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44 (LXX) 2 (2020), 149–162

UDC: 519.175
DOI: 10.37560/matbil2020149k

TOTAL VERTEX STRESS ALTERATION IN CYCLE RELATED GRAPHS

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Abstract. In the main this paper discusses the addition of an edge $uv \in E(\overline{G})$ to a cycle graph C_n to obtain the 1-chorded cycle graph $C_n^{\sim 1}$ such that the total vertex stress of $C_n^{\sim 1}$ compared to the total vertex stress of C_n shows a maximum or minimum alteration over all $uv \in E(\overline{G})$. Furthermore, results for wheel graphs, helm graphs, flower graphs, sunlet graphs, sun graphs and prism graphs are also presented. Finally a heuristic algorithm is proposed which determines the total vertex stress in a general graph G.

1. Introduction

For general notation and concepts in graphs see [1, 2, 5]. Throughout the study only finite, simple connected and undirected graphs will be considered.

In modern network applications such as communication networks, social media networks, neural and artificial intelligence algorithmic networks, the typical structural changes are an increase in order (more nodes) and an increase in edges. The most frequent structural change seems to occur in respect of edge addition to the graphical embodiment of such network. In 1953 the researcher Alfonso Shimbel introduced the notion of *vertex stress* in a graph G denoted by $\mathcal{S}_G(v)$, $v \in V(G)$ (see [3]). Recall that the vertex stress of vertex $v \in V(G)$ is the number of times v is contained as an internal vertex in all shortest paths between all pairs of distinct vertices in $V(G)\backslash v$. Formally stated, $\mathcal{S}_G(v) = \sum\limits_{u \neq w \neq v \neq u} \sigma(v)$ with $\sigma(v)$ the number of shortest paths between vertices u, w which contain v as an internal vertex. See [3, 4]. The *total vertex stress* of G is given by $\mathcal{S}(G) = \sum\limits_{v \in V(G)} \mathcal{S}_G(v)$. The *average vertex stress* of G of order $n \geq 1$ and denoted by $\overline{\mathcal{S}}(G)$ follows naturally as $\overline{\mathcal{S}}(G) = \frac{1}{n} \sum\limits_{v \in V(G)} \mathcal{S}_G(v)$. Note that the trivial graph K_1 is connected. It implies that an isolated vertex v is *inherently* adjacent to itself which in turn implies that

2010 *Mathematics Subject Classification.* Primary: 05C12, 05C30, 05C38.
Key words and phrases. Vertex stress, average vertex stress, total vertex stress.

there exists an *inherent* path vv (not a loop). This inherent path has length zero and therefore contains no internal vertices. Thus, $\mathcal{S}_{K_1}(v) = \mathcal{S}(K_1) = \overline{\mathcal{S}}(K_1) = 0$. In fact, $\mathcal{S}_{K_n}(v) = \mathcal{S}(K_n) = \overline{\mathcal{S}}(K_n) = 0$, $\forall\ n \geq 1$. Since $K_n \cong K_{(1,1,1,\ldots,1)_{n-times}}$ a trivial theorem follows for complete m-partite graphs.

Theorem 1.1. *For a complete m-partite graph* $K_{n_1,n_2,n_3,\ldots,n_m}$*,*

$$\mathcal{S}(K_{n_1,n_2,n_3,\ldots,n_m}) = \sum_{i=1}^{m}[\frac{n_i(n_i-1)}{2}\sum_{\substack{j=1,\\ j\neq i}}^{m} n_j].$$

It is known that for a graph G of order $n \geq 1$ the maximum degree is bounded by $\Delta(G) \leq n-1$. If $deg_G(v) = n-1, \forall v \in V(G)$ then G is said to be a complete (or G is a complete graph). Consider a non-complete graph G with vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. Let $X_1 = \{v \in V(G) : deg_G(v) < n-1\}$. Clearly $X_1 \neq \emptyset$. Select any $v_j \in X_1$ and add all edges v_jv_k, where $v_k \notin N_G[v_j]$ to obtain graph G_1. It is possible to select a similar set $X_2 \subset V(G_1)$. If $X_2 = \emptyset$ then G_1 is complete. Else, the procedure of adding edges to G_1 in respect of $v_j \in X_2$ to obtain G_2 can be repeated and iteratively, so on. In a finite number of iterations say, ℓ-iterations the complete graph $G_\ell \cong K_n$ will be obtained. It can be said that the maximum (or equivalently, minimum) number of edges to be added to G to nullify vertex stress is given by $|\varepsilon(\overline{G})|$. It also means that $\overline{\mathcal{S}}(G_\ell) = 0$. Obviously, either $0 < \overline{\mathcal{S}}(G_i) \leq \overline{\mathcal{S}}(G)$ or $0 < \overline{\mathcal{S}}(G) \leq \overline{\mathcal{S}}(G_i)$, $1 \leq i \leq \ell - 1$. Inevitably the addition of an edge $uv \in E(\overline{G})$ to G results in an *alteration of vertex stress* of some vertices. The aforesaid motivates a study on: "what is the maximum decrease or minimum increase in the average vertex stress, $\overline{\mathcal{S}}(G) - \overline{\mathcal{S}}(G_1)$ by the addition of one edge?" The more general max-min-problem for adding t edges, $1 \leq t < |\varepsilon(\overline{G})|$ remains open.

An immediate application could be in road infrastructure planning. It solves the problem of deciding which cities or intersections should be joined by a road to ensure a maximum decrease in the average traffic congestion throughout a defined road network. From Theorem 1.1 we have a corollary which illustrates this application. Let the partition vertex set corresponding to n_i be $\{v_{n_i,1}, v_{n_i,2}, \ldots, v_{n_i,n_i}\}$.

Corollary 1.1.1. *For a complete m-partite graph* $K_{n_1,n_2,n_3,\ldots,n_m}$ *the maximum and minimum decrease in the total vertex stress are respectively given by:*
(a) *Adding edge* $v_{n_i,j}v_{n_i,k}$*,* $j \neq k$*, and* $1 < n_i = min\{n_\ell : 1 \leq \ell \leq m\}$*,*
(b) *Adding edge* $v_{n_i,j}v_{n_i,k}$*,* $j \neq k$*, and* $1 < n_i = max\{n_\ell : 1 \leq \ell \leq m\}$*.*

Section 2 considers 1-chorded cycle graphs. Section 3 presents results for certain cycle related graphs and a heuristic algorithm is proposed in Section 4. The paper concludes with some open problems and proposals for further research.

2. Maximum or minimum stress decrease in 1-chorded cycle graphs

The notion of stress regular graphs was defined in [4]. Simply put, graphs with vertices having equal vertex stress are stress regular graphs. Recall that conventionally a cycle graph C_n, $n \geq 3$ has vertices $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and edges $E(C_n) = \{v_1v_n\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\}$. For cycle graphs (simply called, cycles) we recall important results from [4].

Theorem 2.1. [4] *The vertex stress of any vertex in a cycle C_{2n}, $n \geq 2$ is* $\mathcal{S}_{C_{2n}}(v) = \frac{n(n-1)}{2}$.

Theorem 2.2. [4] *The vertex stress of any vertex in a cycle C_{2n+1}, $n \geq 1$ is* $\mathcal{S}_{C_{2n+1}}(v) = \frac{n(n-1)}{2}$.

Following from Theorems 2.1 and 2.2 all cycles are stress regular. It is agreed that a q-chorded cycle is denoted by $C_n^{\sim q}$.

It can be said that with the addition of an edge the bound $0 < \overline{\mathcal{S}}(G_1) \leq \overline{\mathcal{S}}(G)$ has an intuitive feel of "true" to it. This intuition follows from the fact that at least one shortest path has been reduced to length one. However, the bound $0 < \overline{\mathcal{S}}(G) \leq \overline{\mathcal{S}}(G_1)$ is perhaps less intuitive and calls for clarity.

By Theorem 2.1 we have that $\mathcal{S}(v) = 3$, $v \in V(C_6)$. Therefore,

$$\sum_{v \in V(C_6)} \mathcal{S}(v) = 18 \Leftrightarrow \overline{\mathcal{S}}(C_6) = 3.$$

Consider G_1 in the figure below.

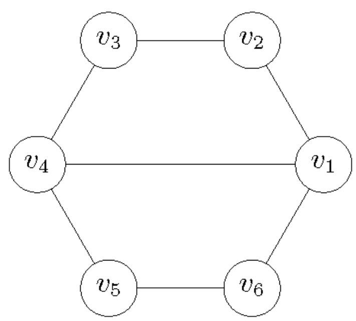

FIGURE 1. Graph G_1 which is a 1-chorded C_6.

It is straightforward to specify all shortest path between all pairs of distinct vertices in G_1. These are:

v_1v_2, $v_1v_2v_3$, $v_1v_4v_3$, v_1v_4, $v_1v_4v_5$, $v_1v_6v_5$, v_1v_6;
v_2v_3, $v_2v_3v_4$, $v_2v_1v_4$, $v_2v_3v_4v_5$, $v_2v_1v_6v_5$, $v_2v_1v_4v_5$, $v_2v_1v_6$;
v_3v_4, $v_3v_4v_5$, $v_3v_4v_5v_6$, $v_3v_2v_1v_6$, $v_3v_4v_1v_6$;
v_4v_5, $v_4v_5v_6$, $v_4v_1v_6$;
v_5v_6.

It follows easily that $\sum_{v \in V(G_1)} \mathcal{S}_{G_1}(v) = 22 \Leftrightarrow \overline{\mathcal{S}}(G_1) = \frac{22}{6} > 3$. This serves as example of the bound $0 < \overline{\mathcal{S}}(G) \leq \overline{\mathcal{S}}(G_1)$. Hence, not only does the addition of an edge reduce the length of at least one shortest path to length one, it may create additional shortest paths. Now consider G_1 in the figure below.

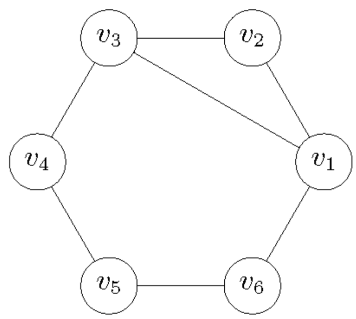

FIGURE 2. Graph G_1 which is a 1-chorded C_6.

Similar to the findings in respect of Figure 1, the 1-chorded cycle C_6 in Figure 2 yields $\sum\limits_{v \in V(G_1)} \mathcal{S}_{G_1}(v) = 11 \Leftrightarrow \overline{\mathcal{S}}(G_1) = \frac{11}{6} < 3$. Since all possible $C_6^{\sim 1's}$ up to isomorphism have been covered it is established that any one of the edges v_1v_3, v_2v_4, v_3v_5, v_4v_6, v_5v_1 or v_6v_2 may be added to C_6 to yield the maximum $\overline{\mathcal{S}}(C_6) - \overline{\mathcal{S}}(C_6^{\sim 1}) = \frac{18}{6} - \frac{11}{6} = \frac{7}{6}$. This result satisfies the bound $0 < \overline{\mathcal{S}}(G_1) \leq \overline{\mathcal{S}}(G)$. It is interesting to note that for some 1-chorded cycles both bounds cannot be satisfied. Consider $C_5^{\sim 1}$ below.

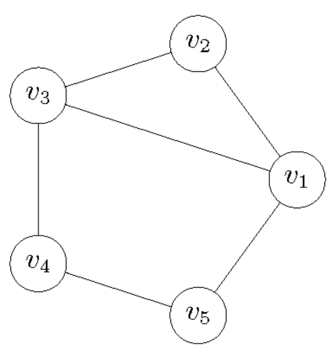

FIGURE 3. Graph G_1 which is $C_5^{\sim 1}$.

Clearly, up to isomorphism the depicted $C_5^{\sim 1}$ is the only case to consider. By Theorem 2.2 we have that for C_5, $\mathcal{S}(v) = 1$, $v \in V(C_5)$. Therefore $\sum\limits_{v \in V(C_5)} \mathcal{S}(v) = 5 \Leftrightarrow \overline{\mathcal{S}}(C_5) = 1$. Consider same for $C_5^{\sim 1}$ it follows that $\sum\limits_{v \in V(C_5^{\sim 1})} \mathcal{S}(v) = 6 \Leftrightarrow \overline{\mathcal{S}}(C_5^{\sim 1}) = \frac{6}{5} > 1$.

Recall a result for paths from [4].

Theorem 2.3. [4] *The vertex stress of v_i in a path P_n, $n \geq 1$ is $\mathcal{S}(v_i) = (i-1)(n-i)$.*

For a path $v_1v_2v_3\cdots v_n$ the *induced vertex stress* by v_1 on each of v_i, $i = 2,3,4,\ldots,v_n$ is given by $\int_{v_1}(v_i) = n - i$. Therefore the total vertex stress induced by v_1 equals, $\frac{(n-1)(n-2)}{2}$. This observation permits a useful lemma.

Lemma 1. *The total vertex stress in a path P_{k+1} is given by $\mathcal{S}(P_{k+1}) = \mathcal{S}(P_k) + \frac{k(k-1)}{2}$, $\mathcal{S}(P_1) = 0$, $k = 1,2,3,\ldots$*

Proof. For $k = 1$ it follows that $\mathcal{S}(P_2) = 0 = 0 + 0 = \mathcal{S}(P_1) + \frac{1(1-1)}{2}$. So the results holds for $k = 1$. Clearly $\mathcal{S}(P_2) + \frac{2(2-1)}{2} = 1 = \mathcal{S}(P_3)$. Assume the result holds for $k = \ell$. By extending path P_ℓ with one vertex to obtain path $P_{\ell+1}$ we just need to additionally account for the total vertex stress induced by vertex $v_{\ell+1}$ on the vertices of path P_ℓ. Hence, through immediate induction and the appropriate derivative of our observation we have that $\mathcal{S}(P_{\ell+1}) = \mathcal{S}(P_\ell) + \frac{\ell(\ell-1)}{2}$, $\forall\ k \geq 1$, $k \to \ell$. $\square$

Besides the recursive result a closed result is presented next.

Proposition 2.1. *The total vertex stress in a path P_n, $n \geq 1$ is given by $\mathcal{S}(P_n) = \frac{n(n-1)(n-2)}{6}$.*

Proof. The result is obvious for $n = 1, 2$. Assume it holds for $3 \leq n \leq k$. Hence, $\mathcal{S}(P_n) = \frac{n(n-1)(n-2)}{6}$, $3 \leq n \leq k$. Consider path P_{k+1}. It suffices to add the total vertex stress the vertex v_{k+1} induces on the vertices of the path P_k. Therefore

$$\begin{aligned}\mathcal{S}(P_{k+1}) &= \frac{k(k-1)(k-2)}{6} + \frac{k(k-1)}{2} = \frac{(k+1)k(k-1)}{6} \\ &= \frac{(k+1)((k+1)-1)((k+1)-2)}{6}.\end{aligned}$$

Hence, through induction the result holds for P_n, $n \geq 1$. $\square$

2.1. **On odd cycles.** Consider the sequence of odd cycles C_{2n+1}, $n = 2,3,4\ldots.$ Label the vertices consecutively in a particular way, i.e. u_1, v_1, v_2, v_3,..., v_n, w_n, w_{n-1}, w_{n-2},..., w_1. Without loss of generality consider the family of sequences (a sequence relates to a given t) of 1-chorded cycles $C_{2n+1}^{\sim 1}$, $n = 2,3,4,\ldots$ by adding the edge v_tw_t, $1 \leq t \leq n-1$. Clearly up to isomorphism a sequence for any t represents the sequence of 1-chorded odd cycles each of which contains an odd sub-cycle and an even sub-cycle.

Theorem 2.4. *The total vertex stress of a 1-chorded cycle $C^{\sim 1}_{2n+1}$, $n \geq 2$ where the chord is v_tw_t, $1 \leq t \leq n-1$ is given by*

$$\mathcal{S}(C^{\sim 1}_{2n+1}) = \frac{(2n-1)t^2 - (2n^2-1)t + (2n^3+2n^2-4n)}{2}.$$

Proof. Consider a 1-chorded cycle $C^{\sim 1}_{2n+1}$ with chord v_tw_t. Clearly the odd sub-cycle is C_{2t+1} and the even sub-cycle is $C_{2(n-t+1)}$. The total vertex stress of $C^{\sim 1}_{2n+1}$ can be determined by decomposition into appropriate subgraphs.
(a) The total vertex stress of sub-cycle C_{2t+1} is given by $\frac{t(2t+1)(t-1)}{2}$.
(b) The total vertex stress of sub-cycle $C_{2(n-t+1)}$ is given by $(n-t)(n-t+1)^2$.
(c) On the two paths $u_1v_1v_2\cdots v_n$ and $u_1w_1w_2\cdots w_n$ the total vertex stress induced by u_1 is given by $(n-t)(t+n-1)$.
(d) Subcase 1: Now consider the total induced vertex stress by v_i, $i = 1,2,3,\dots,t-1$ on the vertex sets $\{v_{i+1}, v_{i+2}, \dots, v_t\}$ and $\{v_{t+1}, v_{t+2}, \dots, v_n, w_n\}$. This accounts for total vertex stress equal to $(t-i)(n-t+1) + \frac{(n-t)(n-t+1)}{2}$, $i = 1,2,3,\dots,t-1$. The aforesaid reduces to

$$\frac{t(t-1)(n-t+1)}{2} + \frac{(t-1)(n-t+1)(n-t)}{2}.$$

Subcase 2: Since the path $v_1v_2v_3\cdots v_tw_tw_{t+1}\cdots w_n$ is a second shortest path to vertex w_n a similar analysis as in Subcase 1 can be applied. The total vertex stress induced by v_i, $i = 1,2,3,\dots,t-1$ along this path is given by

$$\frac{(t+2)(t-1)(n-t)}{2} + \frac{(t-1)(n-t)(n-t-1)}{2}.$$

(e) Due to symmetry a similar analysis as in (d) in respect of vertices w_i, $i = 1,2,3,\dots,t-1$ yields results, identical to those in (d).

Finally, after adding parts (a) to (e) the result is

$$\mathcal{S}(C^{\sim 1}_{2n+1}) = \frac{(2n-1)t^2 - (2n^2-1)t + (2n^3+2n^2-4n)}{2},$$

with chord v_tw_t, $1 \leq t \leq n-1$. $\square$

Analysis. It follows that

$$\mathcal{S}(C^{\sim 1}_{2n+1}) - \mathcal{S}(C_{2n+1}) = \frac{(2n-1)t^2 - (2n^2-1)t + 3n(n-1)}{2},$$

with chord v_tw_t, $1 \leq t \leq n-1$. For purposes of analysis let $A = (2n-1)$, $B = (2n^2-1)$, $C = 3n(n-1)$, $n \geq 2$ and $f(t) = At^2 - Bt + C$, $t \in \mathbb{R}$. Therefore $A > 0$, $B > 0$ and $C > 0$. The graph corresponding to $f(t)$ is a parabola and the symmetry axis is at $t_1 = \frac{2n^2-1}{2(2n-1)} > 0$ and t_1 is not an integer $\forall n \geq 2$. The discriminant is $(2n^2-1)^2 - 12n(2n-1)(n-1) < 0$, $\forall n \geq 2$. This leads to a result in respect of the total vertex stress alteration. Note that the unique case for cycle C_5 has been dealt with.

Theorem 2.5. *Up to isomorphism the total vertex stress of* $C_{2n+1}^{\sim 1}$, $n \geq 3$ *and* $1 \leq t \leq n-1$ *has:*
(*a*) *A maximum increase for chord* $v_t w_t$, $t = 1$.
(*b*) *A minimum increase for chord* $v_t w_t$, $t = \lceil \frac{n}{2} \rceil$ *if* n *is odd.*
(*c*) *A minimum increase for chord* $v_t w_t$, $t = \frac{n}{2}$ *if* n *is even.*

Proof. (a) It suffices to show that $[(2n-1)-(2n^2-1)+3n(n-1)]_{t=1} = n(n-1) > [(2n-1)(n-1)^2 - (2n^2-1)(n-1) + 3n(n-1)]_{t=n-1} = 2(n-1)$.
(b) The result follows from the fact that

$$\lim_{n\to\infty} \frac{2n^2-1}{2(2n-1)} = \frac{n}{2},$$

implies that the symmetry axis $t_1 = \frac{2n^2-1}{2(2n-1)} = \lfloor \frac{n}{2} \rfloor + \delta$, $0 < \delta < 1$.
(c) Similar to (b) the result follows from the fact that the symmetry axis is $t_1 = \frac{2n^2-1}{2(2n-1)} = \frac{n}{2} + \delta$, $0 < \delta < 1$. □

2.2. **On even cycles.** Consider the sequence of even cycles C_{2n}, $n = 2, 3, 4 \ldots$. Clearly for $n \geq 3$ an even 1-chorded cycle $C_{2n}^{\sim 1}$ can have either two odd sub-cycles or two even sub-cycles. We shall distinguish between the two types by $C_{2n}^{\sim 1\triangle}$ and $C_{2n}^{\sim 1\square}$ respectively.

Type 1. For $C_{2n}^{\sim 1\square}$, $n \geq 3$ label the vertices consecutively as $v_1, v_2, v_3, \ldots, v_n, w_n, w_{n-1}, \ldots, w_1$. Let the chord be $v_t w_t$, $2 \leq t \leq n-1$.

Proposition 2.2. *The total vertex stress of a* 1*-chorded cycle* $C_{2n}^{\sim 1\square}$, $n \geq 3$ *where the chord is* $v_t w_t$, $2 \leq t \leq n-1$ *is given by*

$$\mathcal{S}(C_{2n}^{\sim 1\square}) = (n-1)t^2 - (n^2-1)t + n^3 + n^2 - 2n.$$

Proof. The total vertex stress can be determined by decomposition into appropriate subgraphs. Through similar reasoning found in the proof of Theorem 2.4 we have:
(a) The total vertex stress of sub-cycle C_{2t} is given by $t^2(t-1)$.
(b) The total vertex stress of sub-cycle $C_{2(n-t+1)}$ is given by $(n-t+1)^2(n-t)$.
(c) The total vertex stress on the paths $v_1 w_1 w_2 \cdots w_n$ and $w_1 v_1 v_2 \cdots v_n$ is given by $(n-t)(n+t-1)$.
(d) The total vertex stress on paths $v_1 v_2 v_3 \cdots v_n w_n$ and $w_1 w_2 w_3 \cdots w_n v_n$ is given by $n(t-1)(n-t+1)$.
(e) The total vertex stress on paths $v_1 v_2 \cdots v_t w_t w_{t+1} \cdots w_n$ and $w_1 w_2 \cdots w_t v_t v_{t+1} \cdots v_n$ is given by $(n+1)(n-t)(n+1)$.
Hence, the result

$$\mathcal{S}(C_{2n}^{\sim 1\square}) = (n-1)t^2 - (n^2-1)t + n^3 + n^2 - 2n,$$

where the chord is $v_t w_t$, $2 \leq t \leq n-1$. □

By an analysis of the properties of the parabola similar to that in Theorem 2.5 we have the next corollary.

Corollary 2.5.1. *Up to isomorphism the total vertex stress of* $C_{2n}^{\sim 1\Box}$, $n \geq 3$, $2 \leq t \leq n-1$ *has:*
(a) *A maximum increase for chord* $v_t w_t$, $t = 2$.
(b) *A minimum increase for chord* $v_t w_t$, $t = \lceil \frac{n}{2} \rceil$ *if* n *is odd.*
(c) *A minimum increase for chord* $v_t w_t$, $t = \frac{n}{2}$ *if* n *is even.*

Remark. The solutions (a) $t = n-1$, (c) $t = \frac{n}{2}+1$ are valid. Due to isomorphism these solutions are not included in the corollary. Some readers might hold the philosophical view that they should be included.

Type 2. For $C_{2n}^{\sim 1\triangle}$, $n \geq 3$ label the vertices consecutively as $u_1, v_1, v_2, v_3, \dots, v_{n-1}, u_2, w_{n-1}, w_{n-2}, \dots, w_1$. Let the chord be $v_t w_t$, $1 \leq t \leq n-1$. For C_4 and up to isomorphism, one 1-chorded cycle exists which yields a reduction of 2 in respect of total vertex stress.

Proposition 2.3. *The total vertex stress of a* 1*-chorded cycle* $C_{2n}^{\sim 1\triangle}$, $n \geq 3$ *where the chord is* $v_t w_t$, $1 \leq t \leq n-1$ *is given by*

$$\mathcal{S}(C_{2n}^{\sim 1\triangle}) = \frac{(2n-2)t^2 - (2n^2-2n)t + 2n^3 - 3n^2 + n}{2}.$$

Proof. Similar reasoning as used in the proofs of Theorem 2.4 and Proposition 2.2 it suffices to consider:
(a) Total vertex stress of cycles C_{2t+1} and $C_{2(n-t)+1}$ as well as;
(b) Total vertex stress of paths $u_1 v_1 v_2 \cdots v_{n-1} u_2$ and $u_1 w_{n-1} \cdots w_1 u_2$ as well as;
(c) Total vertex stress of paths $v_1 v_2 \cdots v_t w_t w_{t-1} \cdots, w_1$ and $w_{n-1} w_{n-2} \cdots w_t v_t v_{t+1} \cdots v_{n-1}$.

Therefore, $\mathcal{S}(C_{2n}^{\sim 1\triangle}) = \frac{(2n-2)t^2 - (2n^2-2n)t + 2n^3 - 3n^2 + n}{2}$, where the chord is $v_t w_t$, $1 \leq t \leq n-1$. $\Box$

Corollary 2.5.2. *Up to isomorphism the total vertex stress of* $C_{2n}^{\sim 1\triangle}$, $n \geq 3$, $1 \leq t \leq n-1$ *has:*
(a) *A maximum decrease for chord* $v_t w_t$, $t = 1$.
(b) *A minimum increase for chord* $v_t w_t$, $t = \lceil \frac{n}{2} \rceil$ *if* n *is odd.*
(c) *A minimum increase for chord* $v_t w_t$, $t = \frac{n}{2}$ *if* n *is even.*

Remark. The solutions (a) $t = n-1$, (b) $t = \lfloor \frac{n}{2} \rfloor$ are valid. Due to isomorphism these solutions are not included in the corollary. Some readers might hold the philosophical view that they should be included.

3. Certain cycle related graphs

The previous section showed the difficulties encountered with 1-chorded cycles by using brute force counting. The reason is that for a fixed n the graphical structure can vary between 1-chorded cycles. For a comparison we present results for certain families of cycle related graphs which all have well-defined graphical

structure and elegant symmetry properties. Only the order of these graphs vary. Six well known cycle related graph families will be discussed.
(a) A wheel graph W_n, $n \geq 3$ is obtained from a cycle $C_n = v_1v_2v_3\cdots v_nv_1$ (or rim vertices) by adding a central vertex v_0 together with the edges (or spokes) v_0v_i, $1 \leq i \leq n$.
(b) A helm graph H_n is obtained by attaching a pendent vertex (or leaf) u_i to v_i, for all i.
(c) A flower graph Fl_n is obtained from a helm graph by adding the edges v_0u_i, for all i.
(d) A sunlet graph $S_n^{\ominus}$, $n \geq 3$ is obtained by taking cycle C_n together the isolated vertices u_i, $1 \leq i \leq n$ and adding the pendent edges v_iu_i.
(e) A sun graph $S_n^{\boxtimes}$, $n \geq 3$ is obtained by taking the complete graph K_n on the vertices $v_1, v_2, v_3, \ldots, v_n$ together the isolated vertices u_i, $1 \leq i \leq n$ and adding the edges v_iu_i, u_iv_{i+1} and $n+1 \equiv 1$. The boundary cycle of a sun graph is the cycle $C^b = v_1u_1v_2u_2v_3u_3\cdots u_nv_1$.
(f) A prism graph (or circular ladder) L_n°, $n \geq 3$ is obtained by taking two cycles of equal order n. Label as, $C_n^1 = v_1v_2v_3\cdots v_nv_1$ and $C_n^2 = u_1u_2u_3\cdots u_nu_1$. Add the edges v_iu_i, $1 \leq i \leq n$. A circular ladder can be viewed as $H_n^c - v_0$.

Proposition 3.1. *For a wheel graph W_n it follows that:*
(a) $\mathcal{S}(W_3) = 0$.
(b) $\mathcal{S}(W_n) = \frac{n(n-1)}{2}$, $n \geq 4$.

Proof. (a) W_3 is a complete graph.
(b) Let $n \geq 4$. Without loss of generality consider any vertex v_i. In respect of the vertices v_{i-2}, v_{i+2} (modular counting applies) the vertex v_i induces total vertex stress equal to 4 since there exist two shortest v_iv_{i-2}-paths as well as two shortest v_iv_{i+2}-paths.
Case 1. Let n be even. In respect of vertices $V(W_n)\backslash\{v_0, v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$ vertex v_i induces vertex stress equal to 1 on transit through v_0 only. Therefore, $\mathcal{S}(W_n) = n + \frac{n(n-3)}{2}$. After simplification, the result follows through immediate induction for $n \geq 4$ and even.
Case 2. Let n be odd. The result follows by similar reasoning to that in Case 1. $\square$

Proposition 3.2. *For a helm graph H_n it follows that:*
(a) $\mathcal{S}(H_3) = 15$.
(b) $\mathcal{S}(H_n) = n(4n+1)$, $n \geq 4$.

Proof. (a) The vertex u_i through all shortest u_iu_j-paths, $i \neq j$ induces vertex stress of 2 on vertex v_i. Also in respect of vertices v_0, v_j, $j \neq i$ the vertex u_i induces vertex stress equal to 3 on v_i. Therefore, $\mathcal{S}(H_3) = 3 \times 2 + 3 \times 3 = 15$.
(b) Consider helm graphs H_n, $n \geq 4$.
Due to symmetry of the graphical structure and without loss of generality consider vertex u_i. Note that by considering any vertex u_i the vertex labeling can be

changed through consecutive modular counting say clockwise, to obtain vertices $u_i, u_{i+1}, u_{i+2}, \dots, u_{i+(n-1)}$. The rim vertices are relabeled accordingly as well. In respect of vertices v_{i+1}, v_{i-1} the vertex u_i induces vertex stress of 2 on vertex v_i. In respect of vertices u_{i+1}, u_{i-1} the vertex u_i induces vertex stress of 2 on vertex v_i. In respect of vertices $u_{i+2}, u_{i-2}, v_{i+2}, v_{i-2}$ the vertex u_i induces vertex stress of 8 on vertex v_i. In respect of vertex v_0 vertex u_i induces vertex stress of 1 on vertex v_i. In respect of vertex u_j, v_j, $j \neq i, i+1, i-1, i+2, i-2$ the vertex u_i induces vertex stress of 2 on vertex v_i. There are $n-5$ such paths. Thus the total contribution of u_i to the vertex stress on v_i is $13 + 2(n-5)$. Also, of the shortest $u_{i+1}u_{i-1}$-path, $u_{i+1}v_{i-1}$-path, $u_{i-1}v_{i+1}$-path contribute vertex stress of 3 on v_i. Thus total vertex stress of v_i is $16 + 2(n-5)$. The vertex stress contributed by shortest u_iv_j-paths on v_0 is given by $3[(n-3) + (n-3) + (n-4) + (n-5) + ... + 3 + 2 + 1]$.
Finally, the total vertex stress for the wheel W_n must be accounted for. Hence, total vertex stress in H_n is given by

$$\mathcal{S}(H_n) = n(16 + 2(n-5)) + 3[(n-3) + \frac{(n-3)(n-2)}{2}] + \frac{n(n-1)}{2} = n(4n+1).$$

□

Proposition 3.3. *For a flower graph Fl_n it follows that:*
(a) $\mathcal{S}(Fl_3) = 15$.
(b) $\mathcal{S}(Fl_n) = 2n^2$, $n \geq 4$.

Proof. (a) Since the wheel W_3 is complete the only shortest paths to be considered are, $u_1v_0u_2$, $u_1v_0u_2$, $u_2v_0u_3$ and $u_1v_0v_2$, $u_1v_1v_2$ and $u_1v_0v_3$, $u_1v_1v_3$ and $u_2v_0v_3$, $u_2v_2v_3$ and $u_2v_0v_1$, $u_2v_2v_1$ and $u_3v_0v_1$, $u_3v_3v_1$ and $u_3v_0v_2$, $u_3v_3v_2$. Hence, $\mathcal{S}(Fl_3) = 15$.
(b) Since there is no shortest path passing through u_i, stress of u_i is 0. Consider the vertex v_i. In respect of vertices v_{i+1}, v_{i-1} the vertex u_i induces vertex stress of 2 on vertex v_i. This is the only contribution of the paths u_kv_j and u_ku_j. Now consider the contributions of u_i to the vertex the stress of v_0. For each i, the shortest u_iu_j-path, u_iv_j-path and v_iu_j-path contribute 3 to the stress of v_0 for $j = i+1, i+2, ..., n$. The total vertex stress for the wheel W_n must be accounted for. Thus the total vertex stress

$$\mathcal{S}(Fl_n) = n \times 2 + 3[(n-1) + (n-2) + ... + 2 + 1] + \frac{n(n-1)}{2} = 2n^2.$$

□

Proposition 3.4. *For a sunlet graph $S_n^{\ominus}$, $n \geq 3$ it follows that*

$$\mathcal{S}(S_n^{\ominus}) = 2n\lfloor\frac{n}{2}\rfloor(\lfloor\frac{n}{2}\rfloor + 1).$$

Proof. Since u_i is a pendant vertex its vertex stress is zero. On all shortest paths from a u_i to v_j with $d(u_i, v_j) \leq \lfloor\frac{n}{2}\rfloor$ and from u_i to u_j with $d(u_i, u_j) \leq \lfloor\frac{n}{2}\rfloor$, the vertex v_i is an internal vertex. Hence, u_i induces $4\lfloor\frac{n}{2}\rfloor$ vertex stress in respect v_i. Also u_{i+j} and v_{i+j} for $j = 1, 2, ...\lfloor\frac{n}{2} - 1\rfloor$ cumulatively contribute $4\lfloor\frac{n}{2} - j\rfloor$ to the vertex stress of v_i. Summation over all vertices followed by simplification yield the result

$$\mathcal{S}(S_n^{\ominus}) = 2n\lfloor\frac{n}{2}\rfloor(\lfloor\frac{n}{2}\rfloor + 1).$$

□

Proposition 3.5. *For a sun graph $S_n^{\boxtimes}$, $n \geq 3$ it follows that $\mathcal{S}(S_n^{\boxtimes}) = n(6n-15)$.*

Proof. Clearly there is no shortest path with an internal vertex u_i. The shortest paths $u_iv_{i+1}u_{i+1}$, $i = 1, 2, 3, \ldots, n$ and $n+1 \equiv 1$ contributes exactly 1 to the vertex stress of v_{i+1}. The shortest u_iv_j-paths, $j \neq i, i+1$, and shortest $u_{i+1}v_j$-paths, $j \neq i+1, i+2$ contribute exactly 1 to the vertex stress of v_{i+1}. Furthermore, for $j \neq i-1, i+1$ there exist two shortest $u_{i+1}u_j$-paths with the internal vertex v_{i+1}. Similarly for $j \neq i, i+2$ there exist two shortest $u_{i+1}u_j$-paths with the internal vertex v_{i+1}. In conclusion, $\mathcal{S}_{S_n^{\boxtimes}}(v_{i+1}) = 2(n-2) + 4(n-4) + 1 = 6n - 15$. Hence, $\mathcal{S}(S_n^{\boxtimes}) = n(6n-15)$. □

Proposition 3.6. *For a prism graph L_n°, $n \geq 3$ it follows that*

$$\mathcal{S}(L_n^{\circ}) = 2n\frac{[\lfloor\frac{n}{2}\rfloor(\lfloor\frac{n}{2}\rfloor + 1)(\lfloor\frac{n}{2}\rfloor + 2)]}{3} + 2S(C_n).$$

Proof. Consider the vertex v_i. There is no shortest u_ku_j path having v_i as an internal vertex. Hence, for the vertex stress of v_i the contributions are from the shortest u_kv_j-paths and the shortest v_rv_j-paths. The contribution from the v_rv_j-paths is $S_{C_n}(v_i)$. There are $2\lfloor\frac{n}{2}\rfloor$ number of these u_iv_j-paths and each contribute 1 to the stress of v_i. Now, for $j = 1, 2, ..., (\lfloor\frac{n}{2}\rfloor - 1)$, there are $(\lfloor\frac{n}{2}\rfloor - j)$ number of shortest $u_{i+j}v_k$-paths and each contributes $j+1$ to the vertex stress of v_i. Similarly for the shortest $u_{i-j}v_k$-paths. Thus through 'mirror image' summation,

$$\mathcal{S}_{L_n^{\circ}}(v_i) = 2[\lfloor\frac{n}{2}\rfloor + 2(\lfloor\frac{n}{2}\rfloor - 1) + 3(\lfloor\frac{n}{2}\rfloor - 2) + ... + (\lfloor\frac{n}{2}\rfloor - 1)2 + \lfloor\frac{n}{2}\rfloor] + S_{C_n}(v_i).$$

That is

$$\mathcal{S}_{L_n^{\circ}}(v_i) = 2\frac{[(\lfloor\frac{n}{2}\rfloor)(\lfloor\frac{n}{2}\rfloor + 1)(\lfloor\frac{n}{2}\rfloor + 2)]}{6} + S_{C_n}(v_i).$$

Since L_n° is vertex transitive the result follows after simplification. □

4. Heuristic algorithm: Total vertex stress

Definition 4.1. *For a set $X = \{x_q, x_k, x_p, x_h, \ldots, x_s\}$ the set of 2-element distance-subsets is defined as*

$$\{\{x_q, x_k\}, \{x_q, x_p\}, \{x_q, x_h\}, \ldots, \{x_q, x_s\}\}.$$

Let graph G be a finite undirected, simple connected graph of order $n \geq 1$ with $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$.

A heuristic algorithm for total vertex stress in G is described as follows:
Step 0: Let $t = n$, $j = 0$, set of vertex-sets $L_j = \{\emptyset\}$, source $V(G)$. Go to Step 1.
Step 1: Let $i = j + 1$. If $i = t$, go to Step 3. Else, let $V_j = \{v_k \in V(G) : j + 1 \leq k \leq n\}$ and $X_i = \{v_i\} \cup V_j \backslash N[v_i]$. If $X_i = \{v_i\}$ then let $X_i = \emptyset$. Go to Step 2.

Step 2: List $L_i = L_j \cup X_i$ and let $j = i$ and go to Step 1.
Step 3: Source L_{t-1}. For $\ell = 1, 2, 3, \ldots, |L_{t-1}|$ relabel the elements of L_{t-1} to $L_{t-1} = \{X_1, X_2, X_3, \ldots, X_\ell\}$. Now generate the corresponding sets Y_ℓ of 2-element distance-subsets for each $X_\ell \in L_{t-1}$. Label the 2-element subsets as $Z_{\ell,q}$, $1 \leq q \leq |X_\ell|-1$. Go to Step 4.
Step 4: Let $\ell = 1, 2, 3, \ldots, |L_{t-1}|$ and $q = 1, 2, 3, \ldots, |X_\ell|-1$. For each $Z_{\ell,q} = \{v_a, v_b\} \in Y_\ell$ utilize the GeeksforGeeks algorithm[1] [6] to determine the set,

$$\mathcal{P}_t(Z_{\ell,q}),\ 1 \leq t \leq \sum_{j=1}^{|L_{t-1}|} |Y_j|,$$

of all shortest $(v_a v_b)$-paths. Go to Step 5.
Step 5: Let $\nabla = \sum\limits_{j=1}^{|L_{t-1}|} |Y_j|$. Let $m = \sum\limits_{t=1}^{\nabla} |\mathcal{P}_t(Z_{\ell,q})|$. Let $Q(G) = \bigcup\limits_{t=1}^{m} \mathcal{P}_t(Z_{\ell,q}) = \{\underbrace{P_1, P_2, P_3, \ldots, P_{|\mathcal{P}_1|}}_{\mathcal{P}_1(Z_{1,1})}, \underbrace{P_{1+|\mathcal{P}_1|}, P_{2+|\mathcal{P}_1|}, P_{3+|\mathcal{P}_1|}, \ldots, P_{|\mathcal{P}_1|+|\mathcal{P}_2|}}_{\mathcal{P}_2(Z_{1,2})}, \ldots, P_m\}$.
Construct a $(m \times n)$-matrix $A = [a_{ij}]$ where an entry,

$$a_{ij} = \begin{cases} 0, & \text{if } v_j \notin V(P_i),\ P_i \in Q(G); \\ 0, & \text{if } v_j \in V(P_i) \text{ and an end-vertex of } P_i \in Q(G); \\ 1, & \text{if } v_j \in V(P_i) \text{ and an internal vertex of } P_i \in Q(G). \end{cases}$$

Go to Step 6.
Step 6: Let the string $S = (\Sigma(c_1), \Sigma(c_2), \ldots, \Sigma(c_n))$ where, $\Sigma(c_j) = \sum\limits_{i=1}^{m} a_{ij}$, $1 \leq j \leq n$. Exit.
Clearly, $\mathcal{S}_G(v_i) = \Sigma(c_i)$ and $\mathcal{S}(G) = \sum\limits_{j=1}^{n} \Sigma(c_j)$.

Claim 4.1. *The Heuristic algorithm: Total vertex stress, converges (exits after finite number of iterations) and yields an unique outcome.*

Proof. Step 0 is well-defined. The commutative property of vertex labeling cannot yield an ambiguous outcome.
Step 1 has a well-defined counting protocol and the test $i = t$ is unambiguous hence, after $n - 1$ iterations Steps 1 and 2 will terminate and Step 3 will initiate. Furthermore, when the test, if $X_i = \{v_i\}$ is true then v_i is adjacent to all other vertices in G, hence, v_i induces zero vertex stress in G and can be discarded. Therefore, let $X_i = \emptyset$ for purposes of the union of sets. The sets V_j, X_i as well as L_i (in Step 2) are well-defined from set theory and subject to the counting protocol of Step 1.
Sourcing L_{t-1} and relabeling to $L_{t-1} = \{X_1, X_2, X_3, \ldots, X_\ell\}$ are well-defined and a finite procedure. Thereafter, Step 3 relies on Definition 4.1 which in itself is

[1] Professor Sandeep Jain is the founder of GeeksforGeeks. It began as a blogging page and evolved into a successful portal to access amongst others, proven algorithms for various applications.

well-defined and the derivatives of the sets Y_ℓ may result from different vertex labeling. The aforesaid in itself does not yield a different outcome on conclusion.
Step 4 utilizes the GeeksforGeeks algorithm [6] together with a well-defined counting protocol. Completion of this step after finite iterations is guaranteed to proceed to Step 5.
Steps 5 and 6 are well-defined and interchanged rows and columns may result from different vertex labeling. However, the number of rows will remain equal amongst derivatives. Furthermore, since the $'+'$ operation is well-defined and the commutative law holds true, both the steps are finite and yield finite and equal results, i.e. $S = (\Sigma(c_1), \Sigma(c_2), \ldots, \Sigma(c_n))$, $\Sigma(c_j) = \sum\limits_{i=1}^{m} a_{ij}$, $1 \leq j \leq n$ and $\mathcal{S}_G(v_i) = \Sigma(c_i)$ and $\mathcal{S}(G) = \sum\limits_{j=1}^{n} \Sigma(c_j)$.
Assume the results are not unique following relabeling of the vertices. This means that there exists a result $S = (\Sigma(c_1), \Sigma(c_2), \ldots, \Sigma(c_n))$, $\Sigma(c_j) = \sum\limits_{i=1}^{m} a_{ij}$ which is not equal to all other results. This implies that a vertex labeling is possible which yields a different number of distinct shortest paths between at least one pair of distinct vertices in G. The aforesaid is a contraction as vertex labeling cannot change the adjacency matrix of graph G. Also can different vertex labeling not invalidate the integrity of the GeeksforGeeks algorithm. Hence, the results are all equal. Therefore, the claim that the outcome is unique. $\square$

5. Conclusion

The study suggests that to calculate total vertex stress for graphs through brute force requires efficient decomposition. Based on this observation the next problem is open.

Problem 1. Can the decomposition of specialised graphs into cycles and paths as used in Theorem 2.4 and Propositions 2.2 and 2.3 be formalized into an efficient algorithm?

Note that the total vertex stress of path P_5 is $\mathcal{S}(P_5) = 10$. By adding the edge v_1v_4 the graph $G = P_5 + v_1v_4$ has $\mathcal{S}(G) = 10$.

Problem 2. If possible, characterize the graphs for which an edge can be added without a change in the total vertex stress.

It is the view of the authors that brute force is unsatisfactory. Studying real world applications frequently requires large and complex graphs. It is suggested that the heuristic algorithm be formalized and coded and that complexity analysis be reported. Evidently Steps 5 and 6 can easily be integrated for efficiency. Such ICT application will assist the study of time dependent graph structures such as neural networks and other social media networks. Studying the converse i.e. "what is the minimum decrease or maximum increase in the total vertex stress by the deletion of one edge" remains open. It is evident that a wide scope for further research exists.

Spontaneous transmission of information A (real or fake) is the mechanism of unintentionally transmitting the information from a vertex v which contains information A, to neighboring vertices when information B is intentionally send and transmits through v. It means that an effective method of "spreading the news" would be to initially send information A to the subset of vertices which all have maximum vertex stress in the network. Thereafter, all other communication through the networks completes the messaging. This observation opens an avenue for applied research.

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Received 2.10.2020
Revised 25.1.2021
Accepted 4.2.2021