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MINIMAL PATH AND CUT VECTORS OF BINARY TYPE MULTI-STATE MONOTONE SYSTEMS

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Abstract. Binary type multi-state systems are systems that can be regarded as a binary systems at any level. We give a characterization by the structure of their minimal path and minimal cut vectors. Also we present some relations between minimal path and minimal cut vectors of different levels of such systems.

1. Introduction

Consider a multi state system with n components, such that, all components and whole system can be found in $M+1$ levels, from the set $S = \{0, 1, \dots, M\}$. The level M is the level of perfect work of the component or the system and the level 0 is the level of their total failure. We assume that for $i > j$, the system or the component works with higher quality in state i then in state j. Let x_i be the state of the i-th component of the system, for $x_i \in S$ and $1 \leq i \leq n$ and $\mathbf{x} = (x_1, x_2, \dots, x_n)$ the state vector. Then $E = \underbrace{S\times, \dots, \times S}_{n}$ is the state set of the system. For a given system we can define a **structure function** $\phi(\mathbf{x}) : E \to S$ which represents the level of the system when it is in state $\mathbf{x}$. The state vector $\mathbf{x}$ is **path vector** of level j iff $\phi(\mathbf{x}) \geq j$, and it is **cut vector** of level j iff $\phi(\mathbf{x}) < j$.

We use the definition for minimal path and cut vectors given in [1].

Definition 1. *The vector* $\mathbf{x}$ *is minimal path vector of level* j *if* $\phi(\mathbf{x}) \geq j$ *and* $\phi(\mathbf{y}) < j$ *for all* $\mathbf{y} < \mathbf{x}$;

The vector $\mathbf{x}$ *is minimal cut vector of level* j *if* $\phi(\mathbf{x}) < j$ *and* $\phi(\mathbf{y}) \geq j$ *for all* $\mathbf{y} > \mathbf{x}$.

For each state vector $\mathbf{x} \in E$ and $\forall j \in S$ we define the vector $\mathbf{x}^j$ by:

$$\mathbf{x}^j = (x_1^j, x_2^j, \dots, x_n^j), \qquad x_i^j = \begin{cases} 1, & x_i \geq j \\ 0, & x_i < j \end{cases}.$$

Similarly let

$$\phi^j(\mathbf{x}) = \begin{cases} 1, & \phi(\mathbf{x}) \geq j \\ 0, & \phi(\mathbf{x}) < j \end{cases}.$$

According [1], a system is a monotone multi state system **MMS** if its structure function satisfies:

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1) $\phi(\mathbf{x})$ is non-decreasing in each argument
2) $\phi(\mathbf{0}) = 0$, $\phi(\mathbf{M}) = M$, where $\mathbf{0} = (0, 0, \dots, 0)$, and $\mathbf{M} = (M, M, \dots, M)$.

Definition 2. *A system is a binary type monotone multi-state system (BTMMS) if* $\forall j \in \{0, 1, \dots, M\}$ *there exists binary structure function* ϕ_j *such that*

$$\phi(\mathbf{x}) \geq j \Leftrightarrow \phi_j(\mathbf{x}^j) = 1. \tag{1}$$

2. Structure of minimal path and cut vectors of Binary type MMS

Theorem 1. *A n-component monotone multi-state system with structure function* $\phi : E \to S$ *is a BTMMS if and only if for any* $j = \overline{1, M}$*, minimal path vectors of level* j *are of the following form:*

$$\mathbf{x} = (x_1, x_2, \dots, x_{n-1}, x_n),\ \forall i = \overline{1, n},\ x_i \in \{0, j\}. \tag{2}$$

Proof. First we will proof that the minimal path vectors of BTMMS are of the form (2).

Suppose that we have BTMMS with structure function ϕ. From Definition 2 there exists a binary type function ϕ_j that satisfies (1). Let $\mathbf{x}$ be a minimal path vector of level j of the system.

1) Suppose that there exists $k \in S$, such that $x_k = i$, for $0 < i < j$. Let

$$\mathbf{y} = (x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_n).$$

Then, $\mathbf{y} < \mathbf{x}$ and $\mathbf{y}^j = \mathbf{x}^j$. From $\mathbf{y}^j = \mathbf{x}^j$, we have that $\phi_j(\mathbf{y}^j) = \phi_j(\mathbf{x}^j) = 1$, and since the system is BTMMS, $\phi(\mathbf{y}) \geq j$. Consequently $\mathbf{y}$ is a smaller path vector of level j then $\mathbf{x}$, which is in contradiction with the assumption that $\mathbf{x}$ is minimal path vector of level j. So, we have that the vector $\mathbf{x}$ has no coordinates between 0 and j.

2) Suppose that there exists $k \in S$, such that $x_k = i$, where $j < i < M$. Let

$$\mathbf{y} = (x_1, \dots, x_{k-1}, j, x_{k+1}, \dots, x_n).$$

We have that $\mathbf{y} < \mathbf{x}$. Also, $\mathbf{y}^j = \mathbf{x}^j$, so $\phi_j(\mathbf{y}^j) = \phi_j(\mathbf{x}^j) = 1$. From the definition of BTMMS, $\phi(\mathbf{y}) \geq j$. Again we obtain that $\mathbf{y}$ is a smaller path vector of level j than $\mathbf{x}$, which is in contradiction with the assumption that $\mathbf{x}$ is minimal path vector of level j. Therefore the vector $\mathbf{x}$ has no coordinates between j and M.

From 1) and 2) we have that the coordinates of $\mathbf{x}$ are either 0 or j.

To proof the converse, i.e. that if the minimal path vectors of a MMS are of the form (2), then the system is of binary type, we define binary function ϕ_j: $\{0, 1\}^n \to \{0, 1\}$ by:

$$\phi_j(\mathbf{x}^j) = \phi^j(\mathbf{x}). \tag{3}$$

The proof will be complete if we proof that this function is well defined and that $\phi(\mathbf{x}) \geq j \Leftrightarrow \phi_j(\mathbf{x}^j) = 1$.

Let $\mathbf{x}^j = \mathbf{y}^j$ such that $\phi_j(\mathbf{x}) \neq \phi_j(\mathbf{y})$. From (3) we have that $\phi^j(\mathbf{x}) \neq \phi^j(\mathbf{y})$. Without lose of generality we can assume $\phi^j(\mathbf{x}) = 1$ and $\phi^j(\mathbf{y}) = 0$. Let

$$\mathbf{z} = \begin{cases} j, & \mathbf{x}_k \geq j \\ 0, & \mathbf{x} < j \end{cases} = \begin{cases} j, & \mathbf{y}_k \geq j \\ 0, & \mathbf{y} < j \end{cases}.$$

Since $\phi(\mathbf{x}) \geq j$, there is a minimal path vector $\mathbf{v}$ of level j, of the form (2), such that $\mathbf{v} \leq \mathbf{x}$. Then $\mathbf{v} \leq \mathbf{z}$. This implies $\phi(\mathbf{z}) \geq \phi(\mathbf{v}) \geq j$, i.e. $\mathbf{z}$ is path vector of level j.

On the other hand, $\mathbf{z} \leq \mathbf{y} \Rightarrow \phi(\mathbf{z}) \leq \phi(\mathbf{y}) < j$ which is a contradiction. So, the function ϕ_j is well defined.

At the and we will proof that (1) holds.

$$\phi(\mathbf{x}) \geq j \Leftrightarrow \phi^j(\mathbf{x}) = 1 \Leftrightarrow \phi_j(\mathbf{x}^j) = 1.$$

□

Theorem 2. *A n-component MMS with structure function $\phi : E \to S$ is a BT-MMS if and only if for every $j = \overline{1, M}$, the minimal cut vectors of level j are of the following form:*

$$\mathbf{x} = (x_1, x_2, \dots, x_{n-1}, x_n), \ \forall i = \overline{1, n}, \ x_i \in \{M, j-1\}. \qquad (4)$$

Proof. First, we will proof that the minimal cut vectors of BTMMS are of the form (4).

Suppose that we have BTMMS with structure function ϕ. From Definition 2 there exists a binary type function ϕ_j that satisfies (1). Let $\mathbf{x}$ be a minimal cut vector of level j.

1) Suppose that there is $k \in S$ such that $x_k = i$, for $i < j - 1$. Let

$$\mathbf{y} = (x_1, \dots, x_{k-1}, j-1, x_{k+1}, \dots, x_n).$$

It is clear that $\mathbf{y} > \mathbf{x}$ and $\mathbf{y}^j = \mathbf{x}^j$. Since $\mathbf{y}^j = \mathbf{x}^j$

$$\phi_j(\mathbf{y}^j) = \phi_j(\mathbf{x}^j) = 0 \Rightarrow \phi(\mathbf{y}) < j \Rightarrow \phi^j(\mathbf{y}) = 0.$$

Therefore, $\mathbf{y}$ is a cut vector of level j, bigger then $\mathbf{x}$, which is in contradiction with the assumption that $\mathbf{x}$ is a minimal cut vector. So, the vector $\mathbf{x}$ has no coordinates smaller then $j - 1$.

2) Now, suppose that there is $k \in S$, such that $j - 1 < x_k < M$. Let

$$\mathbf{y} = (x_1, \dots, x_{k-1}, M, x_{k+1}, \dots, x_n) > \mathbf{x}.$$

Then, $\mathbf{y}^j = \mathbf{x}^j$, so

$$\phi_j(\mathbf{y}^j) = \phi_j(\mathbf{x}^j) = 0 \Rightarrow \phi(\mathbf{y}) < j \Rightarrow \phi^j(\mathbf{y}) = 0.$$

We get that $\mathbf{y}$ is a cut vector of level j, bigger then $\mathbf{x}$, which is not possible since $\mathbf{x}$ is a minimal cut vector. So the vector $\mathbf{x}$ has no coordinates x_k, such that $j - 1 < x_k < M$.

From 1) and 2) we have that all coordinates of $\mathbf{x}$ are either M or $j-1$.

Now suppose that if the minimal cut vectors of MMS are of the form (4). Define a function ϕ_j by:

$$\phi_j(\mathbf{x}^j) = \phi^j(\mathbf{x}). \tag{5}$$

First we will proof that this function is well defined. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors such that $\mathbf{x}^j = \mathbf{y}^j$ and $\phi_j(\mathbf{x}) \neq \phi_j(\mathbf{y}) \Leftrightarrow \phi^j(\mathbf{x}) \neq \phi^j(\mathbf{y})$. We can assume that $\phi^j(\mathbf{x}) = 1$ and $\phi^j(\mathbf{y}) = 0$. Let $\mathbf{z} = (z_1, \dots, z_n)$ sach that:

$$z_k = \begin{cases} j-1, & x_k < j \\ M, & x_k \geq j \end{cases} = \begin{cases} j-1, & y_k < j \\ M, & y_k \geq j \end{cases}$$

Since $\phi(\mathbf{y}) \leq j$, we have that there exists a minimal cut vector $\mathbf{v}$ for level j that satisfies (4), such that $\mathbf{v} \geq \mathbf{y}$. This vector also satisfies $\mathbf{v} > \mathbf{z}$. Consequently, $j > \phi(\mathbf{v}) \geq \phi(\mathbf{z})$.

On the other hand, $\mathbf{z} \geq \mathbf{x} \Rightarrow \phi(\mathbf{z}) \geq \phi(\mathbf{x}) \geq j$. We obtain that $j > \phi(\mathbf{x}) \geq j$, which is not possible, so the function is well defined.

To complete the proof we will show that (1) is true.

$$\phi(\mathbf{x}) \geq j \Leftrightarrow \phi^j(\mathbf{x}) = 1 \Leftrightarrow \phi_j(\mathbf{x}^j) = 1,$$

so the system is a BTMMS. □

3. Relations between minimal path and cut vectors of different levels

From Theorem 2 and Theorem 1 in the case of BTMMS we can talk about minimal path set and minimal cut set.

Definition 3. *The set $A \subseteq S$ is a minimal path set of level j if and only if there is a minimal path vector of level j such that $\forall i \in A$, $x_i = j$, and $\forall i \in A^c$, $x_i = 0$.*

The set $A \subseteq S$ is a minimal cut set of level j if and only if there is a minimal cut vector of level j such that $\forall i \in A$, $x_i = j-1$, and $\forall i \in A^c$, $x_i = M$.

Proposition 1. *For a given BTMMS with structure function ϕ and a minimal path set A of level j, for all $k > j$, there is no minimal path set B of level k, such that $B \subset A$.*

Proof. Suppose the opposite, i.e. that in this BTMMS, there is a minimal path set A of level j and a minimal path set B of level k, for $j < k$, so that $B \subset A$. We define vectors $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ and $\mathbf{z} = (z_1, \dots, z_n)$ as:

$$x_i = \begin{cases} j, & i \in A \\ 0, & i \in A^c \end{cases}, \quad y_i = \begin{cases} k, & i \in B \\ 0, & i \in B^c \end{cases}, \quad z_i = \begin{cases} j, & i \in B \\ 0, & i \in B^c \end{cases}.$$

It is clear that $\mathbf{x}$ is a minimal path vector of level j. Also $\mathbf{y}$ is a minimal path vector of level $k > j$, so $\phi(\mathbf{y}) > j$. Since $\mathbf{z} < \mathbf{x}$ we have that $\phi(\mathbf{z}) < j$. On the other hand $\mathbf{z}^j = \mathbf{y}^j$, so $\phi_j(\mathbf{z}^j) = \phi_j(\mathbf{y}^j)$, which is no true. These means that our assumption is not true, i.e. there is no minimal path set B of level $k > j$, such that $B \subset A$. □

Proposition 2. *For a given BTMMS with structure function ϕ and a minimal path set B of level k, for all $j < k$, there is a minimal path set A of level j, such that $A \subset B$.*

Proof. Let B be the minimal path set of level k, and $j < k$. We define vectors $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ as:

$$x_i = \begin{cases} j, & i \in B \\ 0, & i \in B^c \end{cases}, \quad y_i = \begin{cases} k, & i \in B \\ 0, & i \in B^c \end{cases}.$$

It is clear that $\mathbf{y}$ is a minimal path vector of level k, so $\phi(\mathbf{y}) = k > j$. Also we have that $\mathbf{x}^j = \mathbf{y}^j$, so $\phi_j(\mathbf{x}^j) = \phi_j(\mathbf{y}^j)$. There of $\phi(\mathbf{x}) \geq j$, so $\mathbf{x}$ is a path vector of level j. Sequentially, there is a minimal path vector $\mathbf{z}$ of level j smaller or equal to $\mathbf{x}$. From Theorem 1, the coordinates of $\mathbf{z}$ are either 0 or j, so there is a set $A \subseteq B$ such that $x_i = \begin{cases} j, & i \in A \\ 0, & i \in A^c \end{cases}$. This set A is a minimal path set of level j. □

Proposition 3. *For a given BTMMS with structure function ϕ and a minimal cut set A of level j, for all $k < j$, there is no minimal cut set B of level k, such that $B \subset A$.*

Proof. Suppose the opposite, i.e. that in this BTMMS, there is a minimal cut set A of level j and minimal cut set B of level k, for $j > k$, such that $B \subset A$. We define vectors $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ and $\mathbf{z} = (z_1, \dots, z_n)$ as:

$$x_i = \begin{cases} j-1, & i \in A \\ M, & i \in A^c \end{cases}, \quad y_i = \begin{cases} k-1, & i \in B \\ M, & i \in B^c \end{cases}, \quad z_i = \begin{cases} j-1, & i \in B \\ M, & i \in B^c \end{cases}$$

It is clear that $\mathbf{x}$ is a minimal cut vector of level j and $\mathbf{y}$ is a minimal cut vector of level k. Since $\mathbf{y}$ is a cut vector of level k, $\phi(\mathbf{y}) < k < j$. There of $\phi_j(\mathbf{y}) = 0$. On the other side, $\mathbf{z}^j = \mathbf{y}^j$, so $\phi_j(\mathbf{z}^j) = \phi_j(\mathbf{y}^j) = 0 \Rightarrow \phi(\mathbf{z}) < j$. This means that $\mathbf{z}$ is bigger cut vector of level j then the vector $\mathbf{x}$, which is in contradiction with our assumption that $\mathbf{x}$ is a minimal cut vector of level j. □

Proposition 4. *For a given BTMMS with structure function ϕ and a minimal cut set B of level k, for all $j > k$, there is a minimal cut set A of level j, such that $A \subseteq B$.*

Proof. Let B be the minimal cut set of level k, and $j > k$. We define vectors $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ as:

$$x_i = \begin{cases} j-1, & i \in B \\ M, & i \in B^c \end{cases}, \quad y_i = \begin{cases} k-1, & i \in B \\ M, & i \in B^c \end{cases}.$$

It is clear that $\mathbf{y}$ is a minimal cut vector of level k, so $\phi(\mathbf{y}) < k < j$. Also we have that $\mathbf{x}^j = \mathbf{y}^j$, so $\phi_j(\mathbf{x}^j) = \phi_j(\mathbf{y}^j)$. There of we have that $\phi(\mathbf{x}) < j$, so $\mathbf{x}$ is a cut vector of level j. Sequentially, there is a minimal cut vector $\mathbf{z}$ of level j bigger or equal then $\mathbf{x}$. From Theorem 2, the coordinates of $\mathbf{z}$ are either M or $j-1$, so there is a set $A \subseteq B$ such that $x_i = \begin{cases} j-1, & i \in A \\ M, & i \in S \setminus A \end{cases}$. This set A is a minimal cut set of level j. □

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