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A NOTE: RELATION BETWEEN TOTAL VERTEX STRESS AND WIENER INDEX

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Abstract. This note introduces results in respect of a relation between the total vertex stress and the Wiener index of a graph. In respect of the total vertex stress $\mathcal{S}(G)$ and the Wiener index $W(G)$ of a non-trivial geodetic graph, it is established that $W(G) = \mathcal{S}(G) + \binom{n}{2}$. The aforesaid holds for even cycles and uni-Joost graphs as well. Avenues for further research are also suggested.

1. Introduction

Unless mentioned otherwise, only finite, undirected and connected simple graphs are considered. It is assumed that the reader is familiar with the basic notions and notation of graph theory. However, useful definitions will be recalled as is necessary. Vertices with different labels will be assumed to be distinct.

The notion of *vertex stress* in a graph was introduced by the researcher Alfonso Shimbel in [11]. This parameter is denoted by $\mathcal{S}_G(v)$, $v \in V(G)$. The vertex stress of vertex $v \in V(G)$ is the number of times v is contained as an internal vertex in all shortest paths between all pairs of distinct vertices in $V(G)\backslash v$. Formally stated, $\mathcal{S}_G(v) = \sum\limits_{u \neq w \neq v} \sigma(v)$ with $\sigma(v)$ the number of shortest paths between vertices u, w which contain v as an internal vertex. Such a shortest uw-path is also called a uw-distance path. See [11, 12]. The *total vertex stress* of G is given by $\mathcal{S}(G) = \sum\limits_{v \in V(G)} \mathcal{S}_G(v)$, [9].

Unless stated otherwise, reference to vertices u, v will mean that u and v are distinct vertices. A widely studied topological index is the Wiener index of a graph. For a graph G it is defined by $W(G) = \sum\limits_{u,v \in V(G)} d(u,v)$ where $d(u,v)$ is the length of a shortest path between vertices u and v in G. See [1, 2, 3, 4, 5, 6]. Note

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that for the trivial graph K_1 we have $\mathcal{S}(K_1) = 0 = W(K_1)$ and for a complete graph K_n, $n \geq 2$ we have that $\mathcal{S}(K_n) = 0 < \frac{n(n-1)}{2} = W(K_n)$. If connectivity is relaxed, then the universal set of all simple graphs say, set $\mathfrak{G}$ can be partitioned into subsets i.e.

$$\begin{aligned} X_1 &= \{\text{All graphs for which } \mathcal{S}(G) = 0 = W(G)\} \\ &= \{\text{All null graphs(edgeless graphs)}\}, \\ X_2 &= \{\text{Graphs for which } 0 < \mathcal{S}(G) < W(G)\}, \\ X_3 &= \{\text{Graphs for which } 0 < \mathcal{S}(G) = W(G)\}, \\ X_4 &= \{\text{Graphs for which } 0 < W(G) < \mathcal{S}(G)\}. \end{aligned}$$

Note that $K_1 \in X_1$ and $K_n \in X_2$, $n \geq 2$. Figure 1 presents an example of a connected graph $G \in X_3$ and Figure 2 presents an example of a connected graph $H \in X_4$.

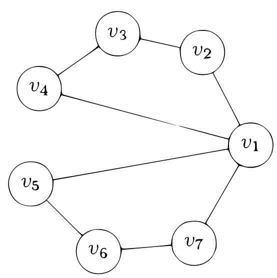

FIGURE 1. Graph $G \in X_3$ having $\mathcal{S}(G) = 40 = W(G)$.

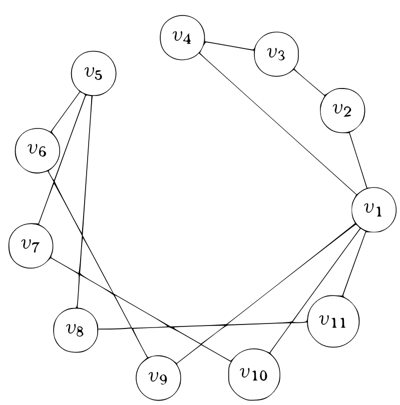

FIGURE 2. Graph $H \in X_4$ having $\mathcal{S}(H) = 136 > 126 = W(H)$.

We identify certain connected graphs for which $0 < \mathcal{S}(G) < W(G)$.

2. Certain connected graphs in X_2

For a path P_n, $n \geq 2$ it is known from [9] that the total vertex stress is given by $\mathcal{S}(P_n) = \frac{n(n-1)(n-2)}{6}$. It is also known that $W(P_n) = \frac{n(n^2-1)}{6}$, [8].

Corollary 0.1. *For $n \geq 2$, $\mathcal{S}(P_n) < W(P_n)$.*

Proof. Clearly, since $n^2 - 1 = (n-1)(n+1) > (n-1)(n-2)$ for $n \geq 2$ it follows that $\mathcal{S}(P_n) < W(P_n)$. □

Proposition 2.1. *$\mathcal{S}(C_n) < W(C_n)$, $n \geq 4$ and even.*

Proof. It is known from [12] that for $n \geq 4$ and even, $\mathcal{S}_{C_n}(v) = \frac{n(n-2)}{8}$. Therefore, $\mathcal{S}(C_n) = \frac{n^2(n-2)}{8}$. Also, $W(C_n) = \frac{n^3}{8}$, [8]. Since $W(C_n) - \mathcal{S}(C_n) = \frac{n^2}{4}$ it follows that $\mathcal{S}(C_n) < W(C_n)$, $n \geq 4$ and even. □

Theorem 1. *For everygeodetic graph G of order $n \geq 2$ it holds that,*

$$W(G) = \mathcal{S}(G) + \binom{n}{2}.$$

Proof. From the definition of the Wiener index, the Wiener index of a geodetic graph is the sum of the length of $\binom{n}{2}$ distance-paths in G. From the definition of total vertex stress, the total vertex stress of a geodetic graph is the sum of total vertex stress of $\binom{n}{2}$ distance-paths in G. Hence, by double counting, $\mathcal{S}(G)$ equal the sum of (length -1) over all distance-paths. The stated equality follows immediately. □

Corollary 1.1. *For a geodetic graph G of order $n \geq 2$ it follows that,*

$$\mathcal{S}(G) < W(G).$$

Theorem 2.3 represents amongst others, trees, complete graphs, block graphs, cycles of odd length as well as the Petersen graph. The value of the result of Theorem 2.3 lies in the proof thereof as it leads to the notion of uni-Joost-g graphs. Recall the definition of a Joost graph from [10].

Definition 2.1. [10] *Consider the paths with common end-vertices u_i, u_j and all other vertices distinct i.e.*

$$u_i v_{1,1} v_{2,1} v_{3,1} \cdots v_{n-2,1} u_j,$$
$$u_i v_{1,2} v_{2,2} v_{3,2} \cdots v_{n-2,2} u_j,$$
$$\vdots$$
$$u_i v_{1,k} v_{2,k} v_{3,k} \cdots v_{n-2,k} u_j,$$

$n \geq 3$ and $k \geq 1$. Such a graph is called a Joost graph and is denoted by $P_n^{(k)}$.

Note that paths $P_n \cong P_n^{(1)}$, $n \geq 3$ are Joost graphs. Even cycles $C_{2(n-1)} \cong P_n^{(2)}$ are Joost graphs as well. It is said that a path is a *1-layered* Joost graph, an even cycle is said to be a *2-layered* Joost graph. In general, a Joost graph $P_n^{(k)}$ is said to be k-layered.

Theorem 2. *For a Joost graph it follows that $\mathcal{S}(P_n^{(k)}) < W(P_n^{(k)})$.*

Proof. Consider all shortest uv-paths of $P_n^{(k)}$. There are $\frac{(n-2)k(k-1)}{2}$ shortest paths having two layers of length $n-1$ and one shortest path having k layers of length $n-1$. The remaining

$$\frac{[k(n-2)+2][k(n-2)+1]}{2} - [\frac{(n-2)k(k-1)}{2} + 1]$$

shortest paths are non-layered paths of different length. The layered shortest paths contribute a total of

$$2\frac{(n-2)k(k-1)}{2}(n-2) + k(n-2)$$

to the count in $\mathcal{S}(P_n^{(k)})$ and

$$\frac{(n-2)k(k-1)}{2}(n-1) + (n-1)$$

to the count in $W(P_n^{(k)})$. Also each non-layered path P_i contributes $l(P_i)-1$ to the count in $\mathcal{S}(P_n^{(k)})$ and $l(P_i)$ to the count in $W(P_n^{(k)})$. Thus,

$$\begin{aligned}\mathcal{S}(P_n^{(k)}) + \tfrac{[k(n-2)+2][k(n-2)+1]}{2} - [\tfrac{(n-2)k(k-1)}{2} + 1] = \\ W(P_n^{(k)}) + \tfrac{(n-2)(n-3)k(k-1)}{2} + (k-1)(n-2) - 1.\end{aligned}$$

By simplification we get $\mathcal{S}(P_n^{(k)}) - W(P_n^{(k)}) = \frac{-(n-1)[k(n-2)+2]}{2} < 0$ for $n > 2$. Hence, the result. □

Consider a geodetic graph G of order $n \geq 2$. Replace any edge say, $v_iv_j \in E(G)$ with a Joost graph $P_n^{(k)}$. The new graph is denoted by $G_{v_i \Diamond v_j}(P_n^{(k)})$ and it is called a *uni-Joost-g* graph.

Corollary 2.1. *A uni-Joost-g graph has*

$$\mathcal{S}(G_{v_i \Diamond v_j}(P_n^{(k)})) < W(G_{v_i \Diamond v_j}(P_n^{(k)})).$$

Proof. The result follows directly from the proofs of Theorems 2.3 and Corollary 2.4. □

3. Worthy observations

Let $G-e$ denote the removal of the edge $e = uv$ from G. Furthermore, $G-e$ must remain connected. An useful folklore result hence, accepted by experts as an established result although it has not been published, is presented next.

Proposition 3.1. $(Folklore)$ *If $e = uv$ is such that $G-e$ is connected, then $W(G) < W(G-e)$.*

Proposition 3.2. *If G is geodetic and $e = uv$ is such that $G-e$ is connected, then $\mathcal{S}(G) < \mathcal{S}(G-e)$.*

Proof. A contributory total vertex stress count of 0 has been lost by removing the edge e. Since $G-e$ is connected a contributory total vertex stress end-to-end tally of at least 1 will yield for some uv-distance path in $G-e$. Hence, $\mathcal{S}(G) < \mathcal{S}(G-e)$. □

Corollary 2.2. *For even cycles $n \geq 6$ (non-geodetic) the result of Proposition 3.2 holds.*

If the cycle C_4 is included (hence, $n \geq 4$) the result in Corollary 3.3 changes to $\mathcal{S}(G) \leq \mathcal{S}(G-e)$. It is easy to see that $W(K_n) = \frac{n(n-1)}{2}$, $n \geq 1$. Since K_n, $n \geq 2$ is a super graph of any connected graph G of equal order n, the next corollary is immediate.

Corollary 2.3. *For a graph G, $W(G) \geq \frac{n(n-1)}{2}$.*

It is easy to verify that $\mathcal{S}(K_n - e) = n-2$. Noting that $n-2 = \binom{2}{2}(n-2)$ and that removing an edge e from K_n is equivalent to deleting a clique Q_2 from K_n leads to a specialized generalization. Consider K_n, $n \geq 2$. Select any $t \leq n-1$ vertices and delete all pairwise edges amongst the selected vertices. It follows easily that for the resultant graph $K_n^{\aleph(t)}$ the total vertex stress is given by $\mathcal{S}(K_n^{\aleph(t)}) = \binom{t}{2}(n-t)$.

Theorem 3. *For G of order $n \geq 2$ and $\varepsilon(G)$ edges it follows that,*

$$\mathcal{S}(G) \geq \binom{n}{2} - \varepsilon(G).$$

Proof. An edge uv has zero contribution to the total vertex stress of graph G. For exactly $\binom{n}{2} - \varepsilon(G)$ pairs of vertices the distance is necessarily $d(w,z) \geq 2$ where $w, z \notin E(G)$. Hence, the contribution of each wz-distance path is at least one. Therefore, $\mathcal{S}(G) \geq \binom{n}{2} - \varepsilon(G)$. $\square$

In [7] it was established that the integers 2 and 5 are not the Wiener index of any graph. We present a result for total vertex stress for the set of non-negative integers, $\mathbb{N}_0$.

Theorem 4. *For $k \in \mathbb{N}_0$ there exists a graph G with $\mathcal{S}(G) = k$.*

Proof. An n-dart graph denoted by K_n^{+1} is a complete graph K_n, $n \geq 1$ with a single pendant vertex attached to a vertex of K_n. The path $P_2 \cong K_1^{+1}$ and $\mathcal{S}(P_2) = \mathcal{S}(K_1^{+1}) = 0$. The path $P_3 \cong K_2^{+1}$ and $\mathcal{S}(P_3) = \mathcal{S}(K_2^{+1}) = 1$. Through immediate induction it follows that since $\mathcal{S}(K_n) = 0$, $\forall n \in \mathbb{N}_0$ any K_n^{+1} having exactly $n-1$ distance-paths of length 2 must have $\mathcal{S}(K_n^{+1}) = n-1$. This settles the result. $\square$

Observe that for K_n^{+1} equality holds in Theorem 3.5. This observation leads to an interesting problem.

Problem 1. Characterize graphs for which $\mathcal{S}(G) = \binom{n}{2} - \varepsilon(G)$.

Motivation: If G of order $n \geq 2$ is complete then $\varepsilon(G) = \binom{n}{2}$. Since $\mathcal{S}(K_{n\geq 2}) = 0$ it follows that $\mathcal{S}(G) = \binom{n}{2} - \varepsilon(G)$. For a star $S_{1,n}$ we have $\mathcal{S}(S_{1,n}) = \frac{n(n-1)}{2}$ and $\varepsilon(S_{1,n}) = n$. Also $\binom{n+1}{2} - n = \frac{n(n-1)}{2}$.

4. Conclusion

Section 2 initiated the study of graphs in subset $X_2 \subset \mathfrak{G}$. The study of the graphs in subsets $X_3 \subset \mathfrak{G}$ and $X_4 \subset \mathfrak{G}$ remains open. The authors consider these avenues of research as worthy endeavors.

Recall from [13] the definition of total induced vertex stress of a vertex denoted by, $\mathfrak{s}_G(v_i)$.

Definition 4.1. [13] *Let $V(G) = \{v_i : 1 \leq i \leq n\}$ and for the ordered vertex pair (v_i, v_j) let there be $k_G(i,j)$ distinct shortest paths of length $l_G(i,j)$ from v_i to v_j. Then, $\mathfrak{s}_G(v_i) = \sum\limits_{j=1, j\neq i}^{n} k_G(i,j)(\ell_G(i,j) - 1)$.*

A similar definition for the total induced Wiener index of a vertex denoted by $\mathfrak{w}_G(v_i)$, follows.

Definition 4.2. *Let $V(G) = \{v_i : 1 \leq i \leq n\}$ and for the ordered vertex pair (v_i, v_j) let any shortest v_iv_j-path be length $l_G(i,j)$. Then, $\mathfrak{w}_G(v_i) = \sum\limits_{j=1, j\neq i}^{n} \ell_G(i,j)$.*

Corollary 4.1. *For a graph G it follows that $W(G) = \frac{1}{2} \sum\limits_{v_i \in V(G)} \mathfrak{w}_G(v_i)$.*

Proof. The result follows from the fact that each ordered vertex pair (v_i, v_j) and (v_j, v_i) are terms of the summation in Definition 4.2. □

The notion of total induced Wiener index of a graph as an auxiliary parameter in studying an alternative enumeration method of the Wiener index remains open for further research.

In a cycle each vertex is an end-vertex of a diam-path and all vertices have equal total induced vertex stress and equal total induced Wiener index. The same observation holds for the two end-vertices of a path. Experimental results suggest that for a graph G in general, the observation holds for all vertices which are end-vertices of a diam-path in G.
Problem 2. Prove or disprove that in a graph G all vertices which are end-vertices of a diam-path in G, have equal total induced vertex stress and equal total induced Wiener index.

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