

SOME NEW INTEGRAL INEQUALITIES FOR ρ -CONVEX FUNCTIONS

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Abstract. The aim of this paper is to generalize the results in [10] using the class of ρ -convex functions. Some special cases are deduced from main results. Applying our ideas and techniques, new interesting inequalities can be obtained in a similar way for different class of functions and operators.

1. INTRODUCTION AND PRELIMINARIES

The class of convex functions is well known in the literature and is defined in the following way:

Definition 1.1. Let I be an interval in \mathfrak{R} . A function $f : I \rightarrow \mathfrak{R}$, is said to be convex on I , if the inequality

$$f(\gamma u + (1 - \gamma)v) \leq \gamma f(u) + (1 - \gamma)f(v) \quad (1.1)$$

holds for all $u, v \in I$ and $\gamma \in [0, 1]$. Also, we say that f is concave, if the inequality in (1.1) holds in the reverse direction.

Theory of convex functions and their variant forms is used to study a wide class of problems that arise in various branches of pure and applied sciences. This theory provides us with a natural, unified and general framework for solving a wide class of optimization problems.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. Let $f : I \subseteq \mathfrak{R} \rightarrow \mathfrak{R}$ be a convex function and $u, v \in I$ with $u < v$. Then the following inequality holds:

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v f(x)dx \leq \frac{f(u)+f(v)}{2}. \quad (1.2)$$

This inequality (1.2) is also known as trapezium inequality.

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The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (1.2) in the premises of newly invented definitions due to motivation of convex function. For applications, generalizations and other aspects of convex functions, Hermite–Hadamard’s inequality and their variant forms, see the references [1]–[10].

The motivation of this paper is not only the generalization of the known results by Wu et al. in [10] but using our new ideas and techniques we can find some new interesting results for different operators and classes of functions as well. About their applications this will be investigate in our next paper.

Now, let us recall the following useful definitions.

Definition 1.2. A set $\mathcal{S} \subseteq \mathfrak{R}$ is said to be convex, if $(1 - \gamma)u + \gamma v \in \mathcal{S}$ for every $u, v \in \mathcal{S}$ and $\gamma \in [0, 1]$.

Recently, Wu et al. [10] introduced the following new classes of ρ -convex sets and ρ -convex functions.

Definition 1.3. A set $\mathcal{S} \subseteq \mathfrak{R}$ is said to be ρ -convex with respect to strictly monotonic continuous function ρ , if

$$\mathcal{M}_{[\rho]}(u, v) := \rho^{-1}((1 - \gamma)\rho(u) + \gamma\rho(v)) \in \mathcal{S}, \quad \forall u, v \in \mathcal{S}, \gamma \in [0, 1].$$

Definition 1.4. A function $\mathcal{S} \subseteq \mathfrak{R}$ is said to be ρ -convex with respect to strictly monotonic continuous function ρ , if

$$f(\mathcal{M}_{[\rho]}(u, v)) \leq (1 - \gamma)f(u) + \gamma f(v), \quad \forall u, v \in \mathcal{S}, \gamma \in [0, 1].$$

Note that the function f is called strictly ρ -convex on \mathcal{S} if the above inequality is true as a strict inequality for each distinct u and v in \mathcal{S} and for each $\gamma \in (0, 1)$. The function f is called ρ -concave (strictly ρ -concave) on \mathcal{S} , if $(-f)$ is ρ -convex (strictly ρ -convex) on \mathcal{S} .

The main objective of this article is to generalize the results by Wu et al. in [10] using the class of ρ -convex functions. We shall also discuss some of its special cases. At the end, a briefly conclusion will be given.

2. MAIN RESULTS

Theorem 2. Suppose that $f : I \rightarrow \mathfrak{R}$ is an integrable ρ -convex function with respect to the function ρ . Then for $\alpha > 0$, the following double integral inequalities hold:

$$\begin{aligned} & f\left(\rho^{-1}\left(\frac{\rho(u) + \rho(v)}{2}\right)\right) \leq \frac{\alpha}{2(\rho(v) - \rho(u))^\alpha} \\ & \times \left[\int_u^v (\rho(x) - \rho(u))^{\alpha-1} f(x)\rho'(x)dx + \int_u^v (\rho(v) - \rho(x))^{\alpha-1} f(x)\rho'(x)dx \right] \\ & \leq \frac{f(u) + f(v)}{2}. \end{aligned} \tag{2.1}$$

Proof. Since f is a ρ -convex function, we have

$$f\left(\rho^{-1}\left(\frac{\rho(u)+\rho(v)}{2}\right)\right) \leq \frac{f(u)+f(v)}{2}. \quad (2.2)$$

Substituting $x = \rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))$ and $y = \rho^{-1}(\gamma\rho(u) + (1-\gamma)\rho(v))$ in (2.2), we get

$$\begin{aligned} & f\left(\rho^{-1}\left(\frac{\rho(u)+\rho(v)}{2}\right)\right) \\ & \leq \frac{f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) + f(\rho^{-1}(\gamma\rho(u) + (1-\gamma)\rho(v)))}{2}. \end{aligned} \quad (2.3)$$

Multiplying both sides of (2.3) with $\gamma^{\alpha-1}$ and integrating with respect to γ on $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 \gamma^{\alpha-1} f\left(\rho^{-1}\left(\frac{\rho(u)+\rho(v)}{2}\right)\right) d\gamma \\ & \leq \int_0^1 \gamma^{\alpha-1} \left[\frac{f(\rho^{-1}((1-\gamma)\rho(u) + t\rho(v))) + f(\rho^{-1}(t\rho(u) + (1-\gamma)\rho(v)))}{2} \right] d\gamma. \end{aligned}$$

Hence

$$\begin{aligned} & f\left(\rho^{-1}\left(\frac{\rho(u)+\rho(v)}{2}\right)\right) \leq \frac{\alpha}{2(\rho(v)-\rho(u))^\alpha} \\ & \times \left[\int_u^v (\rho(u)-\rho(u))^{\alpha-1} f(u)\rho'(u)dx + \int_u^v (\rho(v)-\rho(u))^{\alpha-1} f(u)\rho'(u)dx \right]. \end{aligned}$$

So, the left-hand side of (2.1) is proved. Similarly, since f is a ρ -convex function, we have

$$f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) \leq (1-\gamma)f(u) + \gamma f(v) \quad (2.4)$$

and

$$f(\rho^{-1}(\gamma\rho(u) + (1-\gamma)\rho(v))) \leq \gamma f(u) + (1-\gamma)f(v). \quad (2.5)$$

Multiplying both sides of (2.4) and (2.5) with $\gamma^{\alpha-1}$ and integrating with respect to γ on $[0, 1]$, we get

$$\begin{aligned} & \int_0^1 \gamma^{\alpha-1} [f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) + f(\rho^{-1}(\gamma\rho(u) + (1-\gamma)\rho(v)))] d\gamma \\ & \leq \int_0^1 \gamma^{\alpha-1} [f(u) + f(v)] d\gamma. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{\alpha}{2(\rho(v)-\rho(u))^\alpha} \\ & \times \left[\int_u^v (\rho(u)-\rho(u))^{\alpha-1} f(u)\rho'(u)dx + \int_u^v (\rho(v)-\rho(u))^{\alpha-1} f(u)\rho'(u)dx \right] \\ & \leq \frac{f(u)+f(v)}{2}. \end{aligned}$$

The proof of Theorem 2 is completed. \square

Corollary 2.1. Taking $\alpha = 1$ in Theorem 2, we get ([10], Theorem 3.1).

Theorem 3. Suppose that $f : I \rightarrow \mathfrak{R}$ is an integrable ρ -convex function with respect to the function ρ . Then for $\alpha > 0$, the following double integral inequalities hold:

$$\begin{aligned}
& \frac{2f(u)}{(\rho(v) - \rho(u))^{\alpha+1}} \int_u^v (\rho(x) - \rho(u))^{\alpha-1} (\rho(v) - \rho(x)) f(x) \rho'(x) dx \\
& + \frac{2f(v)}{(\rho(v) - \rho(u))^{\alpha+1}} \int_u^v (\rho(x) - \rho(u))^{\alpha-1} f(x) \rho'(x) dx \\
& \leq \frac{1}{(\rho(v) - \rho(u))^\alpha} \int_u^v (\rho(x) - \rho(u))^{\alpha-1} f^2(x) \rho'(x) dx \\
& \quad + \frac{2f^2(u) + \alpha(\alpha+1)f^2(v) + 2\alpha f(u)f(v)}{\alpha(\alpha+1)(\alpha+2)} \\
& \leq \frac{4f^2(u) + 2\alpha(\alpha+1)f^2(v) + 4\alpha f(u)f(v)}{\alpha(\alpha+1)(\alpha+2)}. \tag{2.6}
\end{aligned}$$

Proof. Using the arithmetic–geometric means inequality, we have

$$\begin{aligned}
& 2f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) ((1-\gamma)f(u) + \gamma f(v)) \\
& \leq [f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v)))]^2 + [(1-\gamma)f(u) + \gamma f(v)]^2 \\
& = [f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v)))]^2 + (1-\gamma)^2 f^2(u) + \gamma^2 f^2(v) + 2\gamma(1-\gamma)f(u)f(v). \tag{2.7}
\end{aligned}$$

Multiplying both sides of (2.7) with $\gamma^{\alpha-1}$ and integrating with respect to γ on $[0, 1]$, we get

$$\begin{aligned}
& 2f(u) \int_0^1 \gamma^{\alpha-1} (1-\gamma) [f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v)))] d\gamma \\
& \quad + 2f(v) \int_0^1 \gamma^{\alpha-1} \gamma [f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v)))] d\gamma \\
& \leq \int_0^1 \gamma^{\alpha-1} [f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v)))]^2 d\gamma + f^2(u) \int_0^1 \gamma^{\alpha-1} (1-\gamma)^2 d\gamma \\
& \quad + f^2(v) \int_0^1 \gamma^{\alpha-1} \gamma^2 d\gamma + 2f(u)f(v) \int_0^1 \gamma^{\alpha-1} \gamma(1-\gamma) d\gamma. \tag{2.8}
\end{aligned}$$

Substituting $x = \rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))$ in (2.8), we get

$$\begin{aligned}
& \frac{2f(u)}{(\rho(v) - \rho(u))^{\alpha+1}} \int_u^v (\rho(u) - \rho(u))^{\alpha-1} (\rho(v) - \rho(u)) f(u) \rho'(u) dx \\
& + \frac{2f(v)}{(\rho(v) - \rho(u))^{\alpha+1}} \int_u^v (\rho(u) - \rho(u))^{\alpha-1} f(u) \rho'(u) dx \\
& \leq \frac{1}{(\rho(v) - \rho(u))^\alpha} \int_u^v (\rho(u) - \rho(u))^{\alpha-1} f^2(u) \rho'(u) dx \\
& \quad + \frac{2f^2(u) + \alpha(\alpha+1)f^2(v) + 2\alpha f(u)f(v)}{\alpha(\alpha+1)(\alpha+2)}. \tag{2.9}
\end{aligned}$$

So the left-hand side of (2.6) is proved. On the other hand, since f is a ρ -convex function, we have

$$f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) \leq (1-\gamma)f(u) + \gamma f(v). \quad (2.10)$$

Hence

$$\begin{aligned} & \frac{1}{(\rho(v) - \rho(u))^\alpha} \int_u^v (\rho(x) - \rho(u))^{\alpha-1} f^2(x) \rho'(x) dx \\ &= \int_0^1 \gamma^{\alpha-1} [f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v)))]^2 d\gamma \\ &\leq \int_0^1 \gamma^{\alpha-1} [(1-\gamma)f(u) + \gamma f(v)]^2 d\gamma \\ &= \frac{2f^2(u) + \alpha(\alpha+1)f^2(v) + 2\alpha f(u)f(v)}{\alpha(\alpha+1)(\alpha+2)}. \end{aligned} \quad (2.11)$$

Combining (2.9) and (2.11) we obtain the right-hand side of (2.6). The proof of Theorem 3 is completed. \square

Corollary 3.1. *Taking $\alpha = 1$ in Theorem 3, we get ([10], Theorem 3.2).*

Theorem 4. *Suppose that $f, h : I \rightarrow \mathbb{R}^+$ are two integrable ρ -convex functions with respect to the function ρ . Then for $\alpha > 0$, the following double integral inequalities hold:*

$$\begin{aligned} & \frac{1}{(\rho(v) - \rho(u))^\alpha} \int_u^v (\rho(x) - \rho(u))^{\alpha-1} f(x)h(x)\rho'(x) dx \\ &\leq \frac{2f(u)h(u)}{\alpha(\alpha+1)(\alpha+2)} + \frac{f(v)h(v)}{\alpha+2} + \frac{N(u,v)}{(\alpha+1)(\alpha+2)} \\ &\leq \frac{F(u)}{\alpha(\alpha+1)(\alpha+2)} + \frac{F(v)}{2(\alpha+2)} + \frac{P(u,v)}{(\alpha+1)(\alpha+2)}, \end{aligned} \quad (2.12)$$

where

$$N(u,v) = f(u)h(v) + f(v)h(u), \quad P(u,v) = f(u)f(v) + h(u)h(v) \quad (2.13)$$

and

$$F(u) = f^2(u) + h^2(u), \quad F(v) = f^2(v) + h^2(v). \quad (2.14)$$

Proof. Since f and h are two integrable ρ -convex functions, we have

$$\begin{aligned} & f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) h(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) \\ &\leq [(1-\gamma)f(u) + \gamma f(v)][(1-\gamma)h(u) + \gamma h(v)]. \end{aligned} \quad (2.15)$$

Multiplying both sides of (2.15) with $\gamma^{\alpha-1}$ and integrating with respect to γ on $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 \gamma^{\alpha-1} f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) h(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) d\gamma \\ &\leq \int_0^1 \gamma^{\alpha-1} [(1-\gamma)f(u) + \gamma f(v)][(1-\gamma)h(u) + \gamma h(v)] d\gamma \end{aligned} \quad (2.16)$$

$$\begin{aligned}
&= f(u)h(u) \int_0^1 \gamma^{\alpha-1}(1-\gamma)^2 d\gamma + N(u, v) \int_0^1 \gamma^{\alpha-1}\gamma(1-\gamma) d\gamma \\
&\quad + f(v)h(v) \int_0^1 \gamma^{\alpha-1}\gamma^2 d\gamma.
\end{aligned}$$

Substituting $x = \rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))$ in (2.16), we obtain the left-hand side of (2.12). For the right-hand side inequality, using the arithmetic–geometric means inequality in (2.16), gives

$$\begin{aligned}
&\int_0^1 \gamma^{\alpha-1}[(1-\gamma)f(u) + \gamma f(v)][(1-\gamma)h(u) + \gamma h(v)] d\gamma \\
&\leq \int_0^1 \gamma^{\alpha-1} \left[\frac{[(1-\gamma)f(u) + \gamma f(v)]^2 + [(1-\gamma)h(u) + \gamma h(v)]^2}{2} \right] d\gamma \\
&= \frac{1}{2} \int_0^1 [F(u)\gamma^{\alpha-1}(1-\gamma)^2 + F(v)\gamma^{\alpha-1}\gamma^2 + 2P(u, v)\gamma^{\alpha-1}\gamma(1-\gamma)] d\gamma \\
&= \frac{F(u)}{\alpha(\alpha+1)(\alpha+2)} + \frac{F(v)}{2(\alpha+2)} + \frac{P(u, v)}{(\alpha+1)(\alpha+2)}.
\end{aligned}$$

The proof of Theorem 4 is completed. \square

Corollary 4.1. *Taking $\alpha = 1$ in Theorem 4, we get ([10], Theorem 3.5).*

Theorem 5. *Suppose that $f, h : I \rightarrow \mathfrak{R}^+$ are two similarly ordered integrable ρ -convex functions with respect to the function ρ . Then for $\alpha > 0$, the following integral inequality holds:*

$$\frac{1}{(\rho(v) - \rho(u))^\alpha} \int_u^v (\rho(x) - \rho(u))^{\alpha-1} f(x)h(x)\rho'(x) dx \leq \frac{M(\alpha, u, v)}{\alpha(\alpha+1)}, \quad (2.17)$$

where

$$M(\alpha, u, v) = f(u)h(u) + \alpha f(v)h(v). \quad (2.18)$$

Proof. Since f and h are two similarly ordered integrable ρ -convex functions with respect to the function ρ , we have

$$\begin{aligned}
&f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) h(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) \\
&\leq [(1-\gamma)f(u) + \gamma f(v)][(1-\gamma)h(u) + \gamma h(v)] \\
&= (1-\gamma)f(u)h(u) + \gamma f(v)h(v) - \gamma(1-\gamma)[(f(u) - f(v))(h(u) - h(v))] \\
&\leq (1-\gamma)f(u)h(u) + \gamma f(v)h(v), \quad (2.19)
\end{aligned}$$

where $(f(u) - f(v))(h(u) - h(v)) \geq 0$. Multiplying both sides of (2.19) with $\gamma^{\alpha-1}$ and integrating with respect to γ on $[0, 1]$, we get

$$\begin{aligned}
&\int_0^1 \gamma^{\alpha-1} f(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) h(\rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))) d\gamma \\
&\leq \int_0^1 \gamma^{\alpha-1} [(1-\gamma)f(u)h(u) + \gamma f(v)h(v)] d\gamma. \quad (2.20)
\end{aligned}$$

Substituting $x = \rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))$ in (2.20), we obtain (2.17). The proof of Theorem 5 is completed. \square

Corollary 5.1. Taking $\alpha = 1$ in Theorem 5, we get ([10], Theorem 3.6).

Theorem 6. Suppose that $f, h : I \rightarrow \mathfrak{R}^+$ are two similarly ordered integrable ρ -convex functions with respect to the function ρ . Then for $\alpha > 0$, the following integral inequality holds:

$$\frac{1}{(\rho(v) - \rho(u))^\alpha} \int_u^v (\rho(v) - \rho(x))^{\alpha-1} f(x)h(x)\rho'(x)dx \leq \frac{R(\alpha, u, v)}{\alpha(\alpha + 1)}, \quad (2.21)$$

where

$$R(\alpha, u, v) = \alpha f(u)h(u) + f(v)h(v). \quad (2.22)$$

Proof. The proof is similar as Theorem 5, so we omit it. \square

Corollary 6.1. Taking $\alpha = 1$ in Theorem 6, we get the following integral inequality:

$$\frac{1}{\rho(v) - \rho(u)} \int_u^v f(x)h(x)\rho'(x)dx \leq \frac{f(u)h(u) + f(v)h(v)}{2}. \quad (2.23)$$

Theorem 7. Suppose that $f, h : I \rightarrow \mathfrak{R}^+$ are two integrable ρ -convex functions with respect to the function ρ . Then for $\alpha > 0$, the following integral inequality holds:

$$\begin{aligned} & 2f\left(\rho^{-1}\left(\frac{\rho(u) + \rho(v)}{2}\right)\right)h\left(\rho^{-1}\left(\frac{\rho(u) + \rho(v)}{2}\right)\right) - \frac{\alpha}{2(\rho(v) - \rho(u))^\alpha} \\ & \times \left[\int_u^v (\rho(x) - \rho(u))^{\alpha-1} f(x)h(x)\rho'(x)dx + \int_u^v (\rho(v) - \rho(x))^{\alpha-1} f(x)h(x)\rho'(x)dx \right] \\ & \leq \frac{\alpha}{(\alpha + 1)(\alpha + 2)}M(1, u, v) + \left[\frac{2 + \alpha(\alpha + 1)}{2\alpha(\alpha + 1)(\alpha + 2)} \right]N(u, v), \end{aligned} \quad (2.24)$$

where $N(u, v)$ is defined from (2.13) and $M(1, u, v)$ is defined from (2.18) for value $\alpha = 1$.

Proof. Since f and h are two integrable ρ -convex functions with respect to the function ρ , by the same way as in the proof of Theorem 2.1, we have

$$\begin{aligned} & f\left(\rho^{-1}\left(\frac{\rho(u) + \rho(v)}{2}\right)\right)h\left(\rho^{-1}\left(\frac{\rho(u) + \rho(v)}{2}\right)\right) \\ & \leq \frac{1}{4}[f(\rho^{-1}((1 - \gamma)\rho(u) + \gamma\rho(v))) + f(\rho^{-1}(\gamma\rho(u) + (1 - \gamma)\rho(v)))] \\ & \quad \times [h(\rho^{-1}((1 - \gamma)\rho(u) + \gamma\rho(v))) + h(\rho^{-1}(\gamma\rho(u) + (1 - \gamma)\rho(v)))] \\ & \leq \frac{1}{4}\{f(\rho^{-1}((1 - \gamma)\rho(u) + \gamma\rho(v)))h(\rho^{-1}((1 - \gamma)\rho(u) + \gamma\rho(v))) \\ & \quad + f(\rho^{-1}(t\rho(u) + (1 - \gamma)\rho(v)))h(\rho^{-1}(t\rho(u) + (1 - \gamma)\rho(v))) \\ & \quad + [(1 - \gamma)f(u) + \gamma f(v)][th(u) + (1 - \gamma)h(v)] \\ & \quad + [\gamma f(u) + (1 - \gamma)f(v)][(1 - \gamma)h(u) + \gamma h(v)]\}. \end{aligned} \quad (2.25)$$

Multiplying both sides of (2.25) with $\gamma^{\alpha-1}$ and integrating with respect to γ on $[0, 1]$, we obtain

$$\begin{aligned} & f\left(\rho^{-1}\left(\frac{\rho(u)+\rho(v)}{2}\right)\right)h\left(\rho^{-1}\left(\frac{\rho(u)+\rho(v)}{2}\right)\right) \\ & \leq \frac{\alpha}{4}\left\{\int_0^1\gamma^{\alpha-1}f\left(\rho^{-1}\left((1-\gamma)\rho(u)+\gamma\rho(v)\right)\right)h\left(\rho^{-1}\left((1-\gamma)\rho(u)+\gamma\rho(v)\right)\right)d\gamma\right. \\ & \quad + \int_0^1\gamma^{\alpha-1}f\left(\rho^{-1}\left(\gamma\rho(u)+(1-\gamma)\rho(v)\right)\right)h\left(\rho^{-1}\left(\gamma\rho(u)+(1-\gamma)\rho(v)\right)\right)d\gamma \\ & \quad \left.+ 2M(1,u,v)\int_0^1\gamma^{\alpha-1}\gamma(1-\gamma)dt + N(u,v)\int_0^1\gamma^{\alpha-1}(\gamma^2+(1-\gamma)^2)d\gamma\right\}. \end{aligned} \tag{2.26}$$

Substituting $x = \rho^{-1}((1-\gamma)\rho(u) + \gamma\rho(v))$ and $y = \rho^{-1}(\gamma\rho(u) + (1-\gamma)\rho(v))$ in (2.26), we obtain (2.24). The proof of Theorem 7 is completed. \square

Corollary 7.1. *Taking $\alpha = 1$ in Theorem 7, we get ([10], Theorem 3.7).*

Theorem 8. *Suppose that $f, h : I \rightarrow \mathfrak{R}^+$ are two integrable ρ -convex functions with respect to the function ρ . Then for $\alpha > 0$, the following integral inequality holds:*

$$\begin{aligned} & f\left(\rho^{-1}\left(\frac{\rho(u)+\rho(v)}{2}\right)\right)h\left(\rho^{-1}\left(\frac{\rho(u)+\rho(v)}{2}\right)\right) \\ & \leq \left[\frac{1}{4} + \frac{\alpha}{(\alpha+1)(\alpha+2)}\right]M(1,u,v) + \left[\frac{2+\alpha(\alpha+1)}{4\alpha(\alpha+1)(\alpha+2)}\right]N(u,v), \end{aligned} \tag{2.27}$$

where $N(u, v)$ is defined from (2.13) and $M(1, u, v)$ is defined from (2.18) for value $\alpha = 1$.

Proof. From Theorems 5, 6 and Theorem 7, we get the desired inequality (2.27). \square

Corollary 8.1. *Taking $\alpha = 1$ in Theorem 8, we get the following integral inequality:*

$$f\left(\rho^{-1}\left(\frac{\rho(u)+\rho(v)}{2}\right)\right)h\left(\rho^{-1}\left(\frac{\rho(u)+\rho(v)}{2}\right)\right) \leq \frac{5M(1,u,v) + 2N(u,v)}{12}. \tag{2.28}$$

Theorem 9. *Suppose that $f, h : I \rightarrow \mathfrak{R}^+$ are two integrable ρ -convex functions with respect to the function ρ . Then for $\alpha > 0$, the following double integral inequalities hold:*

$$\begin{aligned} & \int_u^v \int_u^v \int_0^1 t^{\alpha-1}(\rho(x)-\rho(u))^{\alpha-1}(\rho(y)-\rho(u))^{\alpha-1}f\left(\rho^{-1}\left((1-\gamma)\rho(x)+\gamma\rho(y)\right)\right) \\ & \quad \times h\left(\rho^{-1}\left((1-\gamma)\rho(x)+\gamma\rho(y)\right)\right)\rho'(x)\rho'(y)d\gamma dy dx \\ & \leq \frac{(2+\alpha(\alpha+1))}{\alpha^2(\alpha+1)(\alpha+2)}(\rho(v)-\rho(u))^{2\alpha}D(\alpha,u,v) + \frac{1}{(\alpha+1)(\alpha+2)} \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \int_u^v (\rho(y) - \rho(u))^{\alpha-1} h(y) \rho'(y) dy \int_u^v (\rho(x) - \rho(u))^{\alpha-1} f(x) \rho'(x) dx \right. \\
& \left. + \int_u^v (\rho(y) - \rho(u))^{\alpha-1} f(y) \rho'(y) dy \int_u^v (\rho(x) - \rho(u))^{\alpha-1} h(x) \rho'(x) dx \right\} \\
& \leq \frac{(2 + \alpha(\alpha + 1))}{\alpha^2(\alpha + 1)(\alpha + 2)} (\rho(v) - \rho(u))^{2\alpha} D(\alpha, u, v) \\
& \quad + \frac{(f(u) + f(v))(h(u) + h(v))}{2(\alpha + 1)(\alpha + 2)} (\rho(v) - \rho(u))^{2\alpha}, \tag{2.29}
\end{aligned}$$

where

$$D(\alpha, u, v) = \frac{2f(u)h(u)}{\alpha(\alpha + 1)(\alpha + 2)} + \frac{f(v)h(v)}{\alpha + 2} + \frac{N(u, v)}{(\alpha + 1)(\alpha + 2)} \tag{2.30}$$

and $N(u, v)$ is defined from (2.13).

Proof. Since f and h are two integrable ρ -convex functions with respect to the function ρ , we have

$$\begin{aligned}
& f(\rho^{-1}((1 - \gamma)\rho(u) + \gamma\rho(v))) h(\rho^{-1}((1 - \gamma)\rho(u) + \gamma\rho(v))) \\
& \leq [(1 - \gamma)f(u) + \gamma f(v)][(1 - \gamma)h(u) + \gamma h(v)] \\
& = (1 - \gamma)^2 f(u)h(u) + \gamma(1 - \gamma)[f(u)h(v) + h(u)f(v)] + \gamma^2 f(v)h(v). \tag{2.31}
\end{aligned}$$

Multiplying both sides of (2.31) with $\gamma^{\alpha-1}$ and integrating with respect to γ on $[0, 1]$, we obtain

$$\begin{aligned}
& \int_0^1 \gamma^{\alpha-1} f(\rho^{-1}((1 - \gamma)\rho(u) + \gamma\rho(v))) h(\rho^{-1}((1 - \gamma)\rho(u) + \gamma\rho(v))) d\gamma \\
& \leq f(u)h(u) \int_0^1 \gamma^{\alpha-1} (1 - \gamma)^2 d\gamma + f(v)h(v) \int_0^1 \gamma^{\alpha-1} \gamma^2 d\gamma \\
& \quad + [f(u)h(v) + h(u)f(v)] \int_0^1 \gamma^{\alpha-1} \gamma(1 - \gamma) d\gamma \\
& = \frac{2f(u)h(u)}{\alpha(\alpha + 1)(\alpha + 2)} + \frac{f(v)h(v)}{\alpha + 2} + \frac{f(u)h(v) + h(u)f(v)}{(\alpha + 1)(\alpha + 2)}. \tag{2.32}
\end{aligned}$$

Again, integrating both sides of (2.32) over the plane domain $\{(x, y) : x \in [u, v], y \in [u, v]\}$ and then using the left-hand side of Theorem 4, we deduce the left-hand side of (2.29).

On the other hand, substituting $x = \rho^{-1}((1 - \gamma)\rho(u) + \gamma\rho(v))$ and using again the fact that f and h are two integrable ρ -convex functions with respect to the function ρ , we get

$$\begin{aligned}
& \int_u^v (\rho(x) - \rho(u))^{\alpha-1} f(x) \rho'(x) dx = (\rho(v) - \rho(u))^\alpha \\
& \quad \times \int_0^1 f(\rho^{-1}((1 - \gamma)\rho(u) + \gamma\rho(v))) d\gamma \\
& \leq \frac{(f(u) + f(v))}{2} (\rho(v) - \rho(u))^\alpha. \tag{2.33}
\end{aligned}$$

Similarly, we have

$$\int_u^v (\rho(y) - \rho(u))^{\alpha-1} f(y) \rho'(y) dy \leq \frac{(f(u) + f(v))}{2} (\rho(v) - \rho(u))^\alpha. \quad (2.34)$$

$$\int_u^v (\rho(x) - \rho(u))^{\alpha-1} h(x) \rho'(x) dx \leq \frac{(h(u) + h(v))}{2} (\rho(v) - \rho(u))^\alpha. \quad (2.35)$$

$$\int_u^v (\rho(y) - \rho(u))^{\alpha-1} h(y) \rho'(y) dy \leq \frac{(h(u) + h(v))}{2} (\rho(v) - \rho(u))^\alpha. \quad (2.36)$$

Combining (2.33)–(2.36), we obtain the right-hand side of (2.29). The proof of Theorem 9 is completed. \square

Corollary 9.1. *Taking $\alpha = 1$ in Theorem 9, we get the following double integral inequalities:*

$$\begin{aligned} & \int_u^v \int_u^v \int_0^1 f(\rho^{-1}((1-\gamma)\rho(x) + \gamma\rho(y))) \\ & \quad \times h(\rho^{-1}((1-\gamma)\rho(x) + \gamma\rho(y))) \rho'(x) \rho'(y) d\gamma dy dx \\ & \leq \frac{1}{3} \left\{ 2(\rho(v) - \rho(u))^2 U(u, v) + \int_u^v \int_u^v f(x) h(y) \rho'(x) \rho'(y) dy dx \right\} \\ & \leq \frac{(\rho(v) - \rho(u))^2}{3} \left[2U(u, v) + \frac{(f(u) + f(v))(h(u) + h(v))}{4} \right], \end{aligned} \quad (2.37)$$

where

$$U(u, v) = \frac{f(u)h(u) + f(v)h(v)}{3} + \frac{N(u, v)}{6}. \quad (2.38)$$

3. CONCLUSION

Interested reader can obtain in a similar way new results for different operators such as the k -Riemann–Liouville fractional integral, Katugampola fractional integrals, the conformable fractional integral, Hadamard fractional integrals, etc. and they can be applied to obtain several interesting results in convex analysis, special functions, quantum mechanics, related optimization theory, mathematical inequalities using different class of functions. Our ideas and techniques may stimulate further research in different areas of pure and applied sciences.

Conflict of interest. All the Authors declares that they have no conflict of interest with anyone.

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