \in F \wedge u \to v \equiv_{\preccurlyeq} 1 \in F$ by (F0) so from here, according to (F3), the validity of implication (F2) can be demonstrated. However, the reverse does not have to be valid.

If $\mathfrak{F}(A)$ is the family of all filters in a QRS $\mathfrak{A}$, then $\mathfrak{F}(A)$ is a complete lattice ([11], Theorem 3.1).

In the papers [14, 15], prime and irreducible filters in strong quasi-ordered residuated systems and their mutual relations are considered.

The reader can find several examples of this algebraic system in the articles [11, 12, 15, 16].

2.3. **A few words about hyper structures.** Now, we recall some basic notions of the hypergroup theory from [9]: Let H be a non-empty set. A hypergroupoid is a pair $(H, \circ)$, where $\circ : H \times H \longrightarrow \mathcal{P}(H) \setminus \{\emptyset\}$ is a binary hyperoperation on H. If $a \circ (b \circ c) = (a \circ b) \circ c$ holds, for all $a, b, c \in H$ then $(H, \circ)$ is called a semihypergroup, and it is said to be commutative if $\circ$ is commutative. An element $1 \in H$ is called a unit, if $a \in (1 \circ a) \cap (a \circ 1)$, for all $a \in H$ and it is called a scalar unit, if $\{a\} = 1 \circ a = a \circ 1$, for all $a \in H$. Note that if $A, B \subseteq H$, then $A \circ B = \bigcup_{a \in A,\, b \in B} (a \circ b)$.

In addition to the previous one, in what follows the following notations will also be used ([3]):

Let $(H, \preccurlyeq)$ be a quasi-ordered set and A, B be two subsets of H. Then we write
- $A \lll B$ if there exist $a \in A$ and $b \in B$ such that $a \preccurlyeq b$.
- $A \preccurlyeq B$ if for any $a \in A$, there exists $b \in B$ such that $a \preccurlyeq b$.
- We will write $A \preccurlyeq b$ instead of $A \preccurlyeq \{b\}$.

In light of the foregoing determination, we have $x \preccurlyeq y$ if and only if $\{x\} \preccurlyeq \{y\}$. Also, we will write $a \lll B$ instead of $\{a\} \lll B$.

One can easily conclude that the relation $\preccurlyeq$ is a quasi-order on $\mathcal{P}(H)$. Indeed, since reflexivity is obvious, let's show transitivity. Let $A, B, C \subseteq H$ be such that $A \preccurlyeq B$ and $B \preccurlyeq C$. Then for any $a \in A$ there exists an element $b = b(a) \in B$ such that $a \preccurlyeq b(a)$ and for any $b \in B$ there exists an element $c = c(b) \in C$ such that $b \preccurlyeq c(b)$. So, for any $a \in A$ there exists an element $c = c(b(a)) \in C$ such that $a \preccurlyeq c$. This means that $A \preccurlyeq C$. In the general case, this relation is not antisymmetric.

Also, it is easy to see that $A \preccurlyeq B \implies A \lll B$. In addition to the previous one, the following applies

$$(\forall a \in H)(\forall b \subseteq H)(a \lll B \iff a \preccurlyeq B).$$

In the special case, for $B = \{b\}$, we have $(\forall a, b \in H)(a \lll b \iff a \preccurlyeq b)$. Finally, let's point out that the following holds

$$(\forall a \in H)(\forall B \subseteq H)(a \in B \implies (a \preccurlyeq B \wedge a \lll B)).$$

Also $\emptyset \neq A \subseteq B \implies B \lll A$ holds for $A, B \subseteq H$. Indeed, $A \subseteq B$ means that $(\forall a \in H)(a \in A \implies a \in B)$ holds. Therefore, one can find $b \in B$ such that $b \in A$. Since $b \preccurlyeq b$, we have $B \lll A$.

3. The main results

Section 3 is the main part of this paper. In the subsection 3.1 we introduce the concept of hyper quasi-ordered residuated systems (Definition 3.1) and we prove some of their fundamental properties (Theorem 1 and Theorem 2). Subsection 3.2 is devoted to the concept of filters in this newly determined class of algebraic structures. The concept of filters in quasi-ordered residuated systems, which we introduce in Subsection 3.2, is somewhat different from the concept of filters in structures with which we associate this newly introduced algebraic structure (hyper residuated lattices and hyper hoop-algebras). In this case, in the determination of the (strong) filter F in the hyper quasi-ordered residuated system $\mathfrak{hA}$, we will omit the requirement

$$(\forall x, y \in A)((x \in F \wedge y \in F) \implies x \circ y \in F).$$

What precedes that determination is the analysis of the conditions imposed on the subset F of a hyper quasi-ordered residuated system $\mathfrak{hA}$ in order for it to be a (strong) filter in $\mathfrak{hA}$ as well as their interconnections. The mentioned analysis is presented through several lemmas and propositions that precede the definition of (strong) filters.

3.1. **Concept of hyper QRSs.** The concept of hyper quasi-ordered redissued systems is introduced by the following definition.

Definition 3.1. *A hyper quasi-ordered residuated system*

$$\mathfrak{h}\mathfrak{A} =: (A, \circ, 1, \rightarrow, \preccurlyeq),$$

quasi hyper QRS (by briefly), is a non-empty quasi-ordered set $(A, \preccurlyeq)$ *endowed with two binary hyper operations* $\circ$ *and* $\rightarrow$ *and the element* 1 *such that satisfying the following conditions:*

(H1) $(A, \circ, 1)$ *is a commutative semihypergroup with* 1 *as the unit.*
(H0) $(\forall x \in A)(x \in 1 \circ x)$.
(H2) $(\forall x \in A)(x \preccurlyeq 1)$.
(H3) $(\forall x, y, z \in A)(x \circ y \ll z \iff x \ll y \rightarrow z)$.

We will denote this system of axioms by $\mathfrak{H}$. *With* $[\mathfrak{H}]$ *we will denote everything that has been demonstrated using these axioms (the so-called theory developed over these axioms) up to the place of using this notation.*

Example 3.1. *Any hyper residuated lattice (determined as in the article* [3]*, for example) is a hyper quasi-ordered residuated system.*

Example 3.2. *Any hyper hoop-algebra (determined as in the article* [4]*) is a hyper quasi-ordered system.*

Example 3.3. *Let* $A = \langle -\infty, 1] \, (\subseteq \mathbb{R})$. *Then* $(A, \leqslant)$ *with the natural ordering is a partially ordered set. Define the hyperoperations* $\circ$ *and* $\rightarrow$ *on* A *as follows:* $a \circ b =: min\{a, b\}$ *and* $a \rightarrow b =: \{1\}$ *if* $a \leqslant b$ *and* $a \rightarrow b =: [b, 1]$ *if* $b < a$. *It is not difficult to check that* $(A, \circ, 1, \rightarrow, \leqslant)$ *is a hyper (quasi-)ordered residuated system.*

Example 3.4. *Let* $A = \langle \infty, 1] \, (\subseteq \mathbb{R})$. *Define the hyper operations* $\circ$ *and* $\rightarrow$ *on* A *as follows:*

$(\forall x, y \in A)(x \circ y = \{1, x, y\})$ *and*
$(\forall x, y \in A)((x \leqslant y \implies x \rightarrow y = \{1, y\}) \wedge (y < x \implies x \rightarrow y = \{y\}))$.

Then $(A, \circ, 1, \rightarrow, \leqslant)$ *is a (quasi-)orderd residuated system.*

Example 3.5. *Let* $B =: \{x_i \, : \, i \in \mathbb{N}\}$ *and* $A = B \cup \{1\}$ *with*

$$(\forall i \in \mathbb{N})(x_i \neq 1), \ (\forall i \in \mathbb{N})((x_i \preccurlyeq 1) \wedge (x_i \preccurlyeq x_i)) \text{ and } 1 \preccurlyeq 1.$$

Define binary hyperoperations $\circ$ *and* $\rightarrow$ *on* A *as follows:*

$$(\forall a, b \in A)(a \circ b =: \{x \in A \, : \, a \preccurlyeq x \wedge b \preccurlyeq x\})$$

and

$a \preccurlyeq b \implies a \rightarrow b =: \{1\}$,
$(a = 1 \wedge b \in B) \implies a \rightarrow b =: B$,
$(a = x_i \wedge b = x_j \neq a) \implies a \rightarrow b =: \{x_k : k \in \mathbb{N} \wedge k \leqslant max\{i, j\}\} \cup \{1\}$

for all $a, b \in A$. *With a little more effort, it can be verified that* $(A, \circ, 1, \preccurlyeq, \rightarrow)$ *is a hyper (quasi-)ordered residuated system.*

Example 3.6. *Let* $A =: \{a, b, c, 1\}$ *be a chain such that* $a < b < c < 1$. *Let us defile the hyper operations as follows*

$\to$	1	a	b	c
1	{1}	{a,1}	{b}	{c,1}
a	{1}	{1}	{1}	{1}
b	{1}	{b,c,1}	{b,1}	{1}
c	{1}	{b,1}	{b}	{c,1}

and

$\circ$	1	a	b	c
1	{a,b,c,1}	{a}	{a,b}	{a,b,c}
a	{a}	{a}	{a}	{a}
b	{a,b}	{a}	{a,b}	{a,b}
c	{a,b,c}	{a}	{a,b}	{a,b,c}

Routine calculations show that $(A, \circ, 1, \to, \leqslant)$ *is a hyper (quasi-)ordered residuated system.*

The following two theorems list some of the important fundamental features of this hyper system.

Theorem 1. *In any hyper quasi-ordered residuated system* $(A, \circ, 1, \to, \preccurlyeq)$*, the following holds:*

(a) $(\forall B \subseteq A)(1 \ll B \implies 1 \in B)$.
(b) $(\forall x, y \in A)(x \preccurlyeq y \implies 1 \in x \to y)$.
(c) $(\forall x \in A)(1 \in x \to x)$.
(d) $(\forall x \in A)(1 \in x \to 1)$.
(e) $(\forall B.C, D \subseteq A)(B \ll C \to D \iff B \circ C \ll D)$.
(f) $(\forall x, y \in A)(x \circ y \ll x \wedge x \circ y \ll y)$.
(g) $(\forall B, C \subseteq A)(B \circ C \ll B \wedge B \circ C \ll C)$.
(h) $(\forall x, y \in A)(x \ll y \to x)$.
(i) $(\forall x, y \in A)(1 \in x \to (y \to x))$.
(j) $(\forall B, C \subseteq A)(B \ll C \to B)$.
(k) $(\forall x, y \in A)(x \circ (x \to y) \ll x \ \wedge \ x \circ (x \to y) \ll y)$.
(l) $(\forall x, y, z \in A)(x \to (y \to z) \preccurlyeq (x \circ y) \to z \preccurlyeq x \to (y \to z) \preccurlyeq y \to (x \to z))$.
(m) $(\forall x, y \in A)(x \ll y \to (x \circ y))$.

Proof. (a) If $B \subseteq A$ is such that $1 \ll B$, then there exist an element $b \in B$ such that $1 \preccurlyeq b$. Hence $1 \equiv_{\preccurlyeq} b \in B$ by (H2).

(b) Assume that $x \preccurlyeq y$. From $x \in x \circ 1$ and $x \preccurlyeq y$ it follows that $x \circ 1 \ll y$. Then $1 \ll x \to y$ by (H3). Thus, $1 \preccurlyeq x \to y$ by (a).

(c) The statement (c) follows immediately from the statement (b) by taking $y = x$ and recognizing that $\preccurlyeq$ is a reflexive relation.

(d) The statement (d) follows immediately from the statement (b) and (H2) by taking $y = 1$.

(e) Suppose $B \ll C \to D$. This means that there are elements $b \in B$, $c \in C$, $d \in D$ such that $b \ll c \to d$. Then $b \circ c \ll d$ by (H3). This means $B \circ C \ll D$. Demonstration of the inverse inference is obviously acceptable.

(f) Since $y \preccurlyeq 1$ and $x \preccurlyeq 1$ by (H2), we have $y \preccurlyeq 1 \in x \to x$ and $x \preccurlyeq 1 \in y \to y$ according to (c). This means that $y \ll x \to x$ and $x \ll y \to y$ are valid. From here, we get that $x \circ y \ll x$ and $x \circ y \ll y$ hold according to (H3).

(g) This statement follows from the previous statement.

(h) For arbitrary elements $x, y \in A$, according to (f), $x \circ y \ll x$ holds. From here, $x \ll y \to x$ follows by (H3).

(i) The statement (i) follows immediately from (h) with respect to (b).

(j) The statement (j) is a direct consequence of the statement (g) with respect to formula (e).

(k) The first part of statement (k) follows from the first part of statement (f), if we put $x \to y$ instead of the variable y. From the validity of the formula $x \to y \ll x \to y$ follows the validity of the formula $x \circ (x - y) \ll y$ according to (H3).

(l) Let $x, y, z, u \in A$ be elements such that $u \in x \to (y \to z)$. This means $u \ll x \to (y \to z)$. Then $(u \circ x) \ll y \to z$ by (H3) and, again $(u \circ x) \circ y \ll z$ according to (H3). From here it follows $u \circ (x \circ y) \ll z$ due to the associativity of the operation $\circ$. Thus $u \ll (x \circ y) \to z$ by (H3). Thus means $x \to (y \to z) \preccurlyeq (x \circ y) \to z$. By a similar way, we can prove that $(x \circ y) \to z \preccurlyeq x \to (y \to z)$. From here

$$x \to (y \to z) \preccurlyeq y \to (x \to z)$$

follows due to the commutativity of the operation $\circ$.

(m) It is holds $x \circ y \ll x \circ y$ by reflexivity of the relation $\ll$. Then $x \ll y \to (x \circ y)$ by (H3). □

Theorem 2. *In any hyper quasi-ordered residuated system* $(A, \circ, 1, \to, \preccurlyeq)$*, the following holds:*

(n) $(\forall x, y, z \in A)(x \preccurlyeq y \implies x \circ z \ll y \circ z)$.

(p) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (z \to x \preccurlyeq z \to y \ \wedge \ y \to z \preccurlyeq x \to z))$.

(q) $(\forall x, y, z \in A)(x \to y \preccurlyeq (y \to z) \to (x \to z))$.

(r) $(\forall x, y, z \in A)((x \to y) \circ (y \to z) \ll x \to z)$.

(s) $(\forall x, y, z \in A)(y \to z \ll (x \to y) \to (x \to z))$.

Proof. (n) Let $x, y, z \in A$ be such that $x \preccurlyeq y$. On the other hand, we have $y \ll z \to y \circ z$ according to (m). Hence $x \ll z \to y \circ z$. So $x \circ z \ll y \circ z$ by (H3).

(p) Let $x, y, z, u \in A$ be such that $x \preccurlyeq y$.

Suppose that $u \in z \to x$. From here we have $u \ll z \to x$. Then $u \circ z \ll x$. Thus $u \circ z \ll y$. Hence $u \ll z \to y$. Finally, we have $z \to x \preccurlyeq z \to y$.

Let $t \in y \to z$. Then $t \ll y \to z$. Thus $y \circ t = t \circ y \ll z$ by (H3). Hence $y \ll t \to z$ and $x \ll t \to z$ with respect to $x \preccurlyeq y$. From here, it follows $x \circ t \ll z$ by (H3). Therefore, we have $t \ll x \to z$. Finally, $y \to z \preccurlyeq x \to z$.

(q) Let $x, y, z, u \in A$ be such that $u \in y \to z$. Then $u \ll y \to z$. Thus $u \circ y \ll z$ and $y \ll u \to z$ by (H3). So, there exists an element $t \in u \to z$ such that $y \preccurlyeq t$. Now, $x \to y \preccurlyeq x \to t$ by (p). Further on, we have

$$x \to y \preccurlyeq x \to t \subseteq x \to (u \to z) \preccurlyeq u \to (x \to z) \subseteq (y \to z) \to (x \to z).$$

(r) Statement (r) is obtained directly from statement (q) by reference to statement (e).

(s) Statement (s) is obtained directly from statement (r) by reference to statement (e). □

3.2. **Concept of filters in hyper QRS.** As usual, the concept of filters in hyper residuated lattices ([3], Definition 3.1) and in hyper hoop-algebras ([4], Definition 4.2) is determined as a multiplicative subsemihypergroup of those structures. In this paper, we will not act in that way. In this subsection, we will consider the following conditions imposed on the subset F of the hyper quasi-ordered residuated system $(A.\circ, 1, \to, \preccurlyeq)$:

(HF0) $1 \in F$.
(HF1) $(\forall x, y \in A)(x \circ y \subseteq F \implies (x \in F \wedge y \in F))$.
(HF2) $(\forall x, y \in A)((x \preccurlyeq y \wedge x \in F) \implies y \in F)$.
(HF3) $(\forall x, y \in A)((x \in F \wedge x \rightarrow y \subseteq F) \implies y \in F)$.
(sHF3) $(\forall x, y \in A)((x \in F \wedge F \ll x \rightarrow y) \implies y \in F)$.
(dHF3) $(\forall x, y \in A)(((x \rightarrow y) \cap F \neq \emptyset \wedge x \in F) \implies y \in F)$.
(SH) $(\forall x, y \in A)((x \in F \wedge y \in F) \implies x \circ y \subseteq F)$.
(wSH) $(\forall x, y \in A)((x \in F \wedge y \in F) \implies F \ll x \circ y)$.

In the next few lemmas, some of their mutual relations will be exposed.

Lemma 1. $F \neq \emptyset \vDash (HF2) \implies (HF0)$.

Proof. Let F be a non-empty subset of a hyper quasi-ordered residuated system $\mathfrak{hA}$. Since $F \neq \emptyset$, then there exists an element $x \in F$. According to (H2), $x \preccurlyeq 1$ holds. From here, we get $1 \in F$ by (HF2). □

Lemma 2. $[\mathfrak{H}] \vDash (HF2) \implies (HF1)$.

Proof. Let F be a subset of a hyper quasi-ordered residuated system $\mathfrak{hA}$. Assume that F satisfies condition (HF2). Let $x, y \in A$ be such that $x \circ y \subseteq F$. This means $(\forall u \in A)(u \in x \circ y \implies u \in F)$. On the other hand, by (f), we have $x \circ y \ll x$ and $x \circ y \ll y$. Therefore $v \preccurlyeq x$ for some $v \in x \circ y$ and $t \preccurlyeq y$ for some $t \in x \circ y$. From here it follows that $x \in F$ and $y \in F$ according to (HF2) considering that $v \in F$ and $t \in F$ hold. □

The following lemma relates the conditions (sHF3) to (HF2):

Lemma 3. $[\mathfrak{H}]$, $(HF0) \vDash (sHF3) \implies (HF2)$.

Proof. Let F be a nonempty subset of a hyper quasi-ordered residuated system $\mathfrak{hA}$ satisfy the condition (sHF3). Let $x, y \in A$ be such that $x \in F$ and $x \preccurlyeq y$. Then, by (b), we have $1 \in x \rightarrow y$. Thus $F \ll x \rightarrow y$ since $1 \in F$ and $1 \preccurlyeq 1$. Now, from (sHF3) it follows that $y \in F$. Thus, (HF2) holds. □

Lemma 4. $[\mathfrak{H}]$, $(HF0) \vDash (dHF3) \implies (sHF3)$.

Proof. Let F be a non-empty subset of A satisfying the the conditions (HF0) and (dHF3). Now, let $x \in F$ and $F \ll (x \rightarrow y)$. Then there exist $z \in F$ and $u \in x \rightarrow y$ such that $z \preccurlyeq u$. So, $u \in F$ by (HF2). Hence $F \cap (x \rightarrow y) \neq \emptyset$. Thus $y \in F$ by (dHF3). This means that F satisfies the condition (sHF3). □

Corollary 2.1. $[\mathfrak{H}]$, $(HF0) \vDash (dHF3) \implies (HF2)$.

Proof. We have $(dHF3) \implies (sHF3)$ by Lemma 4 and $(sHF3) \implies (HF2)$ according to Lemma 3. So, $(dHF3) \implies (HF2)$. □

Lemma 5. $[\mathfrak{H}]$, $(HF0) \vDash (sHF3) \implies (wSH)$.

Proof. Let F be a nonempty subset of a hyper quasi-ordered system $\mathfrak{hA}$ that satisfies the condition (sHF3). Then the condition (HF2) holds by Lemma 3. Let $x, y \in A$ be such that $x \in F$ and $y \in F$. On the one hand, according to (m), we have $y \ll x \to x \circ y$. This means that there exists an element $u \in x \to x \circ y$ such that $y \preccurlyeq u$. Since $y \in F$, it follows from this and $y \preccurlyeq u$ that $u \in F$. On the other hand, $u \in x \to x \circ y$ means that there is some $v \in x \circ y$ such that $u \in x \to v$. So, for $y \in F$ there exists $u \in x \to v$ such that $y \preccurlyeq x \to v$, i.e. we have $F \ll x \to v$. Now, from $x \in F$ and $F \ll x \to v$ follows $v \in F$ according to (sHF3). Finally, for $u \in F$ there exists an element $v \in x \circ y$ such that $u \preccurlyeq v$. This means that $F \ll x \circ y$ holds. □

In addition to the previous one, we also have:

Lemma 6. $[\mathfrak{H}] \vDash (SH) \Longrightarrow (wSH)$.

Proof. Let the subset F in a hyper residuated system $\mathfrak{hA}$ satisfy the condition (HS). This means that for each $x, y \in F$, $x \circ y \subseteq F$ holds. Hence $(x \circ y) \cap F \neq \emptyset$, for all $x, y \in F$. Therefore, there exists an element $u \in F$ and $u \in x \circ x$ such that $u \preccurlyeq u$. Thus, F satisfies the condition (wSH). □

Here it is important to point out that, in the general case, $(HF3) \Longrightarrow (HF2)$ cannot be deduced.

Based on the hyper residuated lattices theory (for example, [2]), we introduce the term deductive system in a hyper QRS.

Definition 3.2. *Subset D of a hyper quasi-ordered residuated system $\mathfrak{hA}$ is a deductive system in $\mathfrak{hA}$ if it satisfies the conditions (HF0) and (HF3).*

In context to the mentioned text, here we introduce the term 'strong' deductive system.

Definition 3.3. *Subset D of a hyper quasi-ordered residuated system $\mathfrak{hA}$ is a strong deductive system in $\mathfrak{hA}$ if it satisfies the conditions (HF0) and (sHF3).*

In connection with the previous determination, it should be noted that the condition (HF2) is already contained in the condition (sHF3), and with the mandatory presence of the condition (HF0), according to Lemma 3.

Definition 3.4. *Subset D of a hyper quasi-ordered residuated system $\mathfrak{hA}$ is a reflexive subset of $\mathfrak{hA}$ if it satisfies the following condition*

(R) $(\forall B, C \subseteq A)((B \to C) \cap D \neq \emptyset \Longrightarrow (B \to C) \subseteq D)$.

Example 3.7. *Let $A =: \{a, b, c, 1\}$ be a chain such that $a < b < c < 1$. Let us define the hyper operations as follows*

$\to$	1	a	b	c
1	{1}	{a}	{b}	{c}
a	{1}	{1}	{1}	{1}
b	{1}	{a,b}	{1}	{1}
c	{1}	{a}	{a,b}	{1}

and

$\circ$	1	a	b	c
1	{1}	{a}	{b}	{c}
a	{a}	{a}	{a}	{a}
b	{b}	{a}	{a,b}	{b}
c	{c}	{a}	{b}	{c}

Routine calculations show that $(A, \circ, 1, \to, \preccurlyeq)$ is a hyper (quasi-)ordered residuated system. The subsets $G =: \{1\}$ and $F =: \{c, 1\}$ are reflexive deductive systems in $\mathfrak{hA}$.

The following proposition is immediately valid in accordance with the comment at the end of subsection 2.3.

Proposition 3.1. *Every strong deductive system in a hyper quasi-ordered residuated system* $\mathfrak{h}\mathfrak{A}$ *is a deductive system in* $\mathfrak{h}\mathfrak{A}$ *i.e. the following conclusion is valid*

$$[\mathfrak{H}] \vDash (sHF3) \Longrightarrow (HF3).$$

Example 3.8. *Let* $\mathfrak{h}\mathfrak{A}$ *be a hyper QRS as in Example 3.3. It is easy to show that* $D = [\frac{1}{2}, 1]$ *is a deductive system in* $\mathfrak{h}\mathfrak{A}$.

Example 3.9. *Let* $\mathfrak{h}\mathfrak{A}$ *be a hyper QRS as in Example 3.5. It can be verified that the set* $D_n = \{1, x_1, ..., x_n\}$ *is a deductive system for any* $n \in \mathbb{N}$. *However, the set* D_n *is not a strong deductive system. Indeed, for example, for* $x_n \in D_n$ *and* $D_n \ll x_n \to x_{n+1} = D_{n+1}$, *we have* $x_{n+1} \notin D_n$.

It is obvious that the following lemma holds:

Lemma 7. $[\mathfrak{H}] \vDash (R) \Longrightarrow (dHF3)$.

Proof. If we put $B = \{x\}$ and $C = \{y\}$ in (R), we get (dHF3). $\square$

The following statement shows a conditional connection between the condition (HF3) and the condition (HF2).

Lemma 8. $[\mathfrak{H}], (HF0), (R) \vDash (HF3) \Longrightarrow (HF2)$.

Proof. Let D be a reflexive deductive system of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$. Let $x, y \in A$ be such that $x \preccurlyeq y$ and $x \in D$. Since $1 \in D$ by (HF0) and $1 \in x \to y$ according to (b), we have $(x \to y) \cap D \neq \emptyset$. From here it follows $x \to y \subseteq D$ according to (R). Using this and $x \in D$ it follows $y \in D$ according to (HF3). This proves that (HF2) is a valid formula. $\square$

Proposition 3.2. *Let* D *be a reflexive deductive subset of a hyper quasi-ordered residuated system* $\mathfrak{h}\mathfrak{A}$. *Then*

$$(\forall B, C \subseteq A)(D \ll B \to C \Longleftrightarrow (B \to C) \cap D \neq \emptyset \Longleftrightarrow (B \to C) \subseteq D).$$

Proof. Let D be a reflexive deductive system in a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$.

Since the implication $(B \to C) \subseteq D \Longrightarrow (B \to C) \cap D \neq \emptyset$ is obvious, the equivalence $(B \to C) \cap D \neq \emptyset \Longleftrightarrow (B \to C) \subseteq D$ is valid in the presence of the condition (R).

If $D \ll (B \to C)$, then there exist $b \in B$, $c \in C$, $d \in D$ and $t \in b \to c \subseteq B \to C$ such that $d \preccurlyeq t$. Thus $t \in D$ by (HF2). This means $t \in (b \to c) \cap D$. Hence $(B \to C) \cap D \neq \emptyset$. Conversely, suppose that $(B \to C) \cap D \neq \emptyset$. Then there exist $b \in B$ and $c \in C$ such that $(b \to c) \cap D \neq \emptyset$. So there exists some $t \in (b \to c) \cap D$. Thus, for some $t \in D$ there exists some $t \in b \to c \subseteq B \to C$ such that $t \preccurlyeq t$. Therefore $D \ll B \to C$. $\square$

The concept of filters in a hyper residuated lattice was introduced in [17] and discussed in more detail in [3] as follows: A nonempty subset F of a hyper resuduated lattice L satisfying (HF2) and (SH) is a filter of L. A nonempty subset F of a hyper resuduated lattice L satisfying (HF2) and (wSH) is a weak filter of L. In [3] it was shown (Theorem 3.4) that: A non-empty subset F of a hyper residuated lattice L is a weak filter if and only if it satisfies the condition (HF2) and

(wHF4) $(\forall x, y \in A)((x \in F \wedge y \in F) \implies (x \circ y) \cap F \neq \emptyset)$.

The concept of filters in a hyper hoop-algebra can be found in [4] and is determined in the following way: A nonempty subset F of a hyper hoop-algebra H satisfying (HF2) and (wHF4) is a weak filter of H. A nonempty subset F of a hyper hoop-algebra H satisfying (HF2) and (SH) is a filter of H. In addition, it was shown there: A non-empty subset F of a hyper hoop-algebra H is a weak filter of H if and only if it satisfies conditions (HF2) and (wSH).

Let us show an obvious connection between conditions (SH) and (wHF4).

Lemma 9. $[\mathfrak{H}], \vDash (SH) \implies (wHF4)$.

The reverse implication of the implication in Lemma 9 need not be valid in the general case. Apart from the above, the following implication is also valid:

Lemma 10. $[\mathfrak{H}], (HF2) \vDash (wSH) \implies (wHF4)$.

Proof. Let a subset F of a hyper QRS $\mathfrak{h}\mathfrak{A}$ satisfy the conditions (HF2) and (wSH). Let us prove (wHF4). Let $x, y \in A$ be such that $F \ll x \circ y$. This means that there is some $u \in F$ and there is some $v \in x \circ y$ such that $u \preccurlyeq v$. From $u \in F$ and $u \preccurlyeq v$ it follows $v \in F$ according to (HF2). Therefore, $v \in F \cap (x \circ y)$ holds. So, $F \cap (x \circ y) \neq \emptyset$. This means that the formula (wHF4) is valid. □

Theorem 3. *Let $\mathfrak{h}\mathfrak{A}$ be a hyper quasi-ordered residuated system and $D \subseteq A$. Then*

$$[\mathfrak{H}],\ D \neq \emptyset,\ (HF2) \Vdash (sHF3) \iff (dHF3).$$

Proof. ($\Longrightarrow$) Let $D \subseteq A$ be a strong deductive system in a QRS $\mathfrak{h}\mathfrak{A}$ that also satisfies the condition (HF2). This means that the subset D satisfies the conditions (HF0), (HF2) and (sHF3). Let us prove that it satisfies the condition (dHF3) as well. Let $x, y \in A$ be such that $x \in D$ and $D \cap (x \rightarrow y) \neq \emptyset$. Then there exists an element $d \in A$ such that $d \in D$ and $d \in x \rightarrow y$. Since $d \preccurlyeq d$ is valid, we conclude that $D \ll x \rightarrow y$ is also valid. From here we get that $y \in D$ according to (sHF3). This proves the validity of the formula (dHF3).

($\Longleftarrow$) It follows from Lemma 4 □

In accordance with our earlier orientations - the omission which requires that a filter in a semigroup A be a subsemigroup of the semigroup A (see, for example [11]), we will determine the concept of filters in a hyper QRS as follows:

Definition 3.5. *A subset F of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ is a filter of $\mathfrak{h}\mathfrak{A}$ if it satisfies the conditions (HF2) and (HF3).*

It is not difficult to conclude that $\emptyset$ and A are filters in a hyper quasi-ordered reduced system $\mathfrak{h}\mathfrak{A}$, since the empty set $\emptyset$ and set A satisfy the conditions (HF2) and (HF3).

Besides that, it can be concluded:

Proposition 3.3. *Any non-empty filter in a hyper quasi-ordered residuated system $\mathfrak{hA}$ is a deductive system in $\mathfrak{hA}$.*

Proof. The proof follows directly from Lemma 1. □

Definition 3.6. *A subset F of a hyper quasi-ordered residuated system $\mathfrak{hA}$ is a strong filter of $\mathfrak{hA}$ if it satisfies the conditions (HF2) and (sHF3).*

According to Lemma 3, a nonempty strong deductive system in a hyper quasi-ordered residuated system $\mathfrak{hA}$ is a strong filter in $\mathfrak{hA}$. The reverse is also true (to see Proposition 3.5). It is immediately clear that the following proposition is valid:

Proposition 3.4. *Every non-empty strong filter in a hyper quasi-ordered residuated system $\mathfrak{hA}$ is a filter in $\mathfrak{hA}$.*

Proof. According to Proposition 3.3. □

Proposition 3.5. *Any non-empty strong filter in a hyper quasi-ordered residuated system $\mathfrak{hA}$ is a strong deductive system in $\mathfrak{hA}$.*

Proof. The proof follows directly from Lemma 1. □

Theorem 4. *Let $\mathfrak{hA}$ be a hyper quasi-ordered residuated system and $F \subseteq A$ be a subset which additionally satisfies the condition (HF2). Then F is a strong filter in $\mathfrak{hA}$ if and only if it satisfies the condition (dHF3).*

Proof. The proof is obtained from Theorem 1. □

Example 3.10. *Let A as in Example 3.3. Then the set $F_z = [z, 1]$ is a filter in $\mathfrak{hA}$ for any $z \leqslant 1$. Indeed. Let $x, y \in A$ be such that $x \in [z, 1]$ and $x \rightarrow y \subseteq [z, 1]$ hold. If $x \leqslant y$, then $x \rightarrow y = \{1\}$, so $y \in [z, 1]$ holds. If $y < x$, then $x \rightarrow y = [y, 1]$, so from $[y, 1] \subseteq [z, 1]$ we again conclude that $y \in [z, 1]$ holds. However, $[z, 1]$ is not a strong filter in $\mathfrak{hA}$, because for example, for $y < z < x$ we have $x \in [z, 1]$, $[y, 1] \cap [z, 1] \neq \emptyset$ and $y \notin [z, 1]$.*

Example 3.11. *Let $\mathfrak{hA}$ be as in Example 3.4. The subset $F =: [a, 1]$, for $a \leqslant 1$ is a strong filter in $\mathfrak{hA}$. Indeed:*

(HF3): For $x \in F$ and $x \leqslant y$ the following holds $x \rightarrow y = \{1, x, y\}$, so from $\{1, x, y\} \subseteq F$ it follows $y \in F$. For $x \in F$ and $y < x$ the following holds $x \rightarrow y = \{y\}$, so from $\{y\} \subseteq F$ it follows $y \in F$ again.

(dHF3): For $x \in F$ and $x \leqslant y$ the following holds $x \rightarrow y = \{1, x, y\}$, so from $\{1, x, y\} \cap F \neq \emptyset$ it follows $y \in F$. For $x \in F$ and $y < x$ the following holds $x \rightarrow y = \{y\}$, so from $\{y\} \cap F \neq \emptyset$ it follows $y \in F$ again.

The following theorem shows another conditional connection between the condition (HF3) and the condition (HF2) (in relation to the previous one, see Lemma 8).

Theorem 5. *Let $\mathfrak{hA}$ be a hyper quasi-ordered residuated system and $F \subseteq A$. Then*

$$[\mathfrak{H}] \vDash (HF2) \wedge (SH) \Longrightarrow (HF3).$$

Proof. Let the subset $F \subseteq A$ satisfy the conditions (HF2) and (SH). Let us prove the validity of (HF3). Let's take the elements $x, y \in A$ such that $x \in F$ and $x \to y \subseteq F$ hold. According to (k), we have $x \circ (x \to y) \ll y$. On the other hand, we have $x \circ (x \to y) \subseteq F$ by (SH). This means that there exists an element $u \in x \circ (x \to y)$ such that $u \in F$ and $u \preccurlyeq y$. Thus $y \in F$ by (HF2). This proves the validity of (HF3). □

The previous theorem shows that the filter in the sense of the articles [3, 4] is the same as the filter in our sense.

Theorem 6. *Let $\mathfrak{h}\mathfrak{A}$ be a hyper quasi-ordered residuated system and $F \subseteq A$. Then*

$$[\mathfrak{H}], (HF0) \Vdash (sHF3) \Longleftrightarrow (SH) \wedge (HF2).$$

Proof. ($\Longrightarrow$) Let a subset F of a hyper QRS $\mathfrak{h}\mathfrak{A}$ satisfy the conditions (HF0) and (sHF3). This set also satisfies condition (HF2) the according to Lemma 3. Let us prove that F satisfies the condition (SH). Let $x, y, u \in A$ be such that $x, y \in F$ and $u \in x \circ y$. Then $x \circ y \ll u$ since $u \preccurlyeq u$ holds. Thus $y \ll x \to u$ by (H3). This means $F \ll x \to u$. Hence, due to $x \in F$ and $F \ll x \to u$ it follows $u \in F$ according to (sHF3). Therefore, $x \circ y \subseteq F$.

($\Longleftarrow$) Assume that the nonempty subset F of a hyper QRS $\mathfrak{h}\mathfrak{A}$ satisfies the conditions (SH) and (HF2). Immediately, according to Lemma 1, we conclude that F satisfies condition (HF0). Let us prove that (sHF3) holds. Let $x, y \in A$ be such that $x \in F$ and $F \ll x \to y$. This means that there exist elements $u \in F$ and $v \in x \to y$ such that $u \preccurlyeq v$. From here it follows that $v \in F$ by (HF2). Besides, $x \circ v \subseteq F$ also holds due to (SH). By $v \in x \to y$ we have $v \ll x \to y$. Thus $v \circ x \ll y$ by (H3). So, there exists an element $t \in v \circ x$ such that $t \preccurlyeq y$. Finally, we have $y \in F$ by (HF2). This proves that F satisfies the condition (sHF3). □

From the previous theorem it immediately follows:

Corollary 6.1. *In order for a non-empty subset F in a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ to be a filter in $\mathfrak{h}\mathfrak{A}$, it is sufficient to satisfy the conditions (HF0) and (sHF3).*

Proof. Assume that a nonempty subset F in the hyper QRS $\mathfrak{h}\mathfrak{A}$ satisfies the conditions (HF0) and (sHF3). This set satisfies both conditions (SH) and (HF2) according to the previous theorem. In addition, this set also satisfies condition (HF3) according to Theorem 2. Therefore, F is a filter in $\mathfrak{h}\mathfrak{A}$. □

In what follows, we will deal with minimal and maximal (strong) filters in one hyper quasi-ordered residuated system.

Let F be a proper filter of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$. Then F is said to be a *minimal filter* in $\mathfrak{h}\mathfrak{A}$ if $G \subseteq F$, implies $F = G$ or $G = \emptyset$, for all filters G of $\mathfrak{h}\mathfrak{A}$.

Theorem 7. *Let $\{F_i : i \in I\}$ be a family of non-empty filters in a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ which contain the subset $D \subseteq A$. Then $\bigcap_{i \in I} F_i$ is a filter of $\mathfrak{h}\mathfrak{A}$ which contains the subset $D \subseteq A$.*

Proof. It is obvious that $D \subseteq \bigcap_{i\in I} F_i$ holds.

Let $x, y \in A$ be such $x \in \bigcap_{i\in I} F_i$ and $x \preccurlyeq y$. Then $x \in F_i$ for every $i \in I$. Thus $(\forall i \in I)(y \in F_i)$ by (HF2). Hence $y \in \bigcap_{i\in I} F_i$.

Let $x, y \in A$ be such that $x \in \bigcap_{i\in I} F_i$ and $x \to y \subseteq \bigcap_{i\in I} F_i$. Then $x \in F_i \wedge x \to y \subseteq F_i$ or every $i \in I$. Thus $y \in F_i$ for every $i \in I$ by (HF3). Hence $y \in \bigcap_{i\in I} F_i$. □

Corollary 7.1. *Let D be a subset of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$. Then there exists the minimal filter in $\mathfrak{h}\mathfrak{A}$ which contains the subset D.*

Proof. Let X be the family of all filters in a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ that contain the subset of D. Then $\cap X$ is a minimal filter in $\mathfrak{h}\mathfrak{A}$ that contains D according to the previous theorem. Indeed. If $G\,(\neq \emptyset)$ is a filter in $\mathfrak{h}\mathfrak{A}$ containing D, then $G \in X$. Therefore, $G \supseteq \cap X$. □

Corollary 7.2. *For each element x in a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ there is a minimal filter in $\mathfrak{h}\mathfrak{A}$ that contains x.*

Proof. The proof is obtained from the previous Corollary if we take $D = \{x\}$. □

Example 3.12. *In the hyper quasi-ordered residuated system in Example 3.6, the sets $\{c, 1\}$ and {1} are filters. The set $G =: \{b, c, 1\}$ is not a filter because, for example, $b \to a = \{b, c, 1\} \subseteq G$ and $b \in G$ are valid for the elements $a, b \in A$ but $a \notin G$.*

Example 3.13. *Let $A =: \{a, b, c, 1\}$ be a chain such that*

$$\preccurlyeq =: \{(a,a), (a,b), (a,1), (b,b), (b,1), (c,c), (c,1)\}.$$

Let us define the hyper operations as follows

$\circ$	a	b	c	1
a	{a}	{a}	{a,b,c}	{a}
b	{a}	{a,b}	{b,c}	{a,b}
c	{a,b,c}	{b,c}	{c}	{c}
1	{a}	{a,b}	{c}	{1}

and

$\to$	a	b	c	1
a	{1}	{1}	{c}	{1}
b	{a.b,c}	{1}	{c}	{1}
c	{a,b}	{a,b}	{1}	{1}
1	{a}	{a,b}	{c}	{1}

Routine calculations show that $(A, \circ, 1, \to, \preccurlyeq)$ is a hyper (quasi-)ordered residuated system. The subsets $F_1 =: \{1\}$, $F_2 =: \{c, 1\}$, $F_3 =: \{b, 1\}$ and $F_4 =: \{a, b, 1\}$ are strong filters in $\mathfrak{h}\mathfrak{A}$.

Let F be a proper filter (strong filter, deductive system, strong deductive system) of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$. Then F is said to be a maximal filter (strong filter, deductive system, strong deductive system) in $\mathfrak{h}\mathfrak{A}$ if $F \subseteq G \subseteq A$, implies $F = F$ or $G = A$, for all filters (strong filter, deductive system, strong deductive system, respectively) G in $\mathfrak{h}\mathfrak{A}$.

Theorem 8. *In a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ every proper filter (strong filter, deductive system, strong deductive system) in $\mathfrak{h}\mathfrak{A}$ is contained in a maximal filter (strong filter, deductive system, strong deductive system) in $\mathfrak{h}\mathfrak{A}$.*

Proof. Let F be a proper filter (strong filter, deductive system, strong deductive system) in $\mathfrak{hA}$ and let Y be the family of all proper filters (strong filter, deductive system, strong deductive system) in $\mathfrak{hA}$ containing F. Then $F \in Y$ and $(Y, \subseteq)$ is a poset partially ordered by the inclusion. Let $\{F_i : i \in I\}$ be a chain in Y. Let us prove that $\bigcup_{i\in I} F_i$ is a filter (strong filter, deductive system, strong deductive system, res.) in $\mathfrak{hA}$ that contains F.

Clearly, $1 \in \bigcup_{i\in I} F_i$.

Let $x, y \in A$ be such that $x \in \bigcup_{i\in I} F_i$ and $x \preccurlyeq y$. Then there exists an index $k \in I$ such that $x \in F_k$. Thus $y \in F_k \subseteq \bigcup_{i\in I} F_i$.

Assume that all sets $F_i\,(i \in I)$ satisfy the condition (sHF3). Let $x, y \in A$ be such that $x \in \bigcup_{i\in I} F_i$ and $\bigcup_{i\in I} F_i \ll x \to y$. Then, there exist $j, k \in I$ such that $x \in F_j$ and $F_k \ll x \to y$. Since F_i's forms a chain, we can assume that $F_j \subseteq F_k$. Thus, $F_k \ll x \to y$ and $x \in F_k$ implies that $y \in F_k \subseteq \bigcup_{i\in I} F_i$ by (sHF3). This proves that $\bigcup_{i\in I} F_i$ satisfies the condition (sHF3) and that it contains the set F.

Assume that all sets F_i $(i \in I)$ satisfy the condition (HF3). Let $x, y \in A$ be such that $x \in \bigcup_{i\in I} F_i$ and $x \to y \subseteq \bigcup_{i\in I} F_i$. Then, there exist $j \in I$ and $J \subseteq I$ such that $x \in F_j$ and $x \to y \subseteq \bigcup_{t\in J} F_t$. Since F_i's forms a chain, so we can assume that $F_j \subseteq F_k$ for some $k \in \{j\} \cup J$ and $x \to y \subseteq F_k$. Thus, $x \to y \subseteq F_k$ and $x \in F_k$ imply that $y \in F_k \subseteq \bigcup_{i\in I} F_i$ by (HF3).

So, $\bigcup_{i\in I} F_i \in Y$. Hence any chain of elements of Y has an upper bound in Y. By Zorn's lemma, Y has a maximal element, let's say M. Let us show that M is a maximal filter (strong filter, deductive system, strong deductive system, res.) in $\mathfrak{hA}$ containing F. Let $M \subseteq G \subseteq A$, for some filter (strong filter, deductive system, strong deductive system, res.) G in $\mathfrak{hA}$. If $G \neq A$, then $G \in Y$. Since M is a maximal element of Y, we get $M = G$. Therefore, M is a maximal filter (strong filter, deductive system, strong deductive system, res.) in $\mathfrak{hA}$ which contains F. □

4. Conclusions and future works

In this article, the concept of hyper quasi-ordered residuated system is developed, which is a generalization of the concept of quasi-ordered residuated system, on the one hand, and, on the other hand, it is a generalization of both hyper hoop-algebras and hyper redisuated lattices. The category of hyper quasi-ordered residuated systems, quotient structure, special filters as inductive and conjugative filters, and so on (like for example, the concept of ideals in this structure) could be topics for future research. Finally, but not least, one could and should consider the similarities and differences between the algebraic structures mentioned here.

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