Математички Билтен UDC: 515.122.4:[517.938:517.986
46 (LXXII) 1 (2022), 25–32 DOI: 10.37560/matbil22461025sh

A NOTE ON TOPOLOGICAL TRANSITIVITY OF THE NON-WANDERING SET AND THE CHAIN RECURRENT SET

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Abstract. In this paper we shall use the fundamental theorem of dynamical systems to improve some of the results from [5] concerning topological transitivity of the non-wandering set $\Omega(\varphi)$ and the chain recurrent set $\mathcal{CR}(\varphi)$ in compact metric spaces.

1. Introduction

C.Conley obtained many important results in [3]. One of them considered as a fundamental theorem of dynamical systems states that any flow on a compact metric space decomposes into a chain recurrent part and a gradient-like part. In order to obtain these results he introduced the concept of the chain recurrent set. This paper continues the development of the properties of the chain recurrent set of a given flow. We are going to investigate the properties of sets with different levels of recurrence in the framework of continuous dynamical systems using only topological tools. Recall that a topological group is a set G on which two structures are given, a group structure and a topology such that the group operations are continuous. Specifically, a topology τ on G is said to be a group topology if the map $f : G \times G \to G$ defined by $f(x, y) = xy^{-1}$ is continuous. A topological group is a pair (G, τ) of a group G and a group topology τ on G. A dynamical system is a topological group G together with a topological space X and a continuous group action $\varphi : X \times G \to X$ of G on X. Sometimes we say that X together with the G-action is a G-flow. We denote by $\gamma(x), \alpha(x), \omega(x)$ the trajectory (orbit) and the limit sets of a point x. A fixed point of dynamical system φ, exhibits the simplest type of recurrence. We denote by $Fix(\varphi)$ the set of all fixed points of φ. A point carried back to itself by a dynamical system φ exhibits the next most elementary type of recurrence. For a positive real $T \in G$ a point $x \in X$ is called T-periodic if $\varphi(x, T) = x$. We denote by $Per_T(\varphi)$ the set of all T-periodic points of φ and we set $Per(\varphi) = \bigcup_{T>0} Per_T(\varphi)$. A set $A \subset X$ is said to be positively recursive

2010 *Mathematics Subject Classification.* 54H20, 54C56, 37B20, 37B25.
Key words and phrases. trajectory, limit sets, Lyapunov function, recurrence, group, topological group, topological transitivity, non-wandering set, chain recurrent set.

with respect to a set $B \subset X$ if for each $T \in G$ there is a $t > T$ and an $x \in B$ such that $\varphi(x,t) \in A$. We will say that a set A is self positively recursive whenever it is positively recursive with respect to itself. A point $x \in X$ is said to be non-wandering if every neighborhood U of x is self positively recursive. We denote by $\Omega(\varphi)$ the set of all non-wandering points of φ. The next level of recurrence for a point $x \in X$ is chain recurrence. Let $(x,y) \in X \times X$ and $\varepsilon > 0$, $t > 0$. $(\varepsilon, t, \varphi)$-chain from x to y is a collection $\{x = x_1, x_2, ..., x_n, x_{n+1} = y; t_1, t_2, ..., t_n\}$ such that for all $i \in \{1, 2, ..., n\}$, $t_i \geq t$ and $d\left(\varphi(x_i, t_i), x_{i+1}\right) < \varepsilon$.

$$P(\varphi) = \{(x,y) | \forall \varepsilon, t > 0, \text{ there exists } (\varepsilon, t, \varphi)\text{-chain from } x \text{ to } y\}.$$

Now $\mathcal{CR}(\varphi) = \{x | (x,x) \in P(\varphi)\}$ is the set of all chain recurrent points for φ. It is known that

$$Fix(\varphi) \subseteq Per_T(\varphi) \subseteq Per(\varphi) \subseteq \Omega(\varphi) \subseteq \mathcal{CR}(\varphi).$$

2. Topological transitivity

Topological transitivity is a global characteristic of a dynamical system. Although the local structure of topologically transitive dynamical system fulfills certain conditions, there is a variety of such systems. Some of them have dense periodic points while some of them may be minimal and without any periodic points.

We will present a result concerning the last level of recurrence, chain recurrence in the framework of continuous dynamical systems. Also we will mention few interesting corollaries concerning topological transitivity. In what follows X is a compact metric space and $G = \mathbb{R}$.

Definition 2.1. *A subset $M \subseteq X$ is topologically transitive if it contains a trajectory $\gamma(x)$ which is dense in M.*

Definition 2.2. *A subset $M \subseteq X$ is chain transitive if for all $x, y \in M$ and $\varepsilon, t > 0$ there exists an $(\varepsilon, t, \varphi)$-chain from x to y.*

Recall that a point $x \in X$ is called recurrent if $x \in \omega(x)$. The set of all recurrent points is denoted by $\mathcal{R}(\varphi)$. It is known that $Per(\varphi) \subseteq \mathcal{R}(\varphi) \subseteq \Omega(\varphi) \subseteq \mathcal{CR}(\varphi)$.

Example 2.1. Consider a dynamical system defined on a torus $\mathbb{T}^2$ by means of the planar differential system

$$\frac{d\phi}{dt} = 1, \frac{d\theta}{dt} = \alpha.$$

Let $\alpha > 0$ be irrational. Then every trajectory is dense in the torus and moreover the torus is also the positive and negative limit set of each point. This example describes recurrent trajectories which are not periodic. Also note that this is an example of a topologically transitive manifold.

Example 2.2. Consider the dynamical system defined on the unit square

$$X = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$$

by means of the planar system of differential equations

$$\frac{dx_1}{dt} = 0, \frac{dx_2}{dt} = -x_1 x_2 (1 - x_1)(1 - x_2).$$

Then X is a compact metric space and the phase portrait is shown in Figure 1 below. The rest points consists of the set Q,

$$Q = \{(x_1, x_2) \mid x_1 = 0 \ \ or \ \ x_1 = 1\} \cup \{(x_1, x_2) \mid x_2 = 0 \ \ or \ \ x_2 = 1\}.$$

Let us note that every point in this $\mathbb{R}$-flow is chain recurrent, i.e. $X = \mathcal{CR}(\varphi)$. But the only recurrent points are the rest points from Q, which coincide with the non-wandering set $\Omega(\varphi)$.

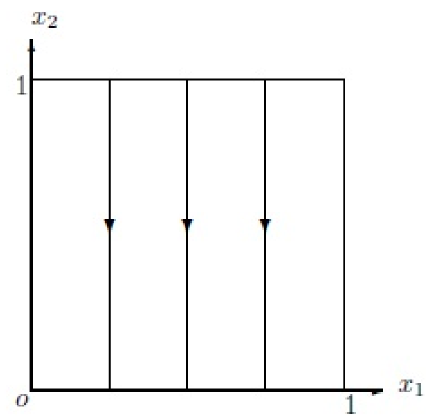

FIGURE 1. Phase portrait of the flow φ on the unit square X

Definition 2.3. *A point $x \in X$ is said to be a flow non-isolated if $U \setminus \varphi(x, [-r, r] \cap G) \neq \emptyset$ for any $r > 0$ and any neighborhood U of x.*

Remark 2.1: Let X be a compact topological manifold. If $dimX > 1$ then every point of X is a flow non-isolated (see [5]).

Proposition 2.1. *Let φ be a $\mathbb{R}$-flow in a compact topological manifold X with $dimX > 1$. If X is topologically transitive i.e. $X = \overline{\gamma(x)}$ for some $x \in X$ then $\Omega(\varphi) = X$. Furthermore the point x is a recurrent point for φ up to reversibility.*

Proof. First let us note that according to remark 2.1 above any compact topological manifold X with $dimX > 1$ is a flow non-isolated i.e. for every $p \in X$ and every neighborhood U of p the following holds: $U \setminus \varphi(p, [-r, r]) \neq \emptyset$ for any $r > 0$. Hence the point x is a flow non-isolated as well. This means that for arbitrary neighborhood U of x and arbitrary $r > 0$ the set $U \setminus \varphi(x, [-r, r])$ is open and non-empty. Using the density of the x-trajectory there exists a $t \in \mathbb{R}$ such that $|t| > r$ and $\varphi(x, t) \in U$. Hence $x \in \Omega(\varphi)$ and $\Omega(\varphi) = X$. By reversing the flow if necessary we can conclude that $x \in \mathcal{R}(\pm\varphi)$ as well. $\square$

Remark 2.2: Note that in example 2.1 the torus $\mathbb{T}^2$ is a compact topological manifold with $dimX > 1$. Also the flow is minimal. Hence every point is a recurrent point.

Corollary 2.1. *The sphere $\mathbb{S}^2$ and the projective plane $\mathbb{RP}^2$ are not topologically transitive manifolds.*

Proof. Let us assume that the sphere $\mathbb{S}^2$ or the projective plane $\mathbb{RP}^2$ are topologically transitive. Then there exists a point x with a dense trajectory. But according to the previous proposition 2.1 every flow on a compact topological manifold with $dimX > 1$ is a flow non-isolated and hence $\Omega(\varphi) = \mathbb{S}^2$ or $\Omega(\varphi) = \mathbb{RP}^2$ accordingly. Furthermore the point x is a recurrent point for φ up to reversibility. But according to the Poincare-Bendixson theorem for two-dimensional manifolds (see Perko [6]) all recurrent motions on the sphere $\mathbb{S}^2$ and on the projective plane $\mathbb{RP}^2$ are trivial. Hence x is a periodic point. This is a contradiction. □

3. Lyapunov functions

In order to prove our main result we will need the following definition of a Lyapunov function for a given family of disjoint compact invariant subsets of X (see [4]).

Definition 3.1. *Let $\mathcal{M} = \{M_j \mid j \in J\}$ be a family of disjoint compact invariant subsets of the phase space X. A Lyapunov function for $\mathcal{M}$ is a continuous function $L_{\mathcal{M}} : X \to \mathbb{R}$ such that:*

i) $L_{\mathcal{M}}(\varphi(x,t)) < L_{\mathcal{M}}(x), \forall t > 0, \forall x \notin \bigcup_{j\in J} M_j$

ii) $L_{\mathcal{M}}(M_j) = c_j, \forall j \in J, (c_j \neq c_i$ *for* $i \neq j)$.

The real numbers c_j are called the critical values of $L_{\mathcal{M}}$.

Example 3.1. Consider a plane flow φ induced by the following system of differential equations:

$$\frac{dr}{dt} = r(1-r)(\sin^2\theta + 1 - r^2),$$
$$\frac{d\theta}{dt} = \sin^2\theta + 1 - r^2.$$

Let's consider the plane flow φ restrictively on the unit disk D. Note that the unit disk D is an invariant subset of the plane. Hence we have a flow defined on a compact metric space. We consider a family of disjoint compact invariant subsets of D given by $\mathcal{M} = \{(0,0), S^1\}$. Let us consider a function $L_{\mathcal{M}} : D \to \mathbb{R}$ defined by $L_{\mathcal{M}}(z) = 1 - |z|$. We shall prove that $L_{\mathcal{M}}$ is a Lyapunov function for $\mathcal{M}$. The continuity is obvious. We choose an arbitrary point $z = (r(t), \theta(t))$ from the annulus $\mathrm{int}(D)\backslash\{(0,0)\}$. Using the inequality $0 < |z| = r(t) < 1$ as well as the inequality $\dot{r} = r(1-r)(\sin^2\theta + 1 - r^2) > 0$ we conclude that $r(t)$ is a strictly monotonically increasing function. Consequently, $L_{\mathcal{M}}(z) > L_{\mathcal{M}}(\varphi(z,t))$ which means that $L_{\mathcal{M}}$ is strictly decreasing along the trajectories in the annulus $\mathrm{int}(D)\backslash\{(0,0)\}$. It only remains to note that $L_{\mathcal{M}}(S^1) = 0$ and $L_{\mathcal{M}}(0,0) = 1$. Hence $L_{\mathcal{M}}$ is a Lyapunov function for $\mathcal{M}$ and $\mathcal{CR}(\varphi) = \{(0,0)\} \cup S^1$.

The following proposition is also from [4].

Proposition 3.1. *Suppose that $\mathcal{M} = \{M_j \mid j \in J\}$ admits a Lyapunov function $L_{\mathcal{M}}$ and $W = \bigcup_{j\in J} M_j$ is compact. Then*

i) For arbitrary $x \in X$ there exists M_s and M_r in $\mathcal{M}$ such that $\alpha(x) \subseteq M_s$ and $\omega(x) \subseteq M_r$.

ii) If $x \notin W$ then $\alpha(x)$ and $\omega(x)$ are nonempty sets and the following inequality holds $L_{\mathcal{M}}(\alpha(x)) > L_{\mathcal{M}}(x) > L_{\mathcal{M}}(\omega(x))$.

An important tool in proving our claim will be the existence of a Lyapunov function provided by the following fundamental theorem of dynamical systems (see Conley [3]).

Theorem 1. *Let (X, d) be a compact metric space with a flow φ. Then there exists a unique family $\mathcal{M} = \{M_j | j \in J\}$ of disjoint, compact and invariant sets which admits Lyapunov function and is maximal with this property. The set $\cup_{j\in J} M_j$ is actually $\mathcal{CR}(\varphi)$, which is compact and M_j are its connected components.*

4. Main result

Before proving our main result we shall make few remarks.

Remark 4.1: Let M be a compact invariant set. Then M contains a compact minimal set.

Remark 4.2: Every compact minimal set contains recurrent points.

Now we are ready to state our main result:

Theorem 2. *The set of all recurrent points $\mathcal{R}(\varphi)$ is contained in a connected component of the chain recurrent set $\mathcal{CR}(\varphi)$ if and only if $X = \mathcal{CR}(\varphi)$ and X is chain transitive. Consequently, if $\mathcal{R}(\varphi)$ is topologically transitive then X is chain transitive.*

Proof. Let us assume that the set of all recurrent points $\mathcal{R}(\varphi)$ is contained in a connected component of $\mathcal{CR}(\varphi)$. Then using the fact that the connected components of $\mathcal{CR}(\varphi)$ coincide with the chain transitive components (see [1]) we can conclude that there exists a chain transitive component M_1 such that $\mathcal{R}(\varphi) \subseteq M_1$ (chain transitive component means that any two points $x, y \in M_1$ can be connected by a finite $(\varepsilon, t, \varphi)$-chain from x to y and M_1 is maximal with that property). Now let us assume that there exists another non-empty chain transitive component $M_2 \neq M_1$ different from M_1. It follows that M_2 is a non-empty compact invariant set disjoint from M_1 i.e. $M_1 \cap M_2 = \emptyset$. Now using the remarks 4.1 and 4.2 (see [2], p.38 and p.41) we conclude that there exists a recurrent point in M_2. But this means that $\mathcal{R}(\varphi) \cap M_2 \neq \emptyset$ which yields that $M_1 \cap M_2 \neq \emptyset$. Hence a contradiction. So we can conclude that there exists only one chain transitive component and hence only one connected chain recurrent component.This means that $M_1 = \mathcal{CR}(\varphi)$ is a connected set. Hence according to theorem 1 the family $\mathcal{M} = \{M_1\}$ with only one element admits a Lyapunov function $L_{\mathcal{M}}$. Now if we assume that there exists $y \in X \setminus \mathcal{CR}(\varphi)$ then according to proposition 3.1 the following inequalities hold:

$$L_{\mathcal{M}}(\alpha(y)) > L_{\mathcal{M}}(y) > L_{\mathcal{M}}(\omega(y)).$$

But from the same proposition 3.1 we can conclude that $\alpha(y), \omega(y) \subseteq M_1$ having in mind that $\mathcal{M}$ is a singleton family. Hence $L_{\mathcal{M}}(\alpha(y)) = L_{\mathcal{M}}(\omega(y))$, surely a contradiction. This means that $X = \mathcal{CR}(\varphi) = M_1$. Furthermore X is connected and chain transitive. For the opposite direction, let us only note that if X is chain transitive then it is connected according to proposition in [1]. Hence the conclusion follows.

□

Now as a corollary of our previous result we obtain the results from [5] concerning the topological transitivity ($\mathcal{TT}$-property) of $\Omega(\varphi)$ and $\mathcal{CR}(\varphi)$.

Corollary 4.1. *If the non-wandering set $\Omega(\varphi)$ or the chain recurrent set $\mathcal{CR}(\varphi)$ has the ($\mathcal{TT}$)-property then the whole space is chain recurrent i.e. $X = \mathcal{CR}(\varphi)$.*

Proof. If the non-wandering set $\Omega(\varphi)$ or the chain recurrent set $\mathcal{CR}(\varphi)$ has the ($\mathcal{TT}$)-property then they are connected. Hence the set of all recurrent points $\mathcal{R}(\varphi)$ as their subset is contained in a connected component of $\mathcal{CR}(\varphi)$. Now the claim follows from theorem 2. □

Remark 4.3: Note that the $\mathcal{TT}$-property of these sets actually is not essential for the recurrent behavior of the whole space X. In example 2.2 non of the sets $\Omega(\varphi)$ or $\mathcal{CR}(\varphi)$ satisfy the $\mathcal{TT}$-property, but $\mathcal{R}(\varphi)$ is contained in a connected component of $\mathcal{CR}(\varphi)$. Hence $X = \mathcal{CR}(\varphi)$.

Corollary 4.2. *If $\mathcal{R}(\varphi) = \omega(x)$ for some $x \in X$ then $X = \mathcal{CR}(\varphi)$. Furthermore X is connected and chain transitive.*

Proof. In compact metric spaces the limit sets are connected (see [2], p.23). Hence the recurrent set $\mathcal{R}(\varphi)$ is also connected. Now the claim follows from theorem 2. □

Remark 4.4: Let us note that in example 2.1 $\mathcal{R}(\varphi) = \omega(x)$ for arbitrary $x \in X$. Hence $X = \mathbb{T}^2 = \mathcal{CR}(\varphi)$. Furthermore the torus $\mathbb{T}^2$ is connected and chain transitive.

Example 4.1. Consider the flow restricted on the Bowen eye shown in Figure 2. If we restrict the flow again on the darken ellipse $X = E$ then the non-wandering set $\Omega(\varphi)$ consists of two rest points, namely $(1, 0)$ and $(1, \pi)$ in polar coordinates (notice that this is not the case if we consider the flow on the whole eye). The chain recurrent set $\mathcal{CR}(\varphi)$ on the other hand is the whole space, i.e. $X = E = \mathcal{CR}(\varphi)$. Note that neither $\Omega(\varphi)$ nor $\mathcal{CR}(\varphi)$ are topologically transitive. But the set of all recurrent points $\mathcal{R}(\varphi)$ which consists of two rest points, namely $(1, 0)$ and $(1, \pi)$ in polar coordinates, is contained in a connected component of the chain recurrent set. Hence $X = \mathcal{CR}(\varphi) = E$.

Corollary 4.3. *Let φ be an arbitrary flow on the sphere $\mathbb{S}^2$ or the projective plane $\mathbb{RP}^2$. Then the chain recurrent set $\mathcal{CR}(\varphi)$ is not topologically transitive.*

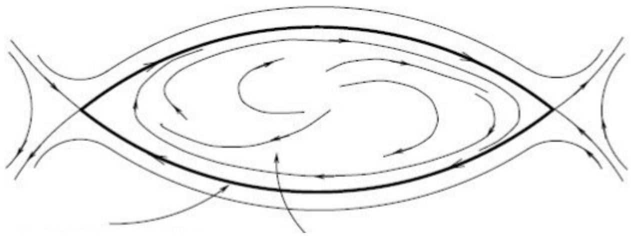

FIGURE 2. Phase portrait of the flow φ on the Bowen eye

Proof. If we assume that the chain recurrent set $\mathcal{CR}(\varphi)$ is topologically transitive then from the previous corollary 4.1 we can conclude that the sphere $\mathbb{S}^2$ or the projective plane $\mathbb{RP}^2$ are topologically transitive as well. But this is a contradiction with corollary 2.1. □

Example 4.2. Consider the flow on the 2-dimensional sphere $\mathbb{S}^2$ with one fixed point x and all other trajectories running along circles as shown in Figure 3. This flow satisfies the $\mathcal{TT}$-property of the non-wandering set $\Omega(\varphi)$, but satisfies neither the $\mathcal{TT}$-property of the chain recurrent set nor the $\mathcal{TT}$-property of the whole space $X = \mathbb{S}^2$. Note that the recurrent set $\mathcal{R}(\varphi) = \Omega(\varphi) = \omega(x)$ consists of only one point x contained in a connected component of $\mathcal{CR}(\varphi)$. Hence the chain recurrent set coincides with the whole space i.e $X = \mathbb{S}^2 = \mathcal{CR}(\varphi)$. Observe that we can obtain this conclusion also from corollary 4.1 using the topological transitivity of $\Omega(\varphi)$ or from corollary 4.2 using the equality $\mathcal{R}(\varphi) = \omega(x)$. Also note that X is connected and chain transitive.

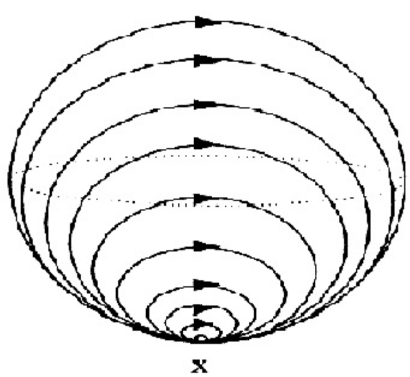

FIGURE 3. Phase portrait of the flow φ on the sphere $\mathbb{S}^2$

Corollary 4.4. *Let C be an arbitrary connected set such that $\mathcal{R}(\varphi) \subseteq C \subseteq \mathcal{CR}(\varphi)$. Then $X = \mathcal{CR}(\varphi)$ and X is connected. Furthermore X is chain transitive.*

Proof. It follows directly from theorem 2. □

Remark 4.5: Let us note that if $\Omega(\varphi)$ or the chain recurrent set $\mathcal{CR}(\varphi)$ are topologically transitive then we can choose $C = \Omega(\varphi)$ or $C = \mathcal{CR}(\varphi)$ respectively and thus obtain the result from [5]. But otherwise we are not limited to that choice. For example in the Bowen eye example 4.1 we can choose C to be the upper-half of the ellipse E^{uh} (including the rest points), i.e. $C = E^{uh}$. Then C is topologically transitive, but is neither $\Omega(\varphi)$ nor $\mathcal{CR}(\varphi)$. The corollary also applies for this choice of C.

Remark 4.6: According to corollary 4.4 the $\mathcal{TT}$-property requirement for $\Omega(\varphi)$ or $\mathcal{CR}(\varphi)$ in [5] is not necessary. The connectedness will do the job quite nicely.

Remark 4.7: In example 2.2 note that the non-wandering set $\Omega(\varphi)$ is not topologically transitive. But on the other hand it is connected which is good enough to use corollary 4.4.

Conclusion. As a finial conclusion we can state that the real reason for the recurrent behavior of the whole space X (in a connected space) is the "*distribution*" of the recurrent points and not the $\mathcal{TT}$-property. The $\mathcal{TT}$-property, when present, actually navigates the recurrent points towards "*distribution*" in a connected component of $\mathcal{CR}(\varphi)$.

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Received 24.3.2022
Revised 25.4.2022
Accepted 26.4.2022