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ON NUMERICAL SOLUTIONS OF LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Fractional differential equations have excited considerable interest recently, both in pure and applied mathematics. In this paper, we apply Fractional Adams-Bashforth Method (FAB), Fractional Adams-Bashforth-Moulton Method (FABM) and Fractional Multistep Differential Transform Method (FMDTM), for obtaining the numerical solutions of two distinct linear systems of fractional differential equations with fractional derivatives described in the Caputo sense. The numerical results for the three methods are compared with the exact solution for each linear system by using the relative difference between the exact and the approximate solution at each integration point. The results are given both graphically and tabularly, concluding that, aside from occasional non-monotoncity for small time values, all three numerical methods gradually diverge from the exact solution with increasing integration time, and the superiority of each numerical method over the others depends on the particular system under investigation.

1. Introduction

Fractional differential equations have become especially interesting in the past decade due to their numerous applications in describing variety of phenomena in many areas of physics and engineering [1, 2]. It has been shown that the spatial-temporal dynamics of variety of physical phenomena can be expressed reliably via equations containing derivatives and integrals of non-integer order. The fractional derivative is not uniquely defined, and in this paper we use the Caputo definition of fractional derivative [1, 2].

Most systems described by fractional differential equations do not have exact solutions, and finding analytical solutions of fractional differential equations could

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be a formidable task. In majority of cases, it is only possible to provide a numerical approximation of the solution [2, 3].

In this paper, we use three different numerical methods to find approximate solutions of two distinct linear differential systems of fractional type. To solve the linear differential equations, we use Fractional Adams-Bashforth Method (FAB), Fractional Adams-Bashforth-Moulton Method (FABM) and Fractional Multistep Differential Transform Method (FMDTM) [4, 5, 6, 7, 8]. The organization of this paper is as follows. The definitions and mathematical preliminaries of fractional calculus theory that we exploit in this paper are provided in Section 2. In Section 3 we give an overview of the used numerical methods and the details of the corresponding algorithms. In Section 4, we present two examples of linear fractional-order systems that are solved analytically in terms of one-parameter Mittag-Leffler functions which are generalizations of an exponential function. The obtained numerical results are then compared with the exact solutions for each example by using the relative difference between the exact and the approximate solution. The time-series are displayed both graphically and tabularly. The conclusions are given in Section 5. All the numerical examples are done by using the Wolfram Mathematica package. The notebook programs are available upon request.

2. Basic definitions

Definition 2.1. A real function $f(t)$, $t > 0$, is said to be in space C_μ, $\mu \in R$ if thre exists a real number $p(> \mu)$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2.2. The Riemman-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(t) \in C_\mu$, $\mu \geq -1$ is defined as:

$$J^\alpha f(t) = \frac{1}{\Gamma\alpha} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau \tag{2.1}$$

Definition 2.3. The Caputo fractional derivative of a function $f(t)$ is defined by:

$$D_t^\alpha f(t) = J^{n-\alpha} D_t^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \tag{2.2}$$

for $n - \alpha < \alpha \leq n$, $n \in N$, $t > 0$, and $f(t) \in C_{-1}^m$.

Remark 1. In the definition for Caputo fractional derivative we first differentiate $f(t)$ n-times, then integrate it $n - \alpha$ times. If $f(t)$ is n-times differentiable, then the α-th order derivative will exist, where $n - \alpha < \alpha \leq n$, otherwise this definition is not applicable. Two main advantages of this definition are: i). fractional derivative of a constant is zero; ii). fractional differential equation of Caputo type has initial conditions of classical non-integer derivative type, in contrast to fractional differential equation of Riemann-Liouville type, where initial conditions are of fractional type.

Two basic properties of Caputo fractional derivative that immediately follow from Definition 2.3 are:

$$J^{\alpha} D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad t > 0, \tag{2.3}$$

$$D_t^{\alpha} J^{\alpha} f(t) = f(t). \tag{2.4}$$

Definition 2.4. The two parameter Mittag-Leffler function is defined (in series form) as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in C, \quad Re(\alpha), \beta > 0 \tag{2.5}$$

If $\beta = 1$ then we have a one-parameter Mittag-Leffler function

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{2.6}$$

The Mittag-Leffler functions are generalizations of the exponential function, and solutions of fractional order linear differential equations are often expressed in terms of Mittag-Leffler functions.

Definition 2.5. The m-th derivative of Mittag-Leffler function, for $m = 0, 1, 2, \ldots$, are given by:

$$E_{\alpha,\beta}^{(m)}(z) = \sum_{k=0}^{\infty} \frac{(k+m)! z^k}{k! \Gamma(\alpha k + \alpha m + \beta)}, \tag{2.7}$$

3. Numerical Methods

For the initial value problem

$$D_t^{\alpha} y(t) = f(t, y(t)), \quad y^{(k)}(0) = y_0^{(k)} \tag{3.1}$$

($k = 0, 1, 2, ..., \lceil \alpha \rceil - 1$), we will construct three numerical methods, FAB, FABM and FMDTM, assuming that a solution of (3.1) is sought on some time interval $[0, T]$.

3.1. **Fractional Adams-Bashforth Method.**

In order to assure the existence and uniqueness of the solution to (3.1), it is assumed that $f(t, y(t))$ is continuous and fulfils the Lipschitz condition with respect to the second variable. On $[0, T]$, for a uniform grid $t_j = hj$ ($j = 0, 1, ..., N$) and a constant time step denoted by $h = \frac{T}{N}$, the goal is to approximate solution values $y_j \approx y(t_j)$ at the grid points. According to the theorem of existence and uniqueness of the solution, initial value problem (3.1) can be reformulated in terms of the weakly-singular Volterra integral equation:

$$y(t) = \sum_{k=0}^{\lceil \alpha \rceil - 1} \frac{y_0^{(k)}}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, y(\tau)) d\tau. \tag{3.2}$$

The method immediately suggests a numerical approach in solving (3.2). The main part of the algorithm for FAB is the iteration formula

$$y[j] = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(jh)^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{k=0}^{j-1} b[j-k] f(kh, y[k]) \tag{3.3}$$

where $b[j-k]$ are the weights which depend only on the difference $(j-k)$ because of the convolution structure of $b_{k,j}$:

$$b_{k,j} = \frac{(j-k)^\alpha - (j-k-1)^\alpha}{\Gamma(\alpha+1)}. \tag{3.4}$$

3.2. **Fractional Adams-Bashforth-Moulton Method.** The discussed FAB method is a natural candidate for a predictor in the process of constructing the predictor-corrector method FABM (the Adams-Moulton method can be constructed in similar way like FAB).

The FABM method is said to be Predict-Evaluate-Correct-Evaluate type because an initial approximation p, the so-called predictor, is evaluated first:

$$p = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(jh)^k}{k!} x_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{k=0}^{j-1} b[j-k] f(kh, y[k]). \tag{3.5}$$

Then the method gives the corrector formula:

$$\begin{aligned} y[j] = & \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(jh)^k}{k!} y_0^{(k)} \\ & + \frac{h^\alpha}{\Gamma(\alpha+2)} (f(jh, p) + ((j-1)^{\alpha+1} - (j-\alpha-1)j^\alpha) \cdot f(0, y[0]) + \\ & + \sum_{k=0}^{j-1} a[j-k] f(kh, y[k])) \end{aligned} \tag{3.6}$$

where p represents FAB, which in this case acts like a predictor. The weight $a[j-k]$ in the corrector $y[j]$ is given by

$$a[j-k] = \begin{cases} \frac{(j-1)^{\alpha+1} - (j-\alpha-1)j^\alpha}{\Gamma(\alpha+2)}, & \text{if } j = 0 \\ \frac{(j-k+1)^{\alpha+1} + (j-k-1)^{\alpha+1} - 2(j-k)^{\alpha+1}}{\Gamma(\alpha+2)}, & \text{if } j \in [1, k-1] \\ 1, & \text{if } j = k \end{cases} \tag{3.7}$$

3.3. **Fractional Multistep Differential Transform Method.** The basic definition and the fundamental theorems of the differential transform method (DTM) and its applicability for various kinds of differential equations are given in the following paragraphs.

The differential transform of the k-th derivative of function $f(t)$ is defined as:

$$F(k) = \frac{1}{k!} [\frac{d^k f(t)}{dt^k}]_{t=t_0} \tag{3.8}$$

where $f(t)$ is the original function and $F(k)$ is the transformed function. The differential inverse transform of $F(k)$ is defined by

$$f(t) = \sum_{k=0}^{\infty} F(k)(t-t_0)^k \tag{3.9}$$

From (3.8) and (3.9), we get $f(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \frac{d^k f(t)}{dt^k}$, which implies that the concept of differential transform is derived from Taylor series expansion.
To describe the FMDTM, we consider the initial value problem (3.1). In actual applications of DTM, approximate solution of the initial value problem (3.1) can be expressed by the finite series in the form

$$y(t) = \sum_{n=0}^{N} a_n t^n, \quad t \in [0, T] \tag{3.10}$$

Assume that the interval $[0, T]$ is divided into M subintervals $[t_{m-1}, t_m]$, $m = 1, 2, ..., M$ of equal step size $h = \frac{T}{M}$ by using the nodes $t_m = mh$. The main idea of the FMDTM is in the following. First, if we apply DTM to (3.1) over the interval $[0, t_1]$, we will obtain the following approximate solution:

$$y_1(t) = \sum_{n=0}^{K} a_{1n} t^n, \quad t \in [0, t_1] \tag{3.11}$$

using the initial conditions $y_1^{(k)}(0) = c_k$.
For $m \geq 2$, at each subinterval $[t_{m-1}, t_m]$ we will use the initial conditions $y_m^{(k)}(t_{m-1}) = y_{m-1}^{(k)}(t_{m-1})$ and apply DTM to (3.1) over the subinterval, where t_0 in (3.8) is replaced by t_{m-1}. The process is repeated and generates a sequence of approximate solutions $y_m(t), m = 1, 2, ..., M$, for the solution $y(t)$:

$$y_m(t) = \sum_{n=0}^{K} a_{mn}(t - t_{m-1})^n, \quad t \in [t_m, t_{m+1}] \tag{3.12}$$

where $N = KM$. In fact, FMDTM assumes the following solution:

$$y(t) = \begin{cases} y_1(t), & t \in [0, t_1] \\ y_2(t), & t \in [t_1, t_2] \\ \vdots & \\ y_M(t), & t \in [t_{M-1}, t_M] \end{cases} \tag{3.13}$$

The new algorithm, FMDTM, is proven simple for computational performance for all values of h.

4. Applications

Example 4.1. As our first example, we consider the homogeneous linear fractional-order differential equation [9]

$$D_t^{\alpha} x(t) = -x(t), \; x(0) = 1, \; t > 0 \tag{4.1}$$

for $0 < \alpha \leq 1$, with its corresponding exact solution

$$x(t) = -E_\alpha(t^\alpha), \tag{4.2}$$

where $E_\alpha(\cdot)$ is the one-parameter Mittag-Leffler function of order α. For the value of the fractional order we take $\alpha = 0.98$ as a particular example, but the conclusions of our numerical analysis holds also for other values of α close to, but less than, 1.

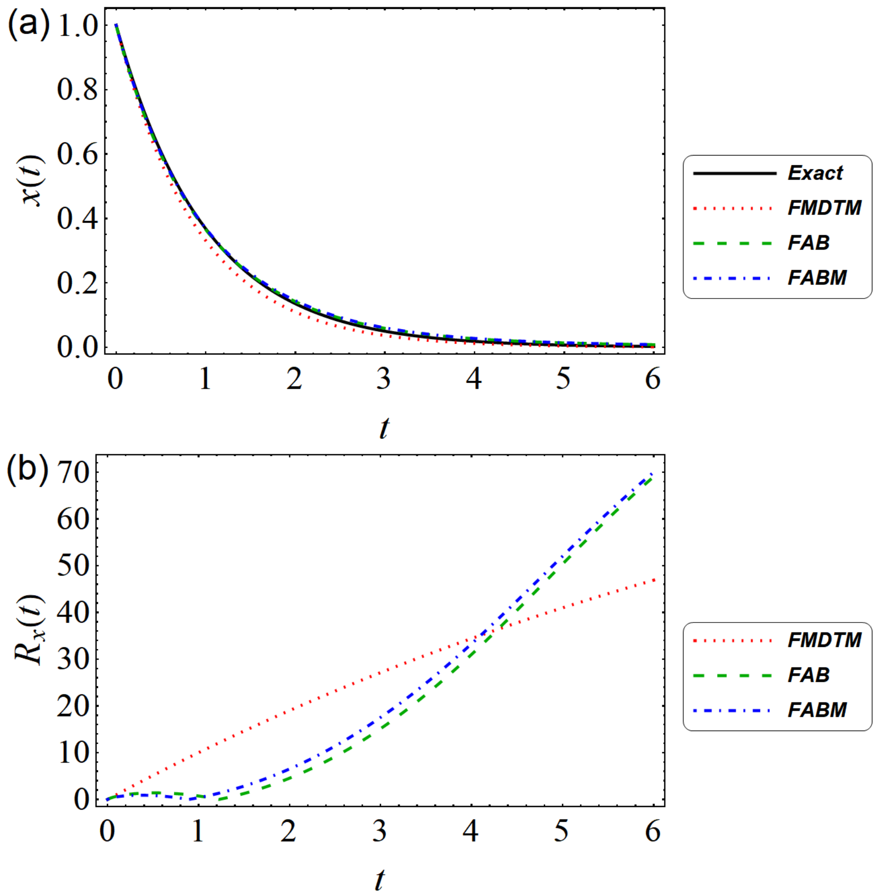

FIGURE 1. (a). Time-series $x(t)$ versus t of the exact integration curve of Eq. (4.1) (black/solid line) and the approximation curves for FMDTM (dotted/red line), FAB (dashed/green line), and FABM (dot-dashed/blue line); (b). Relative difference $R_x(t)$ corresponding to FMDTM (dotted/red line), FAB (dashed/green line), and FABM (dot-dashed/blue line). The fractional-order parameter is $\alpha = 0.98$. In all simulations, initial condition $x(0) = 1$, integration step size $h = 0.01$ and $t \in [0, 6]$.

In Fig. 1a we depict the plots of $x(t)$ versus the time t of the initial value problem (4.1) consisting of the exact integration curve (4.2) as well as the approximation time-series obtained from the three different numerical methods. The exact curve is depicted with a black (solid) line, and the dotted (red) line, the dashed (green) line and the dash-dotted (blue) line are the approximation time-series obtained with FMDTM, FAB, FABM respectively. In all numerical simulations, we take the integration step-size $h = 0.01$, and the integration time-span is $t \in [0, 6]$. It is seen that all the curves are characterized with a typical exponential-like decrease as $t \to \infty$. The deviation of the approximation curves from the exact one can hardly be perceived from the figure. To characterise this deviation, and hence the quality of particular approximation method, we introduce the relative difference parameter $R(t)$ defined as the absolute difference between the exact value x_{exact} of the solution curve $x(t)$ at the fixed time t and the approximate value x_{approx} at the same instant of time divided by the maximum absolute value of these two numbers (expressed in percents % by multiplying with 100):

$$R_x(t) = \frac{|x_{exact}(t) - x_{approx}(t)|}{\max\{|x_{exact}(t)|, |x_{approx}(t)|\}} \cdot 100\%. \tag{4.3}$$

In Fig. 1b we show the calculated plots of the relative difference $R_x(t)$ corresponding to FMDTM (dotted/red line), FAB (dashed/green line), and FABM (dot-dashed/blue line). Apart from a slight non-monotonic behavior at a small time interval at the begining, it is seen that all the curves $R_x(t)$ are monotonically increasing as the integration time becomes larger. This observation suggests that all three numerical methods gradually diverge from the exact solution with increasing integration time, and this divergence quickly becomes significant. Although all three methods approximate the exact solution in a practical range of relative deviations, in this case FMTDM is better approximation than FAB and FABM at large time points. These conclusions can also be extracted by looking at the tabluar values of the exact time-series x_{exact} and the apprroximation time series x_{FMDTM}, x_{FAB} and x_{FABM}, along with the corresponding relative difference parameters R_{FMDTM}, R_{FAB} and R_{FABM}. The results are given in Table 1, where for clarity we show only the time-series at the beginning and at the end of the integration time-span.

Example 4.2. Our second example is the two-dimensional linear fractional differential system [10]

$$\begin{cases} D_t^\alpha x(t) &= 2x(t) - y(t) \\ D_t^\alpha y(t) &= 4x(t) - 3y(t) \end{cases}, \tag{4.4}$$

where $0 < \alpha \le 1$. The system can be re-written in the matrix form

$$\begin{pmatrix} D^\alpha x(t) \\ D^\alpha y(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}. \tag{4.5}$$

The eigenvalues of the matrix $\begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$ are $\lambda_1 = 1$ and $\lambda_2 = -2$, and their corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ respectively. Therefore,

TABLE 1. Partial data values of the time-series $x(t)$ for increasing t of the exact solution of Eq. (4.1) and the numerical approximation values for FMDTM, FAB, and FABM, as well as the corresponding values for the relative difference $R_x(t)$ for each numerical method.

t	x_{exact}	x_{FMDTM}	x_{FAB}	x_{FABM}	R_{FMDTM}	R_{FAB}	R_{FABM}
0	1	1	1	1	0	0	0
0.01	0.99005	0.989006	0.988882	0.989005	0.105467	0.117977	0.105505
0.02	0.980199	0.978132	0.978191	0.978432	0.210822	0.204805	0.180233
0.03	0.970446	0.967378	0.967728	0.968082	0.316067	0.280027	0.243503
0.04	0.960789	0.956743	0.957449	0.957914	0.4212	0.347697	0.29932
0.05	0.951229	0.946224	0.947332	0.947904	0.526223	0.409726	0.349573
0.06	0.941765	0.935821	0.937364	0.938041	0.631135	0.467235	0.395369
0.07	0.932394	0.925532	0.927536	0.928315	0.735936	0.520961	0.437437
0.08	0.923116	0.915356	0.917841	0.91872	0.840626	0.571428	0.476291
0.09	0.913931	0.905293	0.908274	0.909249	0.945207	0.619025	0.512318
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5.9	0.00273944	0.00146987	0.00838373	0.00865573	46.3443	67.3243	68.3511
5.91	0.00271219	0.00145371	0.00834439	0.00861477	46.4009	67.4969	68.517
5.92	0.0026852	0.00143773	0.00830535	0.00857412	46.4574	67.669	68.6825
5.93	0.00265848	0.00142192	0.00826661	0.00853378	46.5139	67.8407	68.8476
5.94	0.00263203	0.00140629	0.00822816	0.00849375	46.5703	68.0119	69.0122
5.95	0.00260584	0.00139082	0.00819001	0.00845402	46.6266	68.1827	69.1763
5.96	0.00257991	0.00137553	0.00815215	0.00841459	46.6829	68.353	69.34
5.97	0.00255424	0.00136041	0.00811458	0.00837547	46.7392	68.5228	69.5033
5.98	0.00252883	0.00134545	0.00807729	0.00833663	46.7953	68.6921	69.6661
5.99	0.00250366	0.00133066	0.00804028	0.00829809	46.8515	68.861	69.8284
6.	0.00247875	0.00131603	0.00800355	0.00825984	46.9075	69.0294	69.9903

the general solution of the system (4.4) is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 v_1 E_\alpha(t^\alpha) + c_2 v_2 E_\alpha(-2t^\alpha), \tag{4.6}$$

where c_1 and c_2 are arbitrary constants. In particular, the initial value problem of system (4.4) for $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1.2 \\ 4.2 \end{pmatrix}$ has the unique solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ 1 \end{pmatrix} E_\alpha(t^\alpha) + \begin{pmatrix} 1 \\ 4 \end{pmatrix} E_\alpha(-2t^\alpha). \tag{4.7}$$

Figure 2a shows the plots of $x(t)$ versus the time t of the system (4.4) of the exact integration curve (black/solid line) and the approximation time-series for FMDTM (red/dotted line), FAB (green/dashed line), and FABM (blue/dot-dashed line). The value of the fractional order is taken $\alpha = 0.98$ as in the previous example. In all numerical simulations, we take the integration step-size $h = 0.01$, and the integration time-span is $t \in [0, 6]$. Contrary to the previous example, the curves are now increasing towards infinity as $t \to \infty$, suggesting that the system is unstable. The deviation of the approximation curves from the exact curve is obvious at high values of t, which is also seen from the depicted relative difference parameter $R_x(t)$ in Fig. 2b. The monotonical increase of the relative difference parameter at large values of t is also observed in the current case, and as before, we have the same non-monotonic behavior at a small time-window in the beginning. The numerical methods gradually diverge from the exact solution as the time rises, and in this case

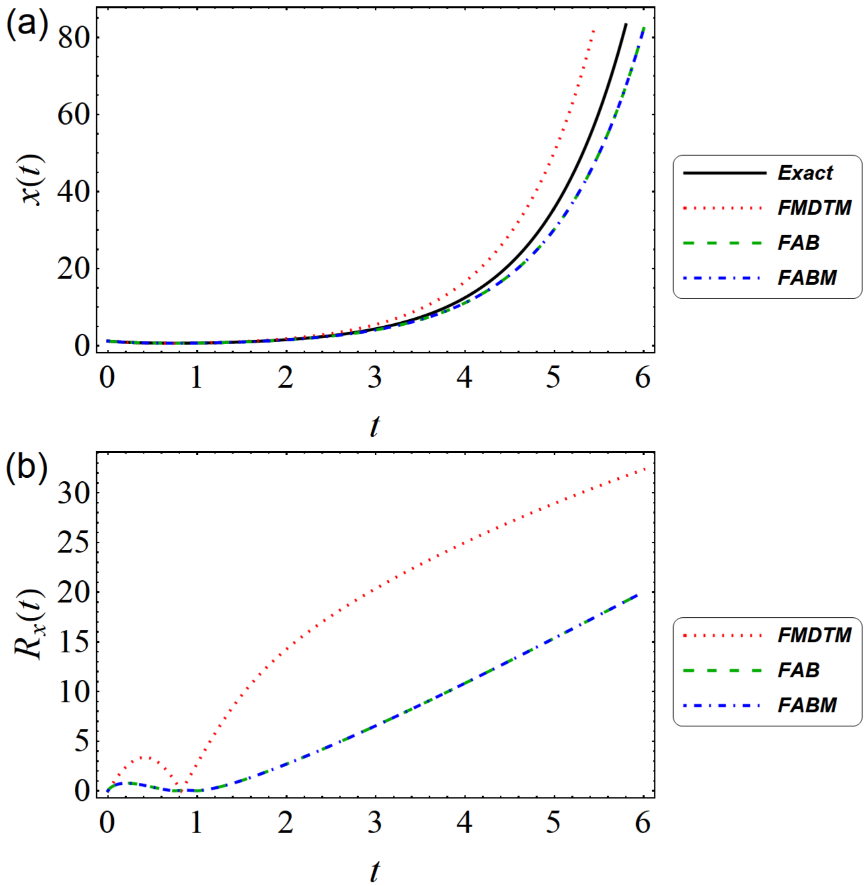

FIGURE 2. (a). Time-series $x(t)$ versus t of the exact integration curve of Eq. (4.4) (black/solid line) and the approximation curves for FMDTM (dotted/red line), FAB (dashed/green line), and FABM (dot-dashed/blue line); (b). Relative difference $R_x(t)$ corresponding to FMDTM (dotted/red line), FAB (dashed/green line), and FABM (dot-dashed/blue line). The fractional-order parameter is $\alpha = 0.98$. Initial conditions $x(0) = 1.2$ and $y(0) = 4.2$, integration step size $h = 0.01$ and $t \in [0, 6]$.

FAB and FABM are slightly better approximations than FMTDM at large time values. The tabluar values of the exact time-series x_{exact} and the apprroximation time series x_{FMDTM}, x_{FAB} and x_{FABM}, along with the corresponding relative difference parameters R_{FMDTM}, R_{FAB} and R_{FABM} are provided in Table 2.

The corresponding plots for the second variable $y(t)$ of the system (4.4) are provided in Figs. 3a and 3b, and the tabular values are given in Table 3. The

TABLE 2. Partial data values of the time-series $x(t)$ for increasing t of the exact solution of system Eq. (4.4) and the numerical approximation values for FMDTM, FAB, and FABM, as well as the corresponding values for the relative difference $R_x(t)$ for each numerical method.

t	x_{exact}	x_{FMDTM}	x_{FAB}	x_{FABM}	R_{FMDTM}	R_{FAB}	R_{FABM}
0	1.2	1.2	1.2	1.2	0	0	0
0.01	1.18207	1.18036	1.18036	1.18036	0.144451	0.144514	0.144514
0.02	1.16456	1.16122	1.16175	1.16175	0.286765	0.241222	0.241222
0.03	1.14747	1.14257	1.1438	1.1438	0.426866	0.319623	0.319623
0.04	1.1308	1.12441	1.12643	1.12643	0.564677	0.385948	0.385948
0.05	1.11452	1.10672	1.10958	1.10958	0.700121	0.443174	0.443174
0.06	1.09865	1.08949	1.09323	1.09323	0.83312	0.493073	0.493073
0.07	1.08316	1.07272	1.07734	1.07734	0.963596	0.536822	0.536822
0.08	1.06805	1.05639	1.06191	1.06191	1.09147	0.575268	0.575268
0.09	1.05332	1.0405	1.0469	1.0469	1.21666	0.609052	0.609052
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5.9	92.612	136.264	74.4903	74.4903	32.0351	19.5673	19.5673
5.91	93.5971	137.78	75.2389	75.2389	32.0675	19.6141	19.6141
5.92	94.5927	139.312	75.995	75.995	32.0999	19.6608	19.6608
5.93	95.5989	140.861	76.7587	76.7587	32.1322	19.7075	19.7075
5.94	96.6158	142.427	77.5301	77.5301	32.1646	19.7542	19.7542
5.95	97.6436	144.01	78.3093	78.3093	32.1968	19.8009	19.8009
5.96	98.6824	145.612	79.0963	79.0963	32.229	19.8476	19.8476
5.97	99.7323	147.231	79.8912	79.8912	32.2612	19.8944	19.8944
5.98	100.793	148.868	80.6941	80.6941	32.2934	19.9411	19.9411
5.99	101.866	150.523	81.505	81.505	32.3255	19.9878	19.9878
6.	102.95	152.197	82.3241	82.3241	32.3576	20.0346	20.0346

conclusions are coincident with the ones we made for the variable $x(t)$ of the system.

5. Conclusions

Numerical solutions of the equations governing fractional-order dynamical systems are often the only approach to study the dynamical behavior of these systems at particular parameter values, since in general the exact solutions of differential equations of fractional-order cannot be sought in practice. Therefore, it is of a great significance to choose the right numerical approximation method to describe the correct behavior of the system under investigation [11, 12, 13].

In this paper, we apply three distinct numerical methods (FAB, FABM, and FMDTM) to numerically approximate two different linear systems of fractional differential equations in which fractional derivatives are taken in the sense of Caputo. The linearity of the systems allowed us to compare the numerical results for the three methods with the exact solution for each linear system. To characterise the successfulness of each numerical method, we use the relative difference parameter between the exact solution of each system at a given time point and the approximate solution at the same instance. From the resulting diagrams and tabular values we conclude that, aside from occasional non-monotoncity for small time values, all three numerical methods gradually diverge from the exact solution with increasing integration time. The superiority of each numerical method over the other methods depends on the particular system under investigation: in the

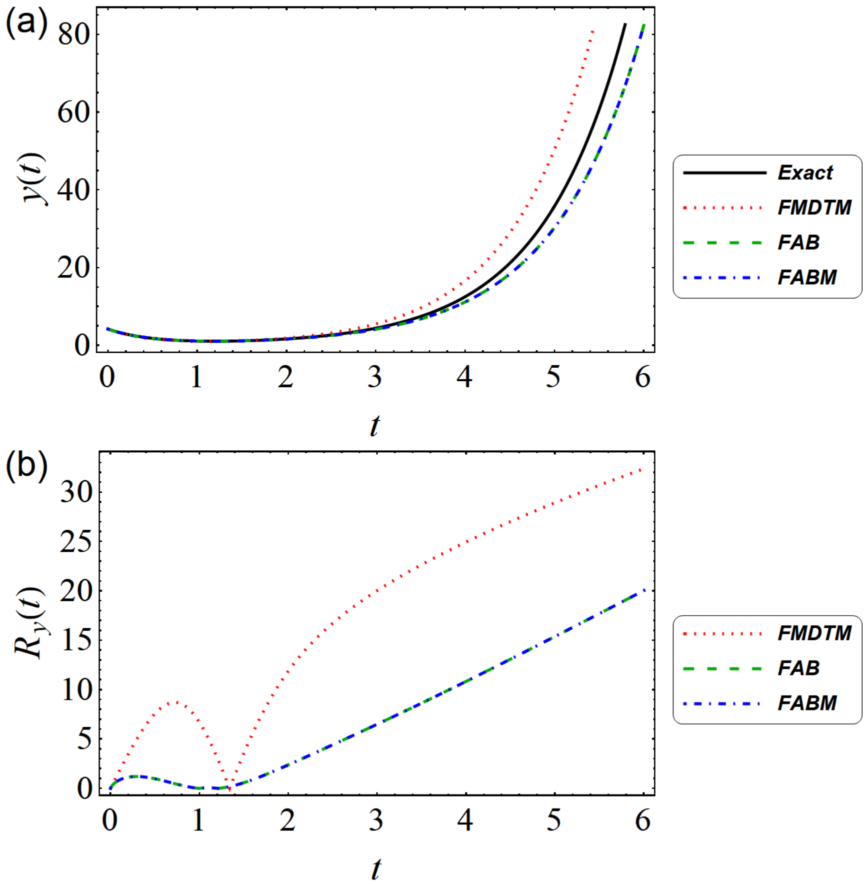

FIGURE 3. (a). Plots of $y(t)$ versus t of the exact integration curve of Eq. (4.4) (black/solid line) and the approximation curves for FMDTM (dotted/red line), FAB (dashed/green line), and FABM (dot-dashed/blue line); (b). Relative difference $R_y(t)$ corresponding to FMDTM (dotted/red line), FAB (dashed/green line), and FABM (dot-dashed/blue line). The fractional-order parameter is $\alpha = 0.98$. Initial conditions $x(0) = 1.2$ and $y(0) = 4.2$, integration step size $h = 0.01$ and $t \in [0, 6]$.

first case, FMTDM was shown to be a better approximation than FAB and FABM, and in the second case the situation has been reversed.

More detailed analysis of the three aforementioned numerical methods for additional fractional-order systems that can be solved analytically, and for different integration and system parameters, is an ongoing work in progress.

TABLE 3. Partial data values of the time-series $y(t)$ for increasing t of the exact solution of system (4.4) and the numerical approximation values for FMDTM, FAB, and FABM, as well as the corresponding values for the relative difference $R_y(t)$ for each numerical method.

t	y_{exact}	y_{FMDTM}	y_{FAB}	y_{FABM}	R_{FMDTM}	R_{FAB}	R_{FABM}
0	4.2	4.2	4.2	4.2	0	0	0
0.01	4.12218	4.11476	4.11476	4.11476	0.180004	0.180063	0.180063
0.02	4.04601	4.03146	4.03377	4.03377	0.359603	0.30257	0.30257
0.03	3.97145	3.95006	3.95542	3.95542	0.538766	0.403653	0.403653
0.04	3.89848	3.87051	3.87934	3.87934	0.717459	0.490837	0.490837
0.05	3.82705	3.79277	3.80532	3.80532	0.895647	0.567682	0.567682
0.06	3.75714	3.71681	3.73323	3.73323	1.07329	0.636283	0.636283
0.07	3.68872	3.64259	3.66297	3.66297	1.25037	0.698022	0.698022
0.08	3.62175	3.57007	3.59445	3.59445	1.42682	0.75388	0.75388
0.09	3.55621	3.49922	3.5276	3.5276	1.60262	0.804593	0.804593
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5.9	92.6183	136.264	74.4969	74.4969	32.0304	19.5657	19.5657
5.91	93.6034	137.78	75.2455	75.2455	32.0629	19.6124	19.6124
5.92	94.599	139.312	76.0016	76.0016	32.0954	19.6591	19.6591
5.93	95.6052	140.861	76.7653	76.7653	32.1278	19.7059	19.7059
5.94	96.6221	142.427	77.5367	77.5367	32.1602	19.7526	19.7526
5.95	97.6499	144.01	78.3159	78.3159	32.1925	19.7994	19.7994
5.96	98.6887	145.612	79.1028	79.1028	32.2248	19.8461	19.8461
5.97	99.7385	147.231	79.8977	79.8977	32.257	19.8928	19.8928
5.98	100.8	148.868	80.7006	80.7006	32.2892	19.9396	19.9396
5.99	101.872	150.523	81.5115	81.5115	32.3214	19.9863	19.9863
6.	102.956	152.197	82.3306	82.3306	32.3535	20.0331	20.0331

REFERENCES

[1] S. K. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, (1993), John Wylei and Sons, Inc., Canada.

[2] D. Baleanu, K. Deithelm, E. Scalas, J. J. Trujillo, *Fractional Calculus: models and numerical methods*, (2017), World Scientific, Singapore.

[3] L. Changpin, Z. Fanhai, *Numerical Methods for Fracional Calculus*, (2015), CRC Press Taylor and Francis Group, New York.

[4] K. Diethelm, N. J. Ford and A. D. Freed, *Detailed error analysis for a fractional Adams method*, Numerical Algorithms, 36, (2004), 31-52.

[5] K. Diethelm, N. J. Ford and A. D. Freed, *A Predictor-Corrector Approach for the Numerical Solution of Fractional Differential Equations*, Nonlinear Dynamics, 29, (2001), 3-22.

[6] C. Bervillier, *Status of the diffrential transformation method*, Applied Mathematics and Computation, 20, (2012), 10158-10170.

[7] S. V. Ertürk, Sh. Momani, *Solving systems of fractional differential equations using differential transform method*, Journal of Computational and Applied Mathematics, 215, (2008), 142-151.

[8] R. Garrappa, *On some explicit Adams multistep methods for fractional differential equations*, Journal of Computational and Applied Mathematics, 229, (2009), 392-399.

[9] Z. M. Odibat and S. Momani, *An algorithm for the numerical solution of differential equations of fractional order*, Journal of Applied Mathematics and Informatics, 26, (2008), 15-27.

[10] Z. M. Odibat, *Analytic study on linear systems of fractional differential equations*, Computers and Mathematics with Applications, 59, (2010), 1171-1183.

[11] G. Markoski, *Bifurcation Analysis of fractional-order Chaotic Rossler System*, Matematichki Bilten, 42, (2018), 28-37.

[12] Y. Seferi, G. Markoski and A. Gjurchinovski, *Comparison of different numerical methods for fractional differential equations*, Matematichki Bilten, 42, (2018), 61-74.

[13] Y. Seferi, G. Markoski and A. Gjurchinovski, *Comparison of two numerical methods for fractional-order Rossler system*, Matematichki Bilten, 44, (2020), 53-60.

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