Математички Билтен
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$\mathcal{G}s_\alpha$-OPEN SETS IN GRILL TOPOLOGICAL SPACES

DHANABAL SARAVANAKUMAR [1], NAGARAJAN KALAIVANI [2],
AND MOHAMED ROWTHER MOHAMED RIYAZDEEN [3]

Abstract. A non-null collection $\mathcal{G}$ of subsets of a topological space (X,τ) is said to be a grill on X if (i) $\emptyset \notin \mathcal{G}$, (ii) $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$, (ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$. A triple $(X,\tau,\mathcal{G})$ is called a grill topological space. A new class $\mathcal{G}s_\alpha O(X)$ of $\mathcal{G}s_\alpha$-open sets in a grill topological space with respect to $\mathcal{G}$ is introduced. Also, we define $\mathcal{G}s_\alpha$-continuous and $\mathcal{G}s_\alpha$-open(closed) functions and study some of their important properties. In addition, we introduce a $(\mathcal{G},\mathcal{G}')s_\alpha$-continuous function and investigate its properties.

1. Introduction and preliminaries

Levine[6] introduced the concepts of semiopen set and semicontinuitous in topological spaces. Nijastad[10] defined the concepts of α-open set, α-interior and α-closure in topological spaces. Mashhour et al.[9] introduced the concepts of α-continuous and α-open mappings in topological spaces. Mashhour et al., [8], introduced the concepts of preopen set, pre-interior and pre-closure in a topological spaces. A subset A in X is said to be α-open (resp. semiopen, preopen) if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$ (resp. $\mathrm{cl}(\mathrm{int}(A))$, $\mathrm{int}(\mathrm{cl}(A))$) and τ^α (resp. $SO(X)$, $PO(X)$) denotes the family of α-open (resp. semiopen, preopen) sets. For any subset A of X, (i) $\alpha\mathrm{int}(A)$ (resp. $\mathrm{sint}(A)$, $\mathrm{pint}(A)$) $= \cup\{U : U \in \tau^\alpha$ (resp. $SO(X)$, $PO(X)$) and $U \subseteq A\}$, (ii) $\alpha\mathrm{cl}(A)$ (resp. $\mathrm{scl}(A)$, $\mathrm{pcl}(A)$) $= \cap\{F : X - F \in \tau^\alpha$ (resp. $SO(X)$, $PO(X)$) and $A \subseteq F\}$.

Choquet, [2], introduced the concept of grill on a topological space and the idea of grills has shown to be a essential tool for studying some topological concepts. A non-null collection $\mathcal{G}$ of subsets of a topological space (X,τ) is called a *grill* on X if (i) $\emptyset \notin \mathcal{G}$, (ii) $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$, (iii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$. A triple $(X,\tau,\mathcal{G})$ is called a *grill topological space*.

Roy and Mukherjee, [12], defined the mappings Φ and Ψ and introduced a

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unique topology $\tau_{\mathcal{G}}$ by a grill in topological spaces. For any point $x \in X$, $\tau(x)$ denotes the collection of all open neighborhoods of x. A mapping $\Phi : P(X) \to P(X)$ is defined by $\Phi(A) = \{x \in X : A \cap U \in \mathcal{G}$, for all $U \in \tau(x)\}$ and $A \in P(X)$. A mapping $\Psi : P(X) \to P(X)$ is defined by $\Psi(A) = A \cup \Phi(A)$ for all $A \in P(X)$. Kuratowski, [4], defined the concept of Kuratowski closure axioms. For $A, B \subseteq X$, the mapping Ψ satisfies Kuratowski closure axioms: (i) $\Psi(\emptyset) = \emptyset$, (ii) if $A \subseteq B$, then $\Psi(A) \subseteq \Psi(B)$, (iii) $\Psi(\Psi(A)) = \Psi(A)$ and (iv) $\Psi(A \cup B) = \Psi(A) \cup \Psi(B)$. Corresponding to a grill $\mathcal{G}$ on a topological space (X, τ), there exists a unique topology $\tau_{\mathcal{G}}$ on X given by $\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X - U) = X - U\}$, where for any $A \subseteq X$, $\Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}}\mathrm{cl}(A)$ and $\tau \subseteq \tau_{\mathcal{G}}$.

The concept of decomposition of continuity in topological spaces and some classes of sets were defined (see [1, 3, 5, 7, 9, 11, 13, 14] for details). A subset A in X is said to be (i) Φ-open if $A \subseteq \mathrm{int}(\Phi(A))$, (ii) $\mathcal{G}$-α.open if $A \subseteq \mathrm{int}(\Psi(\mathrm{int}(A)))$, (iii) $\mathcal{G}$-preopen if $A \subseteq \mathrm{int}(\Psi(A))$, (iv) $\mathcal{G}$-semiopen if $A \subseteq \Psi(\mathrm{int}(A))$, (v) $\mathcal{G}$-β.open if $A \subseteq \mathrm{cl}(\mathrm{int}(\Psi(A)))$, (vi) $\mathcal{G}s_p$-open if $A \subseteq \Psi(\mathrm{pint}(A))$. A subset A of X is called Φ-*closed* (resp. $\mathcal{G}$-α.*closed*, $\mathcal{G}$-*preclosed*, $\mathcal{G}$-*semiclosed*, $\mathcal{G}$-β.*closed*, $\mathcal{G}s_p$-*closed*) if its complement $X - A$ is Φ-open (resp. $\mathcal{G}$-α.open, $\mathcal{G}$-preopen, $\mathcal{G}$-semiopen, $\mathcal{G}$-β.open, $\mathcal{G}s_p$-open). The family of all Φ-open (resp. $\mathcal{G}$-α.open, $\mathcal{G}$-preopen, $\mathcal{G}$-semiopen, $\mathcal{G}$-β.open, $\mathcal{G}s_p$-open) sets is denoted by $\Phi O(X)$ (resp. $\mathcal{G}\alpha O(X)$, $\mathcal{G}PO(X)$, $\mathcal{G}SO(X)$, $\mathcal{G}\beta O(X)$, $\mathcal{G}s_pO(X)$). A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be $\mathcal{G}$-*semicontinuous* if $f^{-1}(V) \in \mathcal{G}SO(X)$ for each $V \in \sigma$.

In this paper, we define the concept of $\mathcal{G}s_\alpha$-open set in a grill topological space $(X, \tau, \mathcal{G})$. Also, we introduce $\mathcal{G}s_\alpha$-interior and $\mathcal{G}s_\alpha$-closure and we study some of their basic properties. Further, we define $\mathcal{G}s_\alpha$-continuous, $\mathcal{G}s_\alpha$-open, $\mathcal{G}s_\alpha$-closed and $\mathcal{G}S^*$-continuous functions in a grill topological space $(X, \tau, \mathcal{G})$ and we investigate some of their fundamental properties. Moreover, we introduce a $(\mathcal{G}, \mathcal{G}')s_\alpha$-continuous function and we show that every $(\mathcal{G}, \mathcal{G}')s_\alpha$-continuous function is $\mathcal{G}s_\alpha$-continuous, but the converse need not to be true.

Proposition 1.1. [12] *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then for all $A, B \subseteq X$:*

(*i*) *$A \subseteq B$ implies that $\Phi(A) \subseteq \Phi(B)$,*

(*ii*) *$\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$,*

(*iii*) *$\Phi(\Phi(A)) \subseteq \Phi(A) = cl(\Phi(A)) \subseteq cl(A)$.*

2. $\mathcal{G}s_\alpha$-OPEN SET IN GRILL TOPOLOGICAL SPACES

Definition 2.1. Let $(X, \tau, \mathcal{G})$ be a grill topological space and let A be a subset of X. Then A is said to be $\mathcal{G}s_\alpha$-*open* if and only if there exists a $U \in \tau^\alpha$ such that $U \subseteq A \subseteq \Psi(U)$. A set A of X is $\mathcal{G}s_\alpha$-*closed* if its complement $X - A$ is $\mathcal{G}s_\alpha$-open. The family of all $\mathcal{G}s_\alpha$-open (resp. $\mathcal{G}s_\alpha$-closed) sets is denoted by $\mathcal{G}s_\alpha O(X)$ (resp. $\mathcal{G}s_\alpha C(X)$).

Example 2.1. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{b\}\}$ and $\mathcal{G} = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then $\mathcal{G}s_\alpha O(X) = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$.

Theorem 1. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$. Then $A \in \mathcal{G}s_\alpha O(X)$ if and only if $A \subseteq \Psi(\alpha int(A))$.*

Proof. If $A \in \mathcal{G}s_\alpha O(X)$, then there exists a $U \in \tau^\alpha$ such that $U \subseteq A \subseteq \Psi(U)$. But $U \subseteq A$ implies that $U \subseteq \alpha\text{int}(A)$. Hence $\Psi(U) \subseteq \Psi(\alpha\text{int}(A))$. Therefore $A \subseteq \Psi(\alpha\text{int}(A))$. Conversely, let $A \subseteq \Psi(\alpha\text{int}(A))$. To prove that $A \in \mathcal{G}s_\alpha O(X)$, take $U = \alpha\text{int}(A)$, then $U \subseteq A \subseteq \Psi(U)$. Hence $A \in \mathcal{G}s_\alpha O(X)$. □

Corollary 1.1. *If $A \subseteq X$, then $A \in \mathcal{G}s_\alpha O(X)$ if and only if $\Psi(A) = \Psi(\alpha int(A))$.*

Proof. Let $A \in \mathcal{G}s_\alpha O(X)$. Then as Ψ is monotonic and idempotent, $\Psi(A) \subseteq \Psi(\Psi(\alpha\text{int}(A))) = \Psi(\alpha\text{int}(A)) \subseteq \Psi(A)$ implies that $\Psi(A) = \Psi(\alpha\text{int}(A))$. The converse is trivial. □

Corollary 1.2. *If $A \subseteq X$, then $\Psi(\alpha int(A)) \in \mathcal{G}s_\alpha O(X)$.*

Proof. Clearly $\Psi(\alpha\text{int}(A)) = \Psi(\alpha\text{int}(\alpha\text{int}(A)) \subseteq \Psi(\alpha\text{int}(\Psi(\alpha\text{int}(A))))$. Then by Theorem 1, $\Psi(\alpha\text{int}(A)) \in \mathcal{G}s_\alpha O(X)$. □

Theorem 2. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. If $A \in \mathcal{G}s_\alpha O(X)$ and $B \subseteq X$ such that $A \subseteq B \subseteq \Psi(\alpha int(A))$, then $B \in \mathcal{G}s_\alpha O(X)$.*

Proof. Given $A \in \mathcal{G}s_\alpha O(X)$. Then by Theorem 1, $A \subseteq \Psi(\alpha\text{int}(A))$. But $A \subseteq B$ implies that $\alpha\text{int}(A) \subseteq \alpha\text{int}(B)$, hence $\Psi(\alpha\text{int}(A)) \subseteq \Psi(\alpha\text{int}(B))$. Therefore, $B \subseteq \Psi(\alpha\text{int}(A)) \subseteq \Psi(\alpha\text{int}(B))$. Hence, $B \in \mathcal{G}s_\alpha O(X)$. □

Corollary 2.1. *If $A \in \mathcal{G}s_\alpha O(X)$ and $B \subseteq X$ such that $A \subseteq B \subseteq \Psi(A)$, then $B \in \mathcal{G}s_\alpha O(X)$.*

Proof. Follows from the Theorem 2 and Corollary 1.1. □

Theorem 3. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. If $A \in \mathcal{G}SO(X)$, then $A \in \mathcal{G}s_\alpha O(X)$.*

Proof. Let $A \in \mathcal{G}SO(X)$. Then $A \subseteq \Psi(\text{int}(A))$. Since $\text{int}(A) \subseteq \alpha\text{int}(A)$, we have that $\Psi(\text{int}(A)) \subseteq \Psi(\alpha\text{int}(A))$ (by Theorem 2.4, [12]). Hence $A \subseteq \Psi(\alpha\text{int}(A))$ and thus $A \in \mathcal{G}s_\alpha O(X)$. □

Note that from the Definition 2.1, Theorem 3, Corollary 1, [10] and Remark 1, [11], we have that

(*i*) $\tau \subseteq \mathcal{G}\alpha O(X) \subseteq \tau^\alpha \subseteq \mathcal{G}s_\alpha O(X) \subseteq \mathcal{G}s_p O(X)$,

(*ii*) $\mathcal{G}\alpha O(X) \subseteq \mathcal{G}SO(X) \subseteq \mathcal{G}s_\alpha O(X)$.

But the converse need not be true as shown by the examples below.

Example 2.2. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then:

(*i*) $A = \{a, b, c\} \in \mathcal{G}\alpha O(X)$, but $A \notin \tau$,

(*ii*) $B = \{a, c\} \in \mathcal{G}SO(X)$, but $B \notin \mathcal{G}\alpha O(X)$,

(*iii*) $C = \{b, d\} \in \mathcal{G}s_p O(X)$, but $C \notin \mathcal{G}s_\alpha O(X)$, (iv) $D = \{a, c\} \in \mathcal{G}s_\alpha O(X)$, but $D \notin \tau^\alpha$.

Example 2.3. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{a\}\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

Then $A = \{a, b\} \in \mathcal{G}s_\alpha O(X)$, but $A \notin \mathcal{G}SO(X)$.

Theorem 4. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. If $\tau^\alpha = \tau$, then $\mathcal{G}s_\alpha O(X) = \mathcal{G}SO(X)$.*

Proof. By Theorem 3, $\mathcal{G}SO(X) \subseteq \mathcal{G}s_\alpha O(X)$. Let $A \in \mathcal{G}s_\alpha O(X)$. Then by Theorem 1, $A \subseteq \Psi(\alpha\text{int}(A))$. Since $\tau^\alpha = \tau$, we have that $\alpha\text{int}(A) = \text{int}(A)$ implies that $A \subseteq \Psi(\alpha\text{int}(A)) = \Psi(\text{int}(A))$ and hence $A \in \mathcal{G}SO(X)$. Thus $\mathcal{G}s_\alpha O(X) \subseteq \mathcal{G}SO(X)$. □

Theorem 5. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then the following conditions hold:*

(i) for each $\alpha \in J$, if $A_\alpha \in \mathcal{G}s_\alpha O(X)$, then $\bigcup_{\alpha\in J} A_\alpha \in \mathcal{G}s_\alpha O(X)$,
(ii) if $A \in \mathcal{G}s_\alpha O(X)$ and $U \in \tau^\alpha$, then $A \cap U \in \mathcal{G}s_\alpha O(X)$.

Proof. (i) Suppose $A_\alpha \in \mathcal{G}s_\alpha O(X)$, for each $\alpha \in J$. Then $A_\alpha \subseteq \Psi(\alpha\text{int}(A_\alpha))$, for each $\alpha \in J$, implies that $\bigcup_{\alpha\in J} A_\alpha \subseteq \bigcup_{\alpha\in J} \Psi(\alpha\text{int}(A_\alpha)) \subseteq \Psi(\alpha\text{int}(\bigcup_{\alpha\in J} A_\alpha))$. Therefore, $\bigcup_{\alpha\in J} A_\alpha \in \mathcal{G}s_\alpha O(X)$.

(ii) Let $A \in \mathcal{G}s_\alpha O(X)$ and $U \in \tau^\alpha$. Then $A \subseteq \Psi(\alpha\text{int}(A))$. Now, $A \cap U \subseteq \Psi(\alpha\text{int}(A)) \cap U = (\alpha\text{int}(A) \cup \Phi(\alpha\text{int}(A))) \cap U = (\alpha\text{int}(A) \cap U) \cup (\Phi(\alpha\text{int}(A)) \cap U) \subseteq \alpha\text{int}(A \cap U) \cup \Phi(\alpha\text{int}(A) \cap U)$ (by Theorem 2.10, [12]) $= \alpha\text{int}(A \cap U) \cup \Phi(\alpha\text{int}(A \cap U)) = \Psi(\alpha\text{int}(A \cap U))$. Therefore, $A \cap U \in \mathcal{G}s_\alpha O(X)$. □

Remark 2.1: The following example shows that if $A, B \in \mathcal{G}s_\alpha O(X)$, then $A \cap B \notin \mathcal{G}s_\alpha O(X)$.

Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}\}$ and $\mathcal{G} = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $\mathcal{G}s_\alpha O(X) = \{\emptyset, X, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Consider $A = \{a, c\}$ and $B = \{a, d\}$. Then $A, B \in \mathcal{G}s_\alpha O(X)$, but $A \cap B = \{a\} \notin \mathcal{G}s_\alpha O(X)$.

Theorem 6. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$. If $A \in \mathcal{G}s_\alpha C(X)$, then $\alpha int(\Psi(A)) \subseteq A$.*

Proof. Suppose $A \in \mathcal{G}s_\alpha C(X)$. Then $X - A \in \mathcal{G}s_\alpha O(X)$ and hence $X - A \subseteq \Psi(\alpha\text{int}(X - A)) \subseteq \alpha\text{cl}(\alpha\text{int}(X - A)) = X - \alpha\text{int}(\alpha\text{cl}(A)) \subseteq X - \alpha\text{int}\Psi(A))$, implies that $\alpha\text{int}(\Psi(A)) \subseteq A$. □

Remark 2.2: For $A \subseteq X$, the following example shows that:
(i) if $\alpha\text{int}(\Psi(A)) \subseteq A$, then $A \notin \mathcal{G}s_\alpha C(X)$;
(ii) $\alpha\text{int}(\Psi(A)) \notin \mathcal{G}s_\alpha C(X)$.
(i) Take $A = \{a\}$ in Example 2.3. Then $\alpha\text{int}(\Psi(\{a\})) = \{a\} \subseteq \{a\}$. Therefore, $\alpha\text{int}(\Psi(A)) \subseteq A$, but $A \notin \mathcal{G}s_\alpha C(X)$.
(ii) From (i), $\alpha\text{int}(\Psi(\{a\})) = \{a\} \notin \mathcal{G}s_\alpha C(X)$. Thus, $\alpha\text{int}(\Psi(A)) \notin \mathcal{G}s_\alpha C(X)$.

Theorem 7. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$ such that*

$$X - \alpha int(\Psi(A)) = \Psi(\alpha int(X - A)).$$

Then the following conditions hold:
(i) $A \in \mathcal{G}s_\alpha C(X)$ if and only if $\alpha int(\Psi(A)) \subseteq A$;
(ii) $\alpha int(\Psi(A)) \in \mathcal{G}s_\alpha C(X)$.

Proof. (i) Necessary part is proved by Theorem 6. Conversely, suppose that $\alpha\text{int}(\Psi(A)) \subseteq A$. Then $X - A \subseteq X - \alpha\text{int}(\Psi(A)) = \Psi(\alpha\text{int}(X - A))$, implies that $X - A \in \mathcal{G}s_\alpha O(X)$. Hence, $A \in \mathcal{G}s_\alpha C(X)$.
(ii) Follows from (i). □

Theorem 8. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. If $A_\alpha \in \mathcal{G}s_\alpha C(X)$ for each $\alpha \in J$, then $\bigcap_{\alpha\in J} A_\alpha \in \mathcal{G}s_\alpha C(X)$.*

Proof. Let $A_\alpha \in \mathcal{G}s_\alpha C(X)$. Then $X - A_\alpha \in \mathcal{G}s_\alpha O(X)$. By Theorem 5 (i), $\bigcup_{\alpha\in J}(X - A_\alpha) \in \mathcal{G}s_\alpha O(X)$. This implies that $\bigcup_{\alpha\in J}(X - A_\alpha) = X - \bigcap_{\alpha\in J} A_\alpha \in \mathcal{G}s_\alpha O(X)$ and hence $\bigcap_{\alpha\in J} A_\alpha \in \mathcal{G}s_\alpha C(X)$. □

Definition 2.2. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$. Then
(i) *$\mathcal{G}s_\alpha$-interior of A* is defined as union of all $\mathcal{G}s_\alpha$-open sets contained in A. Thus $\mathcal{G}s_\alpha\text{int}(A) = \cup\{U : U \in \mathcal{G}s_\alpha O(X) \text{ and } U \subseteq A\}$,
(ii) *$\mathcal{G}s_\alpha$-closure of A* is defined as intersection of all $\mathcal{G}s_\alpha$-closed sets containing A. Thus $\mathcal{G}s_\alpha\text{cl}(A) = \cap\{F : X - F \in \mathcal{G}s_\alpha O(X) \text{ and } A \subseteq F\}$.

Theorem 9. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$. Then the following conditions hold:*

(i) *$\mathcal{G}s_\alpha int(A)$ is a $\mathcal{G}s_\alpha$-open set contained in A,*
(ii) *$\mathcal{G}s_\alpha cl(A)$ is a $\mathcal{G}s_\alpha$-closed set containing A,*
(iii) *A is $\mathcal{G}s_\alpha$-closed if and only if $\mathcal{G}s_\alpha cl(A) = A$,*
(iv) *A is $\mathcal{G}s_\alpha$-open if and only if $\mathcal{G}s_\alpha int(A) = A$,*
(v) *$\mathcal{G}s_\alpha int(\mathcal{G}s_\alpha int(A)) = \mathcal{G}s_\alpha int(A)$,*
(vi) *$\mathcal{G}s_\alpha cl(\mathcal{G}s_\alpha cl(A)) = \mathcal{G}s_\alpha cl(A)$,*
(vii) *$\mathcal{G}s_\alpha int(A) = X - \mathcal{G}s_\alpha cl(X - A)$,*
$(viii)$ *$\mathcal{G}s_\alpha cl(A) = X - \mathcal{G}s_\alpha int(X - A)$.*

Proof. (i) Follows from the Definition 2.2 (i) and Theorem 5 (i).
(ii) Follows from the Definition 2.2 (ii) and Theorem 8.
(iii) Follows from the condition (ii) and Definition 2.2 (ii).
(iv) Follows from the condition (i) and Definition 2.2 (i).
(v) Follows from the conditions (i) and (iv).
(vi) Follows from the conditions (ii) and (iii).
(vii) and (viii) Follows from the Definitions 2.1 and 2.2 (i), (ii).

□

Theorem 10. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A, B \subseteq X$. Then the following conditions hold:*

(i) *if $A \subseteq B$, then $\mathcal{G}s_\alpha int(A) \subseteq \mathcal{G}s_\alpha int(B)$;*
(ii) *if $A \subseteq B$, then $\mathcal{G}s_\alpha cl(A) \subseteq \mathcal{G}s_\alpha cl(B)$;*
(iii) *$\mathcal{G}s_\alpha int(A \cup B) \supseteq \mathcal{G}s_\alpha int(A) \cup \mathcal{G}s_\alpha int(B)$;*
(iv) *$\mathcal{G}s_\alpha cl(A \cap B) \subseteq \mathcal{G}s_\alpha cl(A) \cap \mathcal{G}s_\alpha cl(B)$;*
(v) *$\mathcal{G}s_\alpha int(A \cap B) \subseteq \mathcal{G}s_\alpha int(A) \cap \mathcal{G}s_\alpha int(B)$;*
(vi) *$\mathcal{G}s_\alpha cl(A \cup B) \supseteq \mathcal{G}s_\alpha cl(A) \cup \mathcal{G}s_\alpha cl(B)$.*

Proof. (i) and (ii) follows from the Definitions 2.2 (i) and 2.2 (ii), respectively.
(iii) and (iv) follows from the condition (i), Theorem 5 (i) and the condition (ii), Theorem 8, respectively.
(v) and (vi) follows from the conditions (i) and (ii), respectively. □

Theorem 11. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$. Then the following conditions hold:*

(i) *If $\Psi(\alpha int(A)) \subseteq A$, then $\Psi(\alpha int(A)) \subseteq \mathcal{G}s_\alpha int(A)$.*
(ii) *If $A \subseteq X$ and $X - \alpha int(\Psi(A)) = \Psi(\alpha int(X - A))$, then $\mathcal{G}s_\alpha cl(A) \subseteq \alpha int(\Psi(A))$.*

Proof. (i) Since $\mathcal{G}s_\alpha\text{int}(A)$ is the greatest $\mathcal{G}s_\alpha$-open set containing A and Corollary 2.1 shows that $\Psi(\alpha\text{int}(A)) \in \mathcal{G}s_\alpha O(X)$. Therefore $\Psi(\alpha\text{int}(A)) \subseteq \mathcal{G}s_\alpha\text{int}(A)$.
(ii) Since $\mathcal{G}s_\alpha\text{cl}(A)$ is the least $\mathcal{G}s_\alpha$-closed set containing A and Theorem 7 (ii) shows that $\alpha\text{int}(\Psi(A)) \in \mathcal{G}s_\alpha C(X)$. Therefore $\mathcal{G}s_\alpha\text{cl}(A) \subseteq \alpha\text{int}(\Psi(A))$. □

Definition 2.3. Let $(X, \tau, \mathcal{G})$ be a grill topological space and (Y, σ) a topological space. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be $\mathcal{G}s_\alpha$*-continuous* if $f^{-1}(V) \in \mathcal{G}s_\alpha O(X)$, for each $V \in \sigma^\alpha$.

Example 2.4. Let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{b\}\}$, $\sigma = \{\emptyset, Y, \{1, 2\}\}$ and $\mathcal{G} = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then $\mathcal{G}s_\alpha O(X) = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma^\alpha = \{\emptyset, Y, \{1, 2\}\}$. Define $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ by $f(a) = 3, f(b) = 1$, and $f(c) = 2$. Then inverse image of every α-open sets in Y is $\mathcal{G}s_\alpha$-open in X. Hence, f is $\mathcal{G}s_\alpha$-continuous.

Remark 2.3: The concepts of $\mathcal{G}$-semicontinuous and $\mathcal{G}s_\alpha$-continuous are independent.

(i) Let $X = \{a, b, c, d\}, Y = \{1, 2, 3, 4\}, \tau = \{\emptyset, X, \{a\}\}, \sigma = \{\emptyset, Y, \{4\}, \{3, 4\}, \{1, 3, 4\}\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $\mathcal{G}SO(X) = \{\emptyset, X, \{a\}\}$, $\mathcal{G}s_\alpha O(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$ and $\sigma^\alpha = \{\emptyset, Y, \{4\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. Define $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ by $f(a) = 4, f(b) = 2, f(c) = 3$ and $f(d) = 1$. Then the function f is $\mathcal{G}s_\alpha$-continuous. Also, $f^{-1}(\{3, 4\}) = \{a, c\}$ is not $\mathcal{G}$-semiopen in X for the open set $\{3, 4\}$ of Y. Hence f is not $\mathcal{G}$-semicontinuous.

(ii) Let $X = \{a, b, c, d\}, Y = \{1, 2, 3, 4\}, \tau = \{\emptyset, X, \{a\}, \{d\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}\}, \sigma = \{\emptyset, Y, \{2\}, \{1, 2\}\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $\mathcal{G}SO(X) = \{\emptyset, X, \{a\}, \{d\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}\}$, $\mathcal{G}s_\alpha O(X) = \{\emptyset, X, \{a\}, \{d\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\sigma^\alpha = \{\emptyset, Y, \{2\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\}$. Define $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ by $f(a) = 2, f(b) = 3, f(c) = 1$ and $f(d) = 4$. Then the function f is $\mathcal{G}$-semicontinuous. Also the inverse image $f^{-1}(\{2, 3\}) = \{a, b\}$ is not $\mathcal{G}s_\alpha$-open in X for the α-open set $\{2, 3\}$ of Y. Hence, f is not $\mathcal{G}s_\alpha$-continuous.

From (i) and (ii), we got the concepts of $\mathcal{G}$-semicontinuous and $\mathcal{G}s_\alpha$-continuous are independent.

Theorem 12. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and (Y, σ) a topological space. For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following statements are equivalent:*

- (i) *f is $\mathcal{G}s_\alpha$-continuous;*
- (ii) *For each $F \in \alpha C(Y), f^{-1}(F) \in \mathcal{G}s_\alpha C(X)$,*
- (iii) *For each $x \in X$ and each $V \in \sigma^\alpha$ containing $f(x)$, there exists a $U \in \mathcal{G}s_\alpha O(X)$ containing x such that $f(U) \subseteq V$.*

Proof. $(i) \Leftrightarrow (ii)$. It is obvious.
$(i) \Rightarrow (iii)$. Let $V \in \sigma^\alpha$ and $f(x) \in V (x \in X)$. Then by (i), $f^{-1}(V) \in \mathcal{G}s_\alpha O(X)$ containing x. Taking $f^{-1}(V) = U$, we have that $x \in U$ and $f(U) \subseteq V$.
$(iii) \Rightarrow (i)$. Let $V \in \sigma^\alpha$ and $x \in f^{-1}(V)$. Then $f(x) \in V \in \sigma^\alpha$ and hence by (iii), there exists a $U \in \mathcal{G}s_\alpha O(X)$ containing x such that $f(U) \subseteq V$. Now $x \in U \subseteq \Psi(\alpha\text{int}(U)) \subseteq \Psi(\alpha\text{int}(f^{-1}(V)))$. This shows that $f^{-1}(V) \subseteq \Psi(\alpha\text{int}(f^{-1}(V)))$. Thus f is $\mathcal{G}s_\alpha$-continuous. □

Theorem 13. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and (Y, σ) a topological space. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is $\mathcal{G}s_\alpha$-continuous if and only if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is $\mathcal{G}s_\alpha$-continuous.*

Proof. Suppose that f is $\mathcal{G}s_\alpha$-continuous. Let $x \in X$ and $W \in \alpha(X \times Y)$ containing $g(x)$. Then there exists a $U \in \tau^\alpha$ and $V \in \sigma^\alpha$ such that $g(x) = (x, f(x)) \in U \times V \subseteq W$. Since f is $\mathcal{G}s_\alpha$-continuous, there exists a $G \in \mathcal{G}s_\alpha O(X)$ containing x such that $f(G) \subseteq V$. By Theorem 4 (ii), $G \cap U \in \mathcal{G}s_\alpha O(X)$ and $g(G \cap U) \subseteq U \times V \subseteq W$. This shows that g is $\mathcal{G}s_\alpha$-continuous. Conversely, suppose that g is $\mathcal{G}s_\alpha$-continuous. Let $x \in X$ and $V \in \sigma^\alpha$ containing $f(x)$. Then $X \times V \in \alpha(X \times Y)$ and by $\mathcal{G}s_\alpha$-continuity of g, there exists a $U \in \mathcal{G}s_\alpha O(X)$ containing x such that $g(U) \subseteq X \times V$. Thus we have that $f(U) \subseteq V$ and hence f is $\mathcal{G}s_\alpha$-continuous. □

Definition 2.4. Let (X, τ) be a topological space and $(Y, \sigma, \mathcal{G})$ a grill topological space. A function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ is said to be *$\mathcal{G}s_\alpha$-open* (resp. *$\mathcal{G}s_\alpha$-closed*) if for each $U \in \tau^\alpha$ (resp. for each $U \in \tau^{\alpha c}$), $f(U)$ is $\mathcal{G}s_\alpha$-open (resp. $\mathcal{G}s_\alpha$-closed) in $(Y, \sigma, \mathcal{G})$.

Theorem 14. *Let (X, τ) be a topological space and $(Y, \sigma, \mathcal{G})$ a grill topological space. A function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ is $\mathcal{G}s_\alpha$-open if and only if for each $x \in X$ and each α-neighborhood U of x, there exists a $V \in \mathcal{G}s_\alpha O(Y)$ such that $f(x) \in V \subseteq f(U)$.*

Proof. Suppose that f is a $\mathcal{G}s_\alpha$-open function and let $x \in X$. Also let U be any α-neighborhood of x. Then there exists $G \in \tau^\alpha$ such that $x \in G \subseteq U$. Since f is $\mathcal{G}s_\alpha$-open, $f(G) = V$ (say) $\in \mathcal{G}s_\alpha O(Y)$ and $f(x) \in V \subseteq f(U)$. Conversely, suppose that $U \in \tau^\alpha$. Then for each $x \in U$, there exists a $V_x \in \mathcal{G}s_\alpha O(Y)$ such that $f(x) \in V_x \subseteq f(U)$. Thus $f(U) = \bigcup\{V_x : x \in U\}$ and hence by Theorem 2.5(i), $f(U) \in \mathcal{G}s_\alpha O(Y)$. This shows that f is $\mathcal{G}s_\alpha$-open. □

Theorem 15. *Let (X, τ) be a topological space, $(Y, \sigma, \mathcal{G})$ a grill topological space and let $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ be a $\mathcal{G}s_\alpha$-open function. If $V \subseteq Y$ and $F \in \tau^{\alpha c}$ containing $f^{-1}(V)$, then there exists a $H \in \mathcal{G}s_\alpha O(Y)$ containing V such that $f^{-1}(H) \subseteq F$.*

Proof. Suppose that f is $\mathcal{G}s_\alpha$-open. Let $V \subseteq Y$ and $F \in \tau^{\alpha c}$ containing $f^{-1}(V)$. Then $X - F \in \tau^\alpha$ and by $\mathcal{G}s_\alpha$-openness of f, $f(X - F) \in \mathcal{G}s_\alpha O(Y)$. Thus $H = Y - f(X - F) \in \mathcal{G}s_\alpha C(Y)$ consequently $f^{-1}(V) \subseteq F$ implies that $V \subseteq H$. Further, we obtain that $f^{-1}(H) \subseteq F$. □

Theorem 16. *Let (X, τ) be a topological space and $(Y, \sigma, \mathcal{G})$ a grill topological space. For any bijection $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ the following statements are equivalent:*

(*i*) *$f^{-1} : (Y, \sigma, \mathcal{G}) \to (X, \tau)$ is $\mathcal{G}s_\alpha$-continuous;*
(*ii*) *f is $\mathcal{G}s_\alpha$-open;*
(*iii*) *f is $\mathcal{G}s_\alpha$-closed.*

Proof. It is obvious. □

Definition 2.5. Let $(X, \tau, \mathcal{G})$ be a grill topological space and a subset A of X is said to be a *$\mathcal{G}S^*$-set* if $A = U \cap V$, where $U \in \tau^\alpha$, $V \subseteq X$ and $\Psi(\alpha\text{int}(V)) = \alpha\text{int}(V)$.

Theorem 17. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and let $A \subseteq X$. Then $A \in \tau^\alpha$ if and only if $A \in \mathcal{G}s_\alpha O(X)$ and A is a $\mathcal{G}S^*$-set in $(X, \tau, \mathcal{G})$.*

Proof. Let $A \in \tau^\alpha$. Then $A \in \mathcal{G}s_\alpha O(X)$, implies that $A \subseteq \Psi(\alpha\text{int}(A))$. Also A can be expressed as $A = A \cap X$, where $A \in \tau^\alpha$ and $\Psi(\alpha\text{int}(X)) = \alpha\text{int}(X)$. Thus A is a $\mathcal{G}S^*$-set. Conversely, let $A \in \mathcal{G}s_\alpha O(X)$ and A be a $\mathcal{G}S^*$-set. Thus $A \subseteq \Psi(\alpha\text{int}(A)) = \Psi(\alpha\text{int}(U \cap V))$, where $U \in \tau^\alpha$ and $\Psi(\alpha\text{int}(V)) = \alpha\text{int}(V)$. Now $A \subseteq U \cap A \subseteq U \cap \Psi(\alpha\text{int}(U \cap V)) = U \cap \Psi(U \cap \alpha\text{int}(V)) \subseteq U \cap \Psi(U) \cap \Psi(\alpha\text{int}(V)) = U \cap \alpha\text{int}(V) = \alpha\text{int}(A)$. Hence $A \in \tau^\alpha$. □

Definition 2.6. Let $(X, \tau, \mathcal{G})$ be a grill topological space and (Y, σ) a topological space. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is *$\mathcal{G}S^*$-continuous* if for each $V \in \sigma^\alpha$, $f^{-1}(V)$ is a $\mathcal{G}S^*$-set in $(X, \tau, \mathcal{G})$.

Theorem 18. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and (Y, σ) a topological space. Then for a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following statements are equivalent:*

(*i*) *f is α-continuous,*
(*ii*) *f is $\mathcal{G}s_\alpha$-continuous and $\mathcal{G}S^*$-continuous.*

Proof. It is obvious. □

Definition 2.7. Let $(X, \tau, \mathcal{G})$ and $(Y, \sigma, \mathcal{G}')$ be two grill topological spaces. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma, \mathcal{G}')$ is said to be *$(\mathcal{G}, \mathcal{G}')s_\alpha$-continuous* if $f^{-1}(V) \in \mathcal{G}s_\alpha O(X)$ whenever $V \in \mathcal{G}'s_\alpha O(Y)$.

Note that, in the Example 2.4, consider $\mathcal{G}' = \{\{1\},\{2\},\{1,2\},\{1,3\},\{2,3\},Y\}$. Then $\mathcal{G}'s_\alpha O(Y) = \sigma^\alpha$. Hence the function f is $(\mathcal{G},\mathcal{G}')s_\alpha$-continuous.

Remark 2.4: Every $(\mathcal{G},\mathcal{G}')s_\alpha$-continuous function is $\mathcal{G}s_\alpha$-continuous, but the converse need not be true.

Let $X = \{a,b,c,d\}$, $Y = \{1,2,3,4\}$, $\tau = \{\emptyset, X, \{a\},\{b\},\{a,b\}\}$, $\sigma = \{\emptyset, Y, \{3\}, \{4\},\{3,4\},\{1,3,4\}\}$, $\mathcal{G} = \{\{a\},\{c\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},X\}$ and $\mathcal{G}' = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},Y\}$.

Then $\mathcal{G}s_\alpha O(X) = \{\emptyset, X, \{a\},\{b\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\}\}$ and $\mathcal{G}'s_\alpha O(Y) = \{\emptyset, Y, \{3\},\{4\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$. Define a function $f : (X,\tau,\mathcal{G}) \to (Y,\sigma,\mathcal{G}')$ by $f(a) = 3$, $f(b) = 4$, $f(c) = 2$ and $f(d) = 1$. Then f is $\mathcal{G}s_\alpha$-continuous. Since $\{2,4\} \in \mathcal{G}'s_\alpha O(Y)$, but $f^{-1}(\{2,4\}) = \{b,c\} \notin \mathcal{G}s_\alpha O(X)$. Hence f is not $(\mathcal{G},\mathcal{G}')s_\alpha$-continuous.

Definition 2.8. (i) Let $(X,\tau,\mathcal{G})$ be a grill topological space and a subset A of X is said to be a $\mathcal{G}s_\alpha$-*neighborhood* of a point $x \in X$ if there exists a set $U \in \mathcal{G}s_\alpha O(X)$ such that $x \in U \subseteq A$.

Note that $\mathcal{G}s_\alpha$-neighborhood of x may be replaced by $\mathcal{G}s_\alpha$-open neighborhood of x.

(ii) Let $(X,\tau,\mathcal{G})$ be a grill topological space. $A \subseteq X$ and $p \in X$. Then p is called a $\mathcal{G}s_\alpha$-*limit point* of A if $U \cap (A - \{p\}) \neq \emptyset$, for any set $U \in \mathcal{G}s_\alpha O(X)$ containing p. The set of all $\mathcal{G}s_\alpha$-limit points of A is called a $\mathcal{G}s_\alpha$-*derived set* of A and is denoted by $\mathcal{G}s_\alpha \mathrm{d}(A)$. Clearly, if $A \subseteq B$ then $\mathcal{G}s_\alpha \mathrm{d}(A) \subseteq \mathcal{G}s_\alpha \mathrm{d}(B)$.

Remark 2.5: From the Definition 2.8 (ii), it follows that p is a $\mathcal{G}s_\alpha$-limit point of A if and only if $p \in \mathcal{G}s_\alpha \mathrm{cl}(A - \{p\})$.

Theorem 19. *Let $(X,\tau,\mathcal{G})$ be a grill topological space. For any $A, B \subseteq X$, the $\mathcal{G}s_\alpha$-derived sets have the following properties:*

(i) $\mathcal{G}s_\alpha cl(A) \supseteq A \cup \mathcal{G}s_\alpha d(A)$;
(ii) $\cup_i \mathcal{G}s_\alpha d(A_i) = \mathcal{G}s_\alpha d(\cup_i A_i)$;
(iii) $\mathcal{G}s_\alpha d(\mathcal{G}s_\alpha d(A)) \subseteq \mathcal{G}s_\alpha d(A)$;
(iv) $\mathcal{G}s_\alpha cl(\mathcal{G}s_\alpha d(A)) = \mathcal{G}s_\alpha d(A))$.

Proof. Follows from the Definition 2.8 (ii) and Remark 2.5 □

Theorem 20. *Let $(X,\tau,\mathcal{G})$ and $(Y,\sigma,\mathcal{G}')$ be two grill topological spaces. If $f : (X,\tau,\mathcal{G}) \to (Y,\sigma,\mathcal{G}')$ is a function, then the following statements are equivalent:*

(i) *f is $(\mathcal{G},\mathcal{G}')s_\alpha$-continuous,*
(ii) *for each $x \in X$, the inverse of every $\mathcal{G}'s_\alpha$-neighborhood of $f(x)$ is a $\mathcal{G}s_\alpha$-neighborhood of x,*
(iii) *for each $x \in X$ and each $\mathcal{G}'s_\alpha$-neighborhood B of $f(x)$, there is a $\mathcal{G}s_\alpha$-neighborhood A of x such that $f(A) \subseteq B$,*
(iv) *for each $x \in X$ and each set $B \in \mathcal{G}'s_\alpha O(Y)$ contains $f(x)$, there exists a set $A \in \mathcal{G}s_\alpha O(X)$ containing x such that $f(A) \subseteq B$,*
(v) *$f(\mathcal{G}s_\alpha cl(A)) \subseteq \mathcal{G}'s_\alpha cl(f(A))$ holds for every subset A of X,*

(vi) for any set $H \in \mathcal{G}'s_\alpha C(Y)$, $f^{-1}(H) \in \mathcal{G}s_\alpha C(X)$.

Proof. $(i) \Rightarrow (ii)$. Let $x \in X$ and B be a $\mathcal{G}'s_\alpha$-neighborhood of $f(x)$. By Definition 2.8 (i), there exists $V \in \mathcal{G}'s_\alpha O(Y)$ such that $f(x) \in V \subseteq B$. This implies that $x \in f^{-1}(V) \subseteq f^{-1}(B)$. Since f is $(\mathcal{G}, \mathcal{G}')s_\alpha$-continuous, so $f^{-1}(V) \in \mathcal{G}s_\alpha O(X)$. Hence $f^{-1}(B)$ is a $\mathcal{G}s_\alpha$-neighborhood of x.

$(ii) \Rightarrow (i)$. Let $B \in \mathcal{G}'s_\alpha O(Y)$. Put $A = f^{-1}(B)$. Let $x \in A$. Then $f(x) \in B$. Clearly, B (being $\mathcal{G}'s_\alpha$-open) is a $\mathcal{G}'s_\alpha$-neighborhood of $f(x)$. By (ii), $A = f^{-1}(B)$ is a $\mathcal{G}s_\alpha$-neighborhood of x. Hence by Definition 2.8 (i), there exists $A_x \in \mathcal{G}s_\alpha O(X)$ such that $x \in A_x \subseteq A$. This implies that $A = \cup_{x \in A} A_x$. By Theorem 5 (i), we have that $A \in \mathcal{G}s_\alpha O(X)$. Therefore f is $(\mathcal{G}, \mathcal{G}')s_\alpha$-continuous.

$(i) \Rightarrow (iii)$. Let $x \in X$ and B be a $\mathcal{G}'s_\alpha$-neighborhood of $f(x)$. Then, there exists $O_{f(x)} \in \mathcal{G}'s_\alpha O(Y)$ such that $f(x) \in O_{f(x)} \subseteq B$. It follows that $x \in f^{-1}(O_{f(x)}) \subseteq f^{-1}(B)$. By (i), $f^{-1}(O_{f(x)}) \in \mathcal{G}s_\alpha O(X)$. Let $A = f^{-1}(B)$. Then it follows that A is $\mathcal{G}s_\alpha$-neighborhood of x and $f(A) = f(f^{-1}(B)) \subseteq B$.

$(iii) \Rightarrow (i)$. Let $U \in \mathcal{G}'s_\alpha O(Y)$. Take $W = f^{-1}(U)$. Let $x \in W$. Then $f(x) \in U$. Thus U is a $\mathcal{G}'s_\alpha$-neighborhood of $f(x)$. By (iii), there exists a $\mathcal{G}s_\alpha$-neighborhood V_x of x such that $f(V_x) \subseteq U$. Thus it follows that $x \in V_x \subseteq f^{-1}(f(V_x)) \subseteq f^{-1}(U) = W$. Since V_x is a $\mathcal{G}s_\alpha$-neighborhood of x, which implies that there exists a $W_x \in \mathcal{G}s_\alpha O(X)$ such that $x \in W_x \subseteq W$. This implies that $W = \cup_{x \in W} W_x$. By Theorem 5 (i), $W \in \mathcal{G}s_\alpha O(X)$. Thus f is $(\mathcal{G}, \mathcal{G}')s_\alpha$-continuous.

$(iii) \Rightarrow (iv)$. We may replace the $\mathcal{G}s_\alpha$-neighborhood of x as $\mathcal{G}s_\alpha$-open neighborhood of x in condition (iii). Straightforward.

$(iv) \Rightarrow (v)$. Let $y \in f(\mathcal{G}s_\alpha\text{cl}(A))$ and any set $V \in \mathcal{G}'s_\alpha O(Y)$ containing y. Then, there exists a point $x \in X$ and a set $U \in \mathcal{G}s_\alpha O(X)$ such that $x \in U$ with $f(x) = y$ and $f(U) \subseteq V$. Since $x \in \mathcal{G}s_\alpha\text{cl}(A)$, we have that $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. This implies that $y \in \mathcal{G}'s_\alpha\text{cl}(f(A))$. Therefore, we have that $f(\mathcal{G}s_\alpha\text{cl}(A)) \subseteq \mathcal{G}'s_\alpha\text{cl}(f(A))$.

$(v) \Rightarrow (vi)$. Let $H \in \mathcal{G}'s_\alpha C(Y)$. Then $\mathcal{G}'s_\alpha\text{cl}(H) = H$. By condition (v), $f(\mathcal{G}s_\alpha\text{cl}(f^{-1}(H))) \subseteq \mathcal{G}'s_\alpha\text{cl}(f(f^{-1}(H))) \subseteq \mathcal{G}'s_\alpha\text{cl}(H) = H$ holds. Therefore $\mathcal{G}s_\alpha\text{cl}(f^{-1}(H)) \subseteq f^{-1}(H)$ and thus $f^{-1}(H) = \mathcal{G}s_\alpha\text{cl}(f^{-1}(H))$. Hence $f^{-1}(H) \in \mathcal{G}s_\alpha C(X)$.

$(vi) \Rightarrow (i)$. Let $B \in \mathcal{G}s_\alpha O(X)$. We take $H = Y - B$. Then $H \in \mathcal{G}'s_\alpha C(Y)$. By (iv), $f^{-1}(H) \in \mathcal{G}s_\alpha C(X)$. Hence $f^{-1}(B) = X - f^{-1}(Y - B) = X - f^{-1}(H) \in \mathcal{G}s_\alpha O(X)$. □

Theorem 21. *Let* $(X, \tau, \mathcal{G})$ *and* $(Y, \sigma, \mathcal{G}')$ *be two grill topological spaces. If* $f : (X, \tau, \mathcal{G}) \to (Y, \sigma, \mathcal{G}')$ *is a function, then* f *is* $(\mathcal{G}, \mathcal{G}')s_\alpha$*-continuous if and only if* $f(\mathcal{G}s_\alpha d(A)) \subseteq \mathcal{G}'s_\alpha cl(f(A))$, *for all* $A \subseteq X$.

Proof. Let $f : (X, \tau, \mathcal{G}) \to (Y, \sigma, \mathcal{G}')$ be $(\mathcal{G}, \mathcal{G}')s_\alpha$-continuous, $A \subseteq X$ and $x \in \mathcal{G}s_\alpha\text{d}(A)$. Assume that $f(x) \notin f(A)$ and let V denote a $\mathcal{G}'s_\alpha$-neighborhood of $f(x)$. Since f is $(\mathcal{G}, \mathcal{G}')s_\alpha$-continuous and by Theorem 2.20(iii), there exists a $\mathcal{G}s_\alpha$-neighborhood U of x such that $f(U) \subseteq V$. From $x \in \mathcal{G}s_\alpha\text{d}(A)$, it follows that $U \cap A \neq \emptyset$, there exists at least one element $a \in U \cap A$ such that $f(a) \in f(A)$ and $f(a) \in V$. Since $f(x) \notin f(A)$, we have that $f(a) \neq f(x)$. Thus every $\mathcal{G}'s_\alpha$-neighborhood of

$f(x)$ contains an element $f(a)$ of $f(A)$ different from $f(x)$. Consequently, $f(x) \in \mathcal{G}'s_\alpha\mathrm{d}(f(A))$. Conversely, suppose that f is not $(\mathcal{G}, \mathcal{G}')s_\alpha$-continuous. Then by Theorem 2.20(iii), there exists $x \in X$ and a $\mathcal{G}'s_\alpha$-neighborhood V of $f(x)$ such that every $\mathcal{G}s_\alpha$-neighborhood U of x contains at least one element $a \in U$ for which $f(a) \notin V$. Put $A = \{a \in X : f(a) \notin V\}$. Since $f(x) \in V$, therefore $x \notin A$ and hence $f(x) \notin f(A)$. Since $f(A) \cap (V - \{f(x)\}) = \emptyset$, therefore $f(x) \notin \mathcal{G}'s_\alpha\mathrm{d}(f(A))$. It follows that $f(x) \in f(\mathcal{G}s_\alpha\mathrm{d}(A)) - (f(A) \cup \mathcal{G}'s_\alpha\mathrm{d}(f(A))) \neq \emptyset$, which is a contradiction to the given condition. □

Theorem 22. *Let $(X, \tau, \mathcal{G})$ and $(Y, \sigma, \mathcal{G}')$ be two grill topological spaces. If $f : (X, \tau, \mathcal{G}) \to (Y, \sigma, \mathcal{G}')$ is an injective function, then f is $(\mathcal{G}, \mathcal{G}')s_\alpha$-continuous if and only if $f(\mathcal{G}s_\alpha d(A)) \subseteq \mathcal{G}'s_\alpha d(f(A))$, for all $A \subseteq X$.*

Proof. Let $A \subseteq X$, $x \in \mathcal{G}s_\alpha\mathrm{d}(A)$ and V be a $\mathcal{G}'s_\alpha$-neighborhood of $f(x)$. Since f is $(\mathcal{G}, \mathcal{G}')s_\alpha$-continuous, so by Theorem 20 (*iii*), there exists a $\mathcal{G}s_\alpha$-neighborhood U of x such that $f(U) \subseteq V$. But $x \in \mathcal{G}s_\alpha\mathrm{d}(A)$ gives there exists an element $a \in U \cap A$ such that $a \neq x$. Clearly $f(a) \in f(A)$ and since f is injective, $f(a) \neq f(x)$. Thus every $\mathcal{G}'s_\alpha$-neighborhood V of $f(x)$ contains an element $f(a)$ of $f(A)$ different from $f(x)$. Consequently, $f(x) \in \mathcal{G}'s_\alpha\mathrm{d}(f(A))$. Therefore, we have that $f(\mathcal{G}s_\alpha\mathrm{d}(A)) \subseteq \mathcal{G}'s_\alpha\mathrm{d}(f(A))$. Converse follows from the Theorem 21. □

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[1] PG and Research Department of Mathematics,
Thiagarajar College, Madurai, India
and
Department of Mathematics,
M.S.S. Wakf Board College, Madurai, India.
Email address: drdstop@gmail.com, dsktcm1@gmail.com

[2] Department of Mathematics,
Vel Tech Rangarajan Dr. Sagunthala R and
D Institute of Science and Technology, Chennai, India
Email address: kalaivani.rajam@gmail.com

[3] Department of Mathematics,
M.S.S. Wakf Board College, Madurai, India
Email address: mhmdriyazdeen@gmail.com

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