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QUASI-VALUATION MAPS ON QUASI-ORDERED RESIDUATED SYSTEMS

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Abstract. The concept of quasi-ordered residuated systems was introduced in 2018 by S. Bonzio and I. Chajda. The substructures of ideals and filters in such algebraic structures were considered by the author. This paper introduces and analyzes the concept of quasi-valuation maps on quasi-ordered residuated systems based on a filter in it.

1. Introduction

The idea that universal algebra can be analyzed by means of real functions (recognizable as 'pseudo-valuation mapings') was first developed by D. Busneag in 1996 ([3]). That author has expanded the perception of pseudo-valuation on Hilbert's algebras ([4]) previously constructed for the commutative rings. Song, Roh and Jun, in [19] introduced the notion of quasi-valuation maps based on a subalgebra and an ideal in BCK/BCI-algebras, and then they investigated several properties of them. Quasi-valuation maps on KU-algebras and UP-algebras, as a generalization of KU-algebras, were studied in [7, 9, 11, 13]. The application of this idea to hoop-algebras was developed by M.A. Kologani at al. in 2021 ([8]).

The concept of residuated relational systems ordered under a quasi-order relation, or quasi-ordered residuated systems (briefly, QRS), was introduced in 2018 by S. Bonzio and I. Chajda ([1]). Quasi-ordered residauted system, generally speaking, differs from the commutative residuated lattice $\langle A, \cdot, 0, 1, \sqcap, \sqcup, R\rangle$ where R is a lattice quasi-order. First, our observed system does not have to be limited from below. Second, the observed system does not have to be a lattice. Also, this algebraic system differs from hoop-algebras (for example, see [2]). The substructures of filters and ideals in quasi-ordered residuated systems were considered in articles [12, 14, 15]. This algebraic structure has been the focus of this

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author's interest in several of his researches (see, for example [16, 17]). Some more information about this algebraic structure a reader can find in [18].

In this paper, the notion of quasi-valuation map on quasi-ordered residuated systems and its related properties are investigated. This function on a QRS, constructed in this paper, allows us to design a simple algorithm to recognize substructures of filters and ideals in it. This paper presents some of the specificities of this class of algebraic structures.

2. Preliminaries

In this section, the necessary notions and notations and some of their interrelationships are listed in order to enable a reader to comfortably follow the presentation in this report. It should be pointed out here that the notations for logical conjunction and logical implication have a literal meaning.

2.1. **Concept of quasi-ordered residuated systems.** In article [1], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

Definition 2.1 ([1], Definition 2.1)**.** A *residuated relational system* is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and R is a binary relation on A and satisfying the following properties:
(1) $(A, \cdot, 1)$ is a commutative monoid;
(2) $(\forall x \in A)((x, 1) \in R)$;
(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.
We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

Recall that a *quasi-order relation* '$\preccurlyeq$' on a set A is a binary relation which is reflexive and transitive.

Definition 2.2 ([1])**.** A *quasi-ordered residuated system* is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, where $\preccurlyeq$ is a quasi-order relation in the monoid $(A, \cdot)$

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 2.1 ([1], Proposition 3.1)**.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then*
(4) *The operation '$\cdot$' preserves the pre-order in both positions;*
$$(\forall x, y, z \in A)(x \preccurlyeq y \implies (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y));$$
(5) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (y \rightarrow z \preccurlyeq x \rightarrow z \wedge z \rightarrow x \preccurlyeq z \rightarrow y))$;
(6) $(\forall y, z \in A)(x \cdot (y \rightarrow z) \preccurlyeq y \rightarrow x \cdot z)$;
(7) $(\forall x, y, z \in A)(x \cdot y \rightarrow z \preccurlyeq x \rightarrow (y \rightarrow z))$;
(8) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq x \cdot y \rightarrow z)$;
(9) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq y \rightarrow (x \rightarrow z))$;
(10) $(\forall x, yz \in A)((x \rightarrow y) \cdot (y \rightarrow z) \preccurlyeq x \rightarrow z)$;
(11) $(\forall x, y \in A)((x \cdot y \preccurlyeq x) \wedge (x \cdot y \preccurlyeq y))$;
(12) $(\forall x, y, z \in A)(x \rightarrow y \preccurlyeq (y \rightarrow z) \rightarrow (x \rightarrow z))$;

(13) $(\forall x, y, z \in A)(y \to z \preccurlyeq (x \to y) \to (x \to z))$.

It is generally known that a quasi-order relation $\preccurlyeq$ on a set A generates an equivalence relation $\equiv_{\preccurlyeq} := \preccurlyeq \cap \preccurlyeq^{-1}$ on A. Due to properties (4) and (5), this equality relation is compatible with the operations in A. Thus, $\equiv_{\preccurlyeq}$ is a congruence on A.

In the light of the previous note, it is easy to see that the following applies:
(7) and (8) give:
(H3) $(\forall x, y, z \in A)(x \cdot y \to z \equiv_{\preccurlyeq} x \to (y \to z))$.
Due to the universality of formula (9), we have:

$$(\forall x, y, z \in A)(x \to (y \to z) \equiv_{\preccurlyeq} y \to (x \to z)).$$

Example 2.1: By a hoop ([2]) we mean an algebra $(H, \cdot, \to, 1)$ in which $(H, \cdot, 1)$ is a commutative semigroup with the identity and the following assertions are valid:
(H1) $(\forall x \in H)(x \to x = 1)$,
(H2) $(\forall x, y \in H)(x \cdot (x \to y) = y \cdot (y \to x))$ and
(H3) $(\forall x, y, z \in A)(x \cdot y \to z = x \to (y \to z))$.
In this algebra, order is determined as follows:

$$(\forall x, y \in A)(x \leqslant y \iff x \to y = 1).$$

It is easy to see that $(H, \leqslant)$ is a poset. It is easy to see that every hoop is a (quasi-)ordered residuated system and vice versa does not have to be. □

Since, in the general case, the formula

$$(\forall x, y \in A)(x \cdot (x \to y) \equiv_{\preccurlyeq} y \cdot (y \to x))$$

does not have to be valid in a quasi-ordered residuated system, we conclude that this last mentioned system is a generalization of the hoop-algebra.

Example 2.2: For a commutative monoid A, let $\mathfrak{P}(A)$ denote the power set of A ordered by set inclusion and '$\cdot$' the usual multiplication of subsets of A. Then $\langle \mathfrak{P}(A), \cdot, \to, A, \subseteq \rangle$ is a quasi-ordered residuated system in which the residuum is given by

$$(\forall X, Y \in \mathfrak{P}(A))(Y \to X := \{z \in A : Yz \subseteq X\}).$$

□

Example 2.3: Let $A = \{1, 2, 3, 4\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	a	a	a
b	b	a	b	b	b
c	c	a	b	c	b
d	d	a	b	b	d

and

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	a	1	1	1
c	1	a	d	1	d
d	1	a	c	c	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1 \rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows $\preccurlyeq := \{(1,1), (a,1), (a,b), (a,c), (a,d), (b,b), (b.c), (b,d), (b,1), (c,c), (c,1), (d,d), (d,1)\}$. □

2.2. **Concept of filters.**

Definition 2.3 ([12], Definition 3.1)**.** For a subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a *filter* of $\mathfrak{A}$ if it satisfies conditions
(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$, and
(F3) $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \implies v \in F)$.

Let it note that the empty subset of A satisfies the conditions (F2) and (F3). Therefore, $\emptyset$ is a filter in $\mathfrak{A}$. It is shown ([12], Proposition 3.4 and Proposition 3.2), that if a non-empty subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the following conditions
(F0) $1 \in F$ and
(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$.
Also, it can be seen without difficulty that $(F3) \implies (F2)$ is valid. Indeed, if (F3) holds, then the formula $u \in F \wedge u \preccurlyeq v$, can be transformed into the formula $u \in F \wedge u \rightarrow v \equiv_{\preccurlyeq} 1 \in F$ by (F0) so from here, according to (F3) it can be demonstrate the validity of implications (F2). However, the reverse does not have to be valid.

If $\mathfrak{F}(A)$ is the family of all filters in a QRS $\mathfrak{A}$, then $\mathfrak{F}(A)$ is a complete lattice ([12], Theorem 3.1).

Remark 2.1: In implicative algebras, the term 'implicative filter' is used instead of the term 'filter' we use (see, for example [5, 10]) because in the structure we study the concept of filter is determined more complexly than requirement (F3). It is obvious that our filter concept is also a filter in the sense of [5, 6, 10]. The term 'special implicative filter' is also used in the aforementioned sources if the implicative filter in the sense of [10] satisfies some additional condition.

Example 2.4: Let $\mathfrak{A}$ be a quasu-ordered residuated system as in Example 2.3. Then $F_1 := \{1\}$, $F_2 := \{c, 1\}$, $F_3 := \{1, d\}$ and $F_4 := \{1, c, d\}$ and $F_5 := \{1, b, c, d\}$ are filters of $\mathfrak{A}$. □

2.3. **Concept of ideals.** In the article [14], the concepts of pre-ideal and ideal in quasi-ordered residuated systems were analyzed. Before that, the conditions were analyzed
(J1) $(\forall y, v \in A)((u \in J \vee v \in J) \implies u \cdot v \in J)$,
(J2) $(\forall u, v \in A)((u \preccurlyeq v \wedge v \in J) \implies u \in J)$, and
(J3) $(\forall u, v \in A)((u \rightarrow v \notin J \wedge v \in J) \implies u \in J)$.
Furthermore, in that paper it was proved that (J2) $\Longrightarrow$ (J1) holds and that (J3) $\Longrightarrow$ (J2) also holds for the proper subset J. With respect to the above, we have:

Definition 2.4. Let $\mathfrak{A}$ be a quasi-ordered residuated system. For a subset J of the set A we say that it is a pre-ideal in $\mathfrak{A}$ if the condition (J2) is valid. For a subset J of the set A we say that it is an ideal in $\mathfrak{A}$ if $J = A$ or the condition (J3) is valid.

It can be easily seen that if J is a proper (pre-)ideal of $\mathfrak{A}$, then holds
(J0) $1 \notin J$.

3. Quasi-valuation on QRS

The following definition gives the concept of quasi-valuation maps on a quasi-ordered residuated system.

Definition 3.1. Let $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1\rangle$ a quasi-ordered residuated system. A real valued function $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ is called quasi-valuation on $\mathfrak{A}$ if
$(V0)$ $v(1) = 0$ and
$(V1)$ $(\forall x, y \in A)(v(y) \geqslant v(x) + v(x \rightarrow y))$.

In the following proposition, some of the fundamental properties of the mapping $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ designed in this way are given .

Proposition 3.1. *For any quasi-valuation map v on a quasi-ordered residuated system $\mathfrak{A}$, we have the following assertions:*
(14) $(\forall x, y \in A)(x \preccurlyeq y \Longrightarrow v(x) \leqslant v(y))$.
(15) $(\forall x \in A)(v(x) \leqslant 0)$.
(16) $(\forall x, y \in A)(2v(x \cdot y) \leqslant v(x) + v(y))$.
(17) $(\forall x, y \in A)(v(x \rightarrow y) \leqslant v(y) - v(x))$.

Proof. Let $x, y \in A$ be such that $x \preccurlyeq y$. Then $x \rightarrow y \equiv_{\preccurlyeq} 1$. Thus

$$v(y) \geqslant v(x) + v(x \rightarrow y) = v(x) + v(1) = v(x) + 0 = v(x).$$

The statement (15) is a direct consequence of the statement (14) and the axiom (2) with respect to the condition (V0).

Let $x, y \in A$ be arbitrary elements. Then $x \cdot y \preccurlyeq x$ and $x \cdot y \preccurlyeq y$ by (11). Thus $v(x \cdot y) \leqslant v(x)$ and $v(x \cdot y) \leqslant v(y)$ by (14). From here, we get the required inequality.

Then condition (17) is equivalent to the condition (V1). □

In what follows, we will need the following lemma:

Lemma 1. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then:*
(18) $(\forall x \in A)(1 \equiv_{\preccurlyeq} x \rightarrow (x \rightarrow 1))$,
(19) $(\forall y \in A)(1 \equiv_{\preccurlyeq} y \rightarrow y)$.

Proof. Let $x \in A$ be an arbitrary element. Than $x \cdot x \preccurlyeq x \preccurlyeq 1$ by (11) and (2). Thus $x \preccurlyeq x \rightarrow 1$ by (3). Hence $1 \preccurlyeq x \rightarrow (x \rightarrow 1) \preccurlyeq 1$ by (3) and (2). This proves (18).

The claim (19) is a direct consequence of the reflexivity of the relation $\preccurlyeq$ and axioms (2) and (3). □

On the other hand, we have

Proposition 3.2. *For any quasi-valuation map v on a quasi-ordered residuated system $\mathfrak{A}$, we have the following assertions:*
(20) $(\forall x, y \in A)(v(x \rightarrow y) \geqslant v(x) + v(y))$.
(21) $(\forall x, y \in A)(v(x \cdot y) \geqslant v(x) + v(y))$.

Proof. Let $x, y \in A$ be arbitrary elements. Then:
$0 = v(1)$ by (V0)
$= v(x \rightarrow (x \rightarrow 1))$ by (18)

$= v(x \to (x \to (y \to y)))$ by (19)
$= v(x \to (y \to (x \to y)))$ by (9)
$\leqslant v(y \to (x \to y)) - v(x)$ by (17)
$\leqslant v(x \to y) - v(y) - v(x)$ by (17).

This proves the claim (20).

$v(x \cdot y) \geqslant v(y \to x \cdot y) + v(y)$ by (V1)
$\geqslant v(x \to (y \to x \cdot y)) + v(x) + v(y)$ by (V1)
$= v(x \cdot y \to x \cdot y) + v(x) + v(y)$ by (H3)
$= v(1) + v(x) + v(y)$ by (19)
$= 0 + v(x) + v(y)$ by (V0).

This proves the claim (21). □

The following two theorems connect a quasi-valuation $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ on a quasi-ordered residuated system $\mathfrak{A}$ and the concept of filters in $\mathfrak{A}$.

Theorem 1. *If $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ is a quasi-valuation map on a quasi-ordered redisuated system $\mathfrak{A}$, then the set*

$$F_v := \{x \in A : v(x) = 0\}$$

is a filter of $\mathfrak{A}$.

Proof. The set F_v is not empty because $1 \in F_v$. Therefore, it is sufficient to prove the validity of the condition (F3).

Let $x, y \in A$ be such that $x \in F_v$ and $x \to y \in F_v$. Then $v(x) = 0$ and $v(x \to y) = 0$. Thus $v(y) \geqslant v(x) + v(x \to y) = 0 + 0 = 0$. Hence $v(y) = 0$ by (13). So, $y \in F_v$. This proves the validity of the condition (F3). □

Theorem 2. *Let G be a non-empty filter in a quasi-ordered residuated system $\mathfrak{A} = \langle A, \cdot, 1, \to \rangle$. For any negative real number k, let v_G be a real valued function on $A/\equiv_{\preccurlyeq}$ defined by $v_G(x) := 0$ if $x \in G$ and $v_G(x) := k$ if $x \in A \setminus G$. Then v_G is a quasi-valuation on $\mathfrak{A}$ and $F_{v_G} = G$ holds.*

Proof. It is clear that $v_G(1) = 0$.

Let $x, y \in A$ be arbitrary elements.

Assume that $y \in G$. Then

$$v_G(y) = 0 \geqslant v_G(x) + v_G(x \to y)$$

with respect (13).

Assume that $y \notin G$. The contraposition applied to (F3) gives $x \notin G$ or $x - y \notin G$. If $x \in G$ and $x \to y \notin G$, then $v_G(x) = 0$ and $v_G(x \to y) = k$. Hence

$$v_G(y) = k = 0 + k = v_G(x) + v_G(x \to y).$$

If $x \notin G$ and $x \to y \in G$, then $v_G(x) = k$ and $v_G(x \to y) = 0$. Hence

$$v_G(y) = k = k + 0 = v_G(x) = v_G(x \to u).$$

If $x \notin G$ and $x \to y \notin G$, then $v_G(x) = k$ and $v_G(x \to y) = k$. Hence

$$v_G(y) = k \geqslant k + k = v_G(x) + v_G(x \to y).$$

Therefore v_G is a quasi-valuation of $\mathfrak{A}$.

On the other hand, we have $F_{v_G} = \{x \in A : v_G(x) = 0\} = G$. □

Example 3.1: Let $\mathfrak{A}$ be a quasi-ordered residuated system as in Example 2.3. Then the set $F_2 = \{1, c\}$ is a filter of $\mathfrak{A}$. If $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ is defined by $v(1) = v(c) = 0$ and $v(a) = v(b) = v(d) = -7$, then v is a quasi-valuation on $\mathfrak{A}$ according to the Theorem 2. □

Theorem 3. *If $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ is a quasi-valuation map on a quasi-ordered redisuated system $\mathfrak{A}$, then the set*

$$J_v := \{x \in A : v(x) < 0\}$$

is an ideal of $\mathfrak{A}$.

Proof. Let $x, y \in A$ be arbitrary elements of A such that $x \preccurlyeq y$ and $y \in J_v$. Then $v(y) < 0$. Due to the monotonicity of the function v, we have $v(x) \leqslant v(y) < 0$. Thus, $x \in J_v$. This proves the validity of condition (J2).

To prove the validity of condition (J3), let us take $x, y \in A$ such that $x - y \notin J$ and $y \in J_v$. This means $v(x \to y) = 0$ and $v(y) < 0$. On the other hand, according to (17) we have $0 = v(x \to y) \leqslant v(y) - v(x)$. Thus $v(x) \leqslant v(y) < 0$. Hence $x \in J_v$. □

Example 3.2: Let $A = H$ as in article [8], Example 3.3 and let $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ is determined as in Example 3.9 in the same paper. Then v is a quasi-valuation map on A. Then $F_v = \{1\}$ is a filter in A because $v(1) = 0$ and $J_v = \{0, a, b\}$. □

In what follows, we will design a pseudo-metric space on a quasi-ordered residuated system generated by a pseudo-valuation on it.

By a pseudo-metric on a quasi-ordered residuated system $\mathfrak{A}$, we mean a real-valued function

$$d : A/\equiv_{\preccurlyeq} \times A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$$

satisfying the following properties:

$d(x, y) \geqslant 0$, $d(x, x) = 0$,

d$(x, y) = d(y, x)$ and

$d(x, z) \leqslant d(x, y) + d(y, z)$

for every $x, y, z \in A$.

Let us, first, prove a technical lemma.

Lemma 2. *Let v be a quasi-valuation on a quasi-ordered residuated system $\mathfrak{A}$. Then the following holds*

(22) $(\forall x, y, z \in A)(v(x \to z) \geqslant v(x \to y) + v(y \to z))$.

Proof. Let $x, y, z \in A$ be arbitrary elements. Then

$$x \to z \succcurlyeq (x \to y) \cdot (y \to x).$$

by (10). Thus $v(x \to z) \geqslant v((x \to y) \cdot (y \to x))$ by (14). Hence

$$v(x \to z) \geqslant v(x \to y) + v(y \to x)$$

by (21). □

Theorem 4. *Let $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ be a quasi-valuation on a quasi-ordered residuated system $\mathfrak{A}$. Then*

$$d_v : A/\equiv_{\preccurlyeq} \times A/\equiv_{\preccurlyeq} \ni (x,y) \longmapsto d_v(x,y) := -(v(x \to y) + v(y \to x)) \in \mathbb{R}$$

is a pseudo-metric on $\mathfrak{A}$ and so (A, d_v) is a pseudo-metric space.

Proof. Let v be a quasi-valuation on a quasi-ordered residuated system $\mathfrak{A}$.
Then $v(x) \leqslant 0$ for all $x \in A$ by (15). Thus $d_v(x,y) \geqslant 0$ for all $x, y \in A$.
It is clear that $d_v(x,x) = 0$ and $d_v(x,y) = d_v(y,x)$ for all $x, y \in A$.
Let $x, y, z \in A$ be arbitrary elements. Then:

$$\begin{aligned} d_v(x,z) &= -(v(x \to z) + v(z \to x)) \\ &\leqslant -(v(x \to y) + v(y \to z)) - (v(z \to y) + v(y \to x)) \\ &= -(v(x \to y) + v(y \to x)) - (v(y \to z) + v(z \to y)) \\ &= d_v(x,y) + d_v(y,z). \end{aligned}$$

Hence $(A/\equiv_{\preccurlyeq}, d_v)$ is a pseudo-metric space. □

4. Conclusion

In this paper, the notion of quasi-valuation map on a quasi-ordered residual system based on filter in it is introduced. Using the notion of quasi-valuations, a corresponding pseudo-metric space was designed. Judging by the results presented in the articles [2, 7, 8, 9, 11, 19], the idea of constructing the concept of quasi-valuation on an algebraic structure and its connection with the corresponding pseudo-metric space is independent of the specifics of the algebraic support by which this design is realized. On the other hand, the specificities in each case arise from the logical environment in which the concept of quasi-valuation map is developed.

A natural continuation of this report could be the consideration of determinations of quasi-valuation maps w on a quasi-ordered residuated system $\mathfrak{A}$ based on an implicative filter (weak implicative filter, comparative filter and so on...) in it. In that case it is to be expected that the set $F_w = \{x \in A : w(x) = 0\}$ will be an implicative filter (weak implicative filter, comparative filter and so on, respectively) in $\mathfrak{A}$.

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