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# SOME RELATIONS CONCERNING TRIANGLES AND BICENTRIC QUADRILATERALS IN CONNECTION WITH PONCELET'S CLOSURE THEOREM

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**Abstract.** Some relations concerning triangles and bicentric quadrilaterals in connection with Poncelet's closure theorem are found. These relations may be interesting and useful. So. using relations given by (2.4) and (3.15) we give in a very simple way an other proof of the Poncelet's closure theorem for triangles and bicentric quadrilaterals when conics are circles one inside of the other.

## 1. Preliminaries

A polygon which is both chordal and tangential will be called a *bicentric polygon.* The first who concerned with bicentric polygons was German mathematician Nicolaus Fuss (1755 – 1826), a friend of Leonhard Euler. Ho posed himself the following problem (known as Fuss' problem of the bicentric quadrilateral):

*Find the relation between the radii and the line segment joining the centers of the circles of circumscription and inscription of a bicentrical quadrilateral.*

He found that:

$$2\rho^2(r^2+z^2)=(r^2-z^2)^2, \tag{1.1}$$

where $r$ and $\rho$ are radii and $z$ is the distance between the centers of the circles of circumscription and inscription.

This problem is listed and considered in [?, p.188] as one of the 100 great problem of elementary mathematics.

Fuss also found corresponding formulas for bicentric pentagon, hexagon, heptagon and octagon (Nova Acta Petropol., XII, 1798).

The corresponding formula for triangle is

$$r^2-z^2=2r\rho, \tag{1.2}$$

and had already been given by Euler.

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The very remarkable theorem concerning bicentric polygon is given by French mathematician Poncelet (1788-1867). In the formulation of this theorem will be used the so-called Poncelet traverse. In short about this.

Let $C_1$ and $C_2$ be two circles in a plane. If from any point on $C_2$ we draw a tangent to $C_1$, extend the tangent line so that it intersects $C_2$, and draw from the point of intersection a new tangent to $C_1$, extend this tangent similarly to intersect $C_2$ and continue in this way, we obtain the so-called Poncelet traverse which, when it consists of $n$ chord of the circle $C_2$ (circle of circumscription), is called $n$-sided,

The Poncelet's theorem (for circles) can be expressed as follows:

*If on the circle of circumscription there is one point of origin for which $n$-sided Poncelet traverse is closed, then the $n$-sided traverse will also be closed for any other point of origin on the circle.*

The Figure 1 is an illustration for triangle. If $|OM| = z$ and (1.2) is valid, then there are infinitely many triangles whose incircle and circumcircle are $C_1$ and $C_2$.

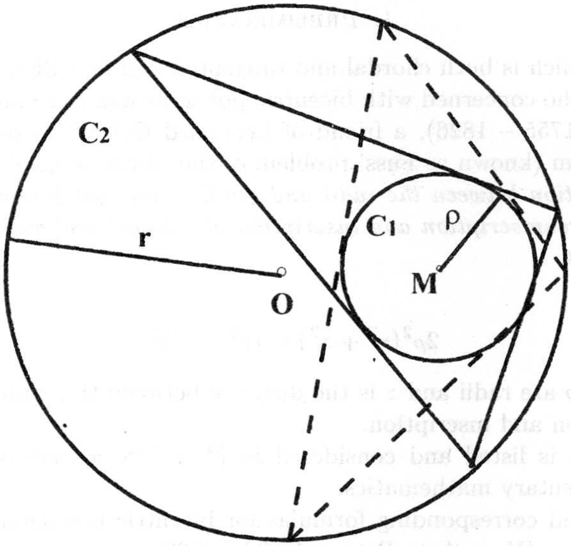


**Figure 1**

Poncelet demonstrated that analogously hold for conic sections so that general theorem reads:

Poncelet's closure theorem. *If an $n$-sided Poncelet traverse constructed for two given conic sections is closed for one position of the point of origin, it is closed for any position of the point of origin.*

A lot of interesting informations concerning Poncelet's closure theorem we have found in [4] and references therein.

## 2. Some relations concerning triangles

First on notation which will be used in this section.

Let $r, \rho$ and $z$ be any given lengths (in fact positive numbers) such that holds Euler's relation (1.2). Let $M$ and $O$ be points and $C_1 = M(\rho)$ and $C_2 = O(r)$ be circles such that holds $|OM| = z$.

If $A_1A_2A_3$ is a considered triangle whose incircle is $C_1$ and circumcircle $C_2$, then by $t_1, t_2, t_3$ will be denoted the lengths of its tangents, that is

$$t_i + t_{i+1} = |A_iA_{i+1}|, \quad i = 1, 2, 3.$$

Of course, indices are calculated *modulo* 3.

As an illustration see Figure 2.

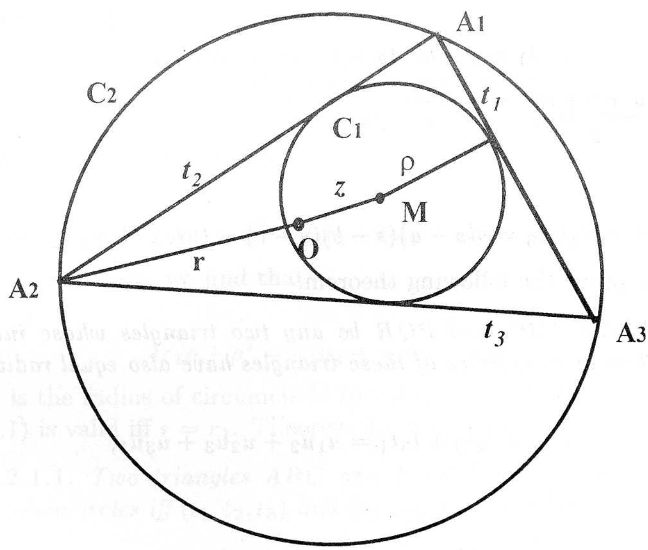


**Figure 2**

Now in short about one well-known relation concerning triangles which will be used in the following theorem. This relation is:

$$(t_1 + t_2 + t_3)\rho^2 = t_1t_2t_3$$

The following proof do not use trigonometry and may be useful in the following considerations.

Since

$$[(t_1+t_2+t_3)\rho]^2 = (\text{area of } A_1A_2A_3)^2,$$

we have to prove that also

$$(t_1+t_2+t_3)t_1t_2t_3 = (\text{area of } A_1A_2A_3)^2,$$

It is easy, since from

$$t_1+t_2=a,\ \ t_2+t_3=b,\ \ t_3+t_1=c,$$

where $a=|A_1A_2|$, $b=|A_2A_3|$, $c=|A_3A_1|$, it follows that

$$2t_1=a-b+c,\ \ 2t_2=a+b-c\ ,2t_3=-a+b+c,$$

or

$$t_1=s-b,\ \ t_2=s-c\ ,t_3=s-a,$$

where $s=\dfrac{a+b+c}{2}$.
Thus

$$(t_1+t_2+t_3)t_1t_2t_3 = s(s-a)(s-b)(s-c) = (\text{area of } A_1A_2A_3)^2.$$

Now we can prove the following theorem.

**Theorem 2.1.** *Let $ABC$ and $PQR$ be any two triangles whose incircles have equal radii. Then circumcircles of these triangles have also equal radii iff*

$$t_1t_2+t_2t_3+t_3t_1 = u_1u_2+u_2u_3+u_3u_1, \tag{2.1}$$

*where*

$$t_1+t_2=|AB|,\ \ t_2+t_3=|BC|,\ \ t_3+t_1=|CA|,$$
$$u_1+u_2=|PQ|,\ \ u_2+u_3=|QR|,\ \ u_3+u_1=|RA|.$$

*Proof.* From $(t_1+t_2+t_3)\rho^2 = t_1t_2t_3$ we have

$$t_3=\frac{\rho^2(t_1+t_2)}{t_1t_2-\rho^2} \tag{2.2}$$

and we can write

$$t_1t_2+t_2t_3+t_3t_1 = t_1t_2+t_2\cdot\frac{\rho^2(t_1+t_2)}{t_1t_2-\rho^2}+\frac{\rho^2(t_1+t_2)}{t_1t_2-\rho^2}\cdot t_1,$$

$$t_1t_2 + t_2t_3 + t_3t_1 = \frac{t_1^2t_2^2 + \rho^2(t_1^2 + t_2^2) + \rho^2 t_1t_2}{t_1t_2 - \rho^2}. \tag{2.3}$$

Now, let by $J$ be denoted the area of $ABC$. Then from

$$J = (t_1 + t_2 + t_3)\rho, \quad r = \frac{abc}{4J}$$

where $a = t_1 + t_2$, $b = t_2 + t_3$, $c = t_3 + t_1$, it follows that

$$4r\rho(t_1 + t_2 + t_3) = (t_1 + t_2)(t_2 + t_3)(t_3 + t_1).$$

The above relation, using (2.2), can be written as

$$4r\rho = \frac{t_1^2t_2^2 + \rho^2(t_1^2 + t_2^2) + \rho^4}{t_1t_2 - \rho^2}, \tag{2.4}$$

from which, adding $\rho^2$ on both sides, we obtain

$$4r\rho + \rho^2 = \frac{t_1^2t_2^2 + \rho^2(t_1^2 + t_2^2) + \rho^2 t_1t_2}{t_1t_2 - \rho^2}.$$

Hence, since (2.3) holds, we have

$$4r\rho + \rho^2 = t_1t_2 + t_2t_3 + t_3t_1. \tag{2.5}$$

In the quite same way we find that

$$4r_1\rho + \rho^2 = u_1u_2 + u_2u_3 + u_3u_1, \tag{2.6}$$

where $r_1$ is the radius of circumcircle to the triangle $PQR$.
Thus, (2.1) is valid iff $r = r_1$. Theorem 2.1 is proved. □

**Corollary 2.1.1.** *Two triangles $ABC$ and $PQR$ have congruent incircles and congruent circumcircles iff $(t_1, t_2, t_3)$ and $(u_1, u_2, u_3)$ are solutions of the equations*

$$(x_1 + x_2 + x_3)\rho^2 = x_1x_2x_3, \quad x_1x_2 + x_2x_3 + x_3x_1 = 4r\rho + \rho^2.$$

In the following theorem will be shown that such triangles there are infinity many, namely, that for every $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ which satisfy the above two equations, there exists a triangle of such kind.

Before we make a statement of the following theorem, in short about some relations which will be used.

Let $ABC$ be a triangle and let $C_1$ and $C_2$ be its incircle and circumcircle as shown in Fig.2. Then

$$(t_1 + t_2 + t_3)\rho = \frac{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}{4r} = \text{area of } ABC. \tag{2.7}$$

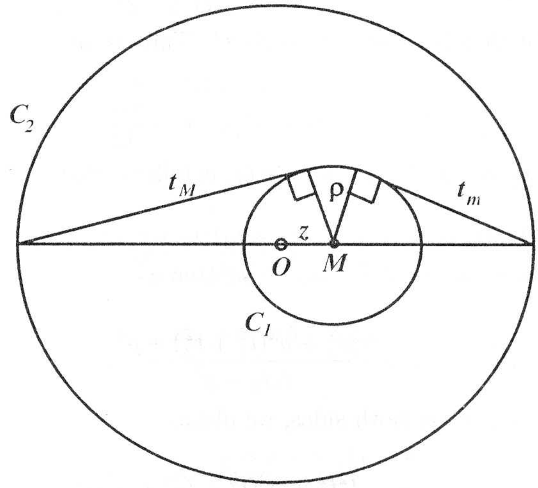


Figure 3

In connection with Fig.3 let us remark that

$$t_m^2 = (r-z)^2 - \rho^2, \quad t_M^2 = (r+z)^2 - \rho^2$$

where $t_m$ is the length of the least tangent that can be drawn from $C_2$ to $C_1$, and $t_M$ is the length of the largest tangent that can be drawn from $C_2$ to $C_1$.

Now we can prove the following theorem.

**Theorem 2.2.** *Let $r$, $\rho$ and $z$ be any given positive numbers that* (1.2) *satisfied and let $t_m$ and $t_M$ be given by*

$$t_m = \sqrt{(r-z)^2 - \rho^2}, \quad t_M = \sqrt{(r+z)^2 - \rho^2}. \tag{2.8}$$

*Then every positive solution $(t_1, t_2, t_3) \in \mathbb{R}_+^3$ of the equations*

$$t_1 t_2 + t_2 t_3 + t_3 t_1 = 4r\rho + \rho^2, \quad (t_1 + t_2 + t_3)\rho^2 = t_1 t_2 t_3 \tag{2.9}$$

*is given by*

$$t_1 \text{ is a positive number such that } t_m \leq t_1 \leq t_M, \tag{2.10}$$

$$t_2 = \frac{2r\rho t_1 + \sqrt{D}}{\rho^2 + t_1^2}, \tag{2.11}$$

$$t_3 = \frac{2r\rho t_1 - \sqrt{D}}{\rho^2 + t_1^2}, \tag{2.12}$$

*where*

$$D = 4r^2\rho^2 t_1^2 - \rho^2(\rho^2 + t_1^2)(4r\rho + \rho^2 + t_1^2). \tag{2.13}$$

*Proof.* The relations (2.9) can be written as

$$t_1(t_2 + t_3) + t_2 t_3 = 4r\rho + \rho^2, \tag{2.14}$$

$$-\rho^2(t_2 + t_3) + t_1(t_2 t_3) = \rho^2 t_1, \tag{2.15}$$

from which it follows that

$$t_2 + t_3 = \frac{4r\rho t_1}{\rho^2 + t_1^2}, \quad t_2 t_3 = \frac{\rho^2(4r\rho + \rho^2 + t_1^2)}{\rho^2 + t_1^2}.$$

It is easy to find that $t_2$ and $t_3$ are given by (2.11) and (2.12).
Now we have to prove that

$$4r^2 t_1^2 - (\rho^2 + t_1^2)(4r\rho + \rho^2 + t_1^2) \geq 0 \tag{2.16}$$

for every $t_1$ such that $t_m \leq t_1 \leq t_M$. For that purpose, of course, it is sufficiently to prove that the left-hand side of (2.16) is equal to zero for $t_1 = t_m$ and $t_1 = t_M$. It
is easy to show that

$$4r^2 t_m^2 - (\rho^2 + t_m^2)(4r\rho + \rho^2 + t_m^2) = 0 \Leftrightarrow (1.2),$$
$$4r^2 t_M^2 - (\rho^2 + t_M^2)(4r\rho + \rho^2 + t_M^2) = 0 \Leftrightarrow (1.2),$$

where (1.2) denotes Euler's relation given by (1.2). So, for $t_1 = t_m$ we can write

$$4r^2 t_m^2 - (\rho^2 + t_m^2)(4r\rho + \rho^2 + t_m^2) = \\ 4(r - z)^2(r^2 - z^2 - 2r\rho) = 0$$

since $r^2 - z^2 - 2r\rho = 0$ by (1.2).

Thus, from (2.11) and (2.13) it is clear that $t_2 > 0$ for every $t_1$ such that $t_m \leq t_1 \leq t_M$.

Also from (2.12), since obviously

$$2r\rho t_1 - \rho\sqrt{4r^2 t_1^2 - (\rho^2 + t_1^2)(\rho^2 + 4r\rho + t_1^2)} > 0,$$

it is clear that $t_3 > 0$ for every $t_1$ such that $t_m \leq t_1 \leq t_M$.

This completes the proof of Theorem 2.2.

□

Although $t_1$ is not given explicitely but by condition $t_m \leq t_1 \leq t_M$, it is easy to check that

$$t_1t_2 + t_2t_3 + t_3t_1 = t_1 \cdot \frac{2r\rho t_1 + \sqrt{D}}{\rho^2 + t_1^2} + \frac{2r\rho t_1 + \sqrt{D}}{\rho^2 + t_1^2} \cdot \frac{2r\rho t_1 - \sqrt{D}}{\rho^2 + t_1^2} + \frac{2r\rho t_1 - \sqrt{D}}{\rho^2 + t_1^2} \cdot t_1$$
$$= \frac{(4r\rho + \rho^2)(\rho^2 + t_1^2)}{\rho^2 + t_1^2} = 4r\rho + \rho^2.$$

Also the second relation in (2.9) can be easily checked.

Before we state some corollaries of Theorem 2.2 here is an example.

Example 1. Let $r = 3$ and $z = 1$. Using (1.2) we find that $\rho = \frac{4}{3}$. Since $t_m = 1.490711985$, $t_M = 3.771236166$ we can take for $t_1$ any number from interval $[1.490711985,\ 3.771236166]$. If we take $t_1 = 1.6$ then, by (2.11) and (2.12), we have

$$t_2 = 2.34075236, \quad t_3 = 3.56088681.$$

The corresponding triangle is shown in Figure 4.

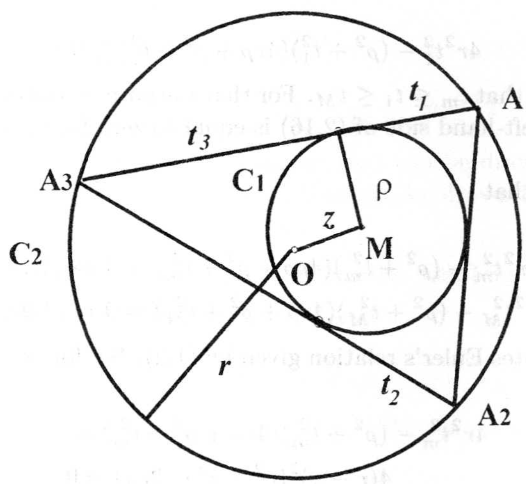


Figure 4

Notice 1. It is easy to see that proving Theorem 2.2 we give in fact another proof of the Poncelet's closure theorem for triangles using very simple and elementary mathematical facts. Therefore this proof may be interesting in itself.

In this, as can be seen, the relation (2.5) has a main role.

**Corollary 2.2.1.** *Let $A_1$ be any given point on $C_2$ and let $t_1$ be length of the tangent $A_1T_1$ drawn from*

*$C_2$ to $C_1$. Then the lengths $t_2$ and $t_3$ of the other two consecutive tangents drawn from $C_2$ to $C_1$ are given by (2.11) and (2.12).*

**Corollary 2.2.2.** *Every positive solution $(t_1, t_2, t_3) \in \mathbb{R}^3_+$ of the equation*

$$4r\rho(t_1 + t_2 + t_3) = (t_1 + t_2)(t_2 + t_3)(t_3 + t_1) \tag{2.17}$$

*is given by (2.10),(2.11) and (2.12).*

*Proof.* See (2.7). If $ABC$ is a triangle whose incircle is $C_1$ and circumcircle $C_2$ then hold relations (2.9). It is easy to see that equation (2.17) for $t_1, t_2, t_3$ given by (2.10),

(2.11) and (2.12) become identity

$$(\rho^2 + t_1^2 + 4r\rho)(\rho^2 + t_1^2) = (\rho^2 + t_1^2 + 4r\rho)(\rho^2 + t_1^2).$$

**Corollary 2.2.3.** *For every tangent drawn from $C_2$ to $C_1$ holds*

$$4r^2t^2 \geq (\rho^2 + t^2)(\rho^2 + t^2 + 4r\rho).$$

*Proof.* The relation (2.13) holds.

□

**Corollary 2.2.4.** *Let $t_1$ be a length of any given tangent drawn from $C_2$ to $C_1$ and let $t_2$ be given by (2.11), that is*

$$t_2 = \frac{2r\rho t_1 + \sqrt{D}}{\rho^2 + t_1^2}. \tag{2.11}$$

*Then holds (2.4), that is*

$$4r\rho = \frac{t_1^2 t_2^2 + \rho^2(t_1^2 + t_2^2) + \rho^4}{t_1 t_2 - \rho^2}. \tag{2.4}$$

*Proof.* From (2.14) and (2.15), that is, from

$$t_1t_2 + t_2t_3 + t_3t_1 = 4r\rho + \rho^2, \quad t_3 = \frac{\rho^2(t_1 + t_2)}{t_1t_2 - \rho^2}$$

we get

$$(\rho^2 + t_1^2)t_2^2 - 4r\rho t_1 t_2 + \rho^2(t_1^2 + 4r\rho + \rho^2) = 0, \tag{2.18}$$

which can be written as (2.4). Now, from (2.18) we get

$$(t_2)_{1,2} = \frac{2r\rho t_1 \pm \sqrt{D}}{\rho^2 + t_1^2},$$

where $D$ is the same as that given by (2.13).

Let us remark that (in the case of a triangle) both of $(t_2)_1$ and $(t_2)_2$ are consecutive to $t_1$ since $(t_2)_2 = t_3$.

Thus, using relation (2.4) instead of relations (2.14) and (2.15) we also obtain the solutions given by (2.10) – (2.12).

□

**Corollary 2.2.5.** *For every* $t_1, t_2, t_3$ *given by* (2.10) – (2.12) *it holds*

$$\sum_{i=1}^{3}(\rho^2 + t_i^2)(\rho^2 + t_{i+1}^2) = 8r\rho^2(2r - \rho). \tag{2.19}$$

*Proof.* From (2.4) it follows

$$t_1t_2 = \rho^2 + \frac{(\rho^2 + t_1^2)(\rho^2 + t_2^2)}{4r\rho}.$$

Analogously we have

$$t_2t_3 = \rho^2 + \frac{(\rho^2 + t_2^2)(\rho^2 + t_3^2)}{4r\rho},$$

$$t_3t_1 = \rho^2 + \frac{(\rho^2 + t_3^2)(\rho^2 + t_1^2)}{4r\rho}.$$

By adding, since $t_1t_2 + t_2t_3 + t_3t_1 = 4r\rho + \rho^2$, we get (2.19).

□

**Corollary 2.2.6.** *Let* $t_m$ *and* $t_M$ *be given by* (2.8). *Then the relation* (2.5) *can be written as*

$$t_1t_2 + t_2t_3 + t_3t_1 = \left(\frac{t_mt_M}{\rho}\right)^2.$$

*Proof.* Since from the Euler's relation (1.2) it follows that

$$(r^2 - z^2)^2 = 4r^2\rho^2, \quad z^2 = r^2 - 2r\rho$$

we can write

$$\begin{aligned} t_m^2t_M^2 &= (r^2 - z^2)^2 - \rho^2((r - z)^2 + (r + z)^2) + \rho^4 \\ &= 4r^2\rho^2 - 2\rho^2(r^2 + z^2) + \rho^4 \\ &= 4r^2\rho^2 - 2\rho^2(r^2 + r^2 - 2r\rho) + \rho^4 = 4r\rho^3 + \rho^4. \end{aligned}$$

In connection with relation(2.19) and the Euler's relation (1.2) the following remark will be made.

Let $A_1A_2A_3$ and $B_1B_2B_3$ be axial symmetric triangles in relation to line $OM$ as shown in Figure 5. Then, concerning triangle $A_1A_2A_3$, we have

$$t_1 = t_m = \sqrt{(r - z)^2 - \rho^2}, \qquad t_2 = t_3 = \sqrt{r^2 - (\rho - z)^2} \tag{2.20}$$

where

$$t_i + t_{i+1} = |A_i A_{i+1}|, \quad i = 1, 2, 3.$$

Using, $t_1, t_2, t_3$ given by (2.20), the equality (2.19) can be written as

$$(r^2 - 2r\rho - z^2)(3r^2 + r(6\rho - 4z) - 4\rho^2 + z^2) = 0.$$

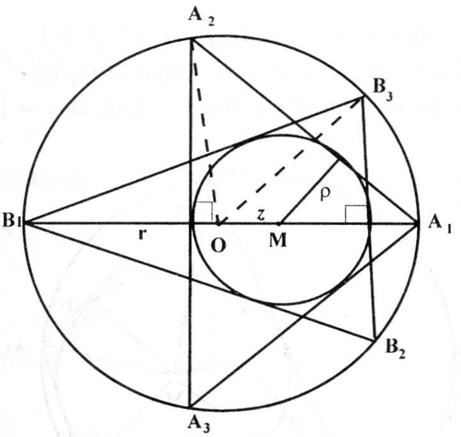


Figure 5

Concerning triangle $B_1B_2B_3$ we have

$$t_1 = t_M = \sqrt{(r+z)^2 - \rho^2}, \qquad t_2 = t_3 = \sqrt{r^2 - (\rho + z)^2}, \tag{2.21}$$

where

$$t_i + t_{i+1} = |B_i B_{i+1}|, \quad i = 1, 2, 3.$$

Using $t_1, t_2, t_3$ given by (2.21), the equality (2.19) can be written as

$$(r^2 - 2r\rho - z^2)(3r^2 + r(6\rho + 4z) - 4\rho^2 + z^2) = 0.$$

Similarly holds for every $t_1, t_2, t_3$ given by (2.10) – (2.12). The reason for this lie in
the fact that (2.19) holds for every $t_1, t_2, t_3$ given by (2.10) – (2.12) and that $r^2 - 2r\rho - z^2 = 0$. Therefore the equality (2.19) can be written as

$$(r^2 - 2r\rho - z^2)\, f(r, \rho, z, t_1) = 0,$$

where $f(r, \rho, z, t_1)$ is a polynomial in $r, \rho, z, t_1$.

□

## 3. Some relations concerning bicentric quadrilaterals

First about notation which will be used in this section.

Let $A_1A_2A_3A_4$ be a given bicentric quadrilateral. See Figure 6. Then by $C_1$ and $C_2$ will be denoted its incircle and circumcircle. By $t_1, t_2, t_3, t_4$ will be denoted the lengths of its tangents such that

$$t_i + t_{i+1} = |A_iA_{i+1}|, \quad i = 1, 2, 3, 4.$$

The radius of $C_1$ will be denoted by $\rho$ and that of $C_2$ by $r$. By $z$ will be denoted the distance between the centers of $C_1$ and $C_2$, that is, $z = |OM|$.

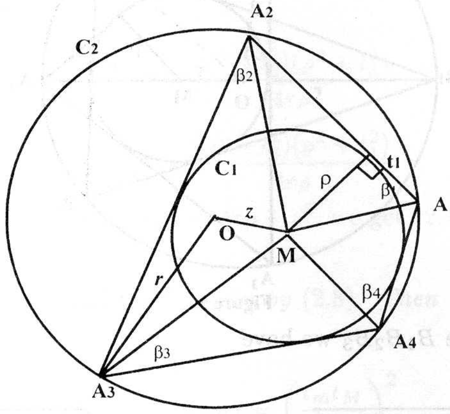


**Figure 6**

Now about some relations which will be used in this section.

First about relation

$$(t_1 + t_2 + t_3 + t_4)\rho^2 = t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2, \tag{3.1}$$

which has an important role in the following considerations. In short about its proof.

Let $\beta_i$ = measure of $\measuredangle MA_iA_{i+1}$, $i = 1, 2, 3, 4$ (see Fig.6). Then obviously

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = \pi.$$

Thus

$$tan(\beta_1 + \beta_2) = -tan(\beta_3 + \beta_4)$$

or

$$\frac{\tau_1 + \tau_2}{1 - \tau_1\tau_2} = -\frac{\tau_3 + \tau_4}{1 - \tau_3\tau_4} \tag{3.2}$$

where $\tau_i = \tan\beta_i$, $i = 1,2,3,4$. Since $\tan\beta_i = \dfrac{\rho}{t_i}$, $i = 1,2,3,,$ the equality (3.2) can be written as (3.1).

Also, in the following will be used the well-known elementary fact concerning area of a bicentric quadrilateral, namely, that it holds

$$\rho s = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{abcd}, \tag{3.3}$$

where

$$a = t_1 + t_2, \quad b = t_2 + t_3, \quad c = t_3 + t_4, \quad d = t_4 + t_1$$

$$s = \frac{a+b+c+d}{2} = t_1 + t_2 + t_3 + t_4.$$

Using the above relations, the following theorem can be proved.

**Theorem 3.1.** *Let $A_1A_2A_3A_4$ be a tangential quadrilateral and let $t_1, t_2, t_3, t_4$ be lengths such that*

$$t_i + t_{i+1} = |A_iA_{i+1}|, \quad i = 1,2,3,4.$$

*Then this quadrilateral is also a chordal one, that is a bicentric one, iff*

$$t_3 = \frac{\rho^2}{t_1}, \quad t_4 = \frac{\rho^2}{t_2} \tag{3.4}$$

*where $\rho$ is the radius of the inscribed circle into $A_1A_2A_3A_4$.*

*Proof.* The tangential quadrilateral $A_1A_2A_3A_4$, according to (3.3.), will also be chordal one iff

$$(t_1 + t_2 + t_3 + t_4)\rho = \sqrt{(t_1+t_2)(t_2+t_3)(t_3+t_4)(t_4+t_1)}. \tag{3.5}$$

Since according to (3.1) it holds

$$\rho^2 = \frac{t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2}{t_1 + t_2 + t_3 + t_4}, \tag{3.6}$$

the relation (3.5) can be written as

$$\begin{aligned}&(t_1 + t_2 + t_3 + t_4)(t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2)\\ &= (t_1+t_2)(t_2+t_3)(t_3+t_4)(t_4+t_1)\end{aligned}$$

from which it follows that

$$t_1^2t_3^2 - 2t_1t_2t_3t_4 + t_2^2t_4^2 = 0$$

or

$$(t_1t_3 - t_2t_4)^2 = 0.$$

Thus

$$t_1 t_3 = t_2 t_4. \tag{3.7}$$

Now, from (3.6), putting $t_4 = \dfrac{t_1 t_3}{t_2}$, we find that

$$\rho^2 = \frac{t_1 t_3 (t_1 + t_2)(t_2 + t_3)}{(t_1 + t_2)(t_2 + t_3)}$$

or

$$\rho^2 = t_1 t_3 \tag{3.8}$$

Also is valid $\rho^2 = t_2 t_4$ since (3.7) is valid.
Theorem (3.1) is proved.

□

In this connection let us remark that (3.1), putting $t_3 = \dfrac{\rho^2}{t_1}$ and $t_4 = \dfrac{\rho^2}{t_2}$, become an identity

$$(t_1 + t_2 + \frac{\rho^2}{t_1} + \frac{\rho^2}{t_2})\rho^2 = t_1\rho^2 + t_2\rho^2 + \frac{\rho^4}{t_1} + \frac{\rho^4}{t_2}.$$

**Theorem 3.2.** *Let $ABCD$ and $PQRS$ be two bicentric quadrilaterals whose inscribed circles have equal radii. Then circumcircles of these quadrilaterals have also equal radii iff*

$$t_1 t_2 + t_2 t_3 + t_3 t_4 + t_4 t_1 = u_1 u_2 + u_2 u_3 + u_3 u_4 + u_4 u_1, \tag{3.9}$$

*where $t_1, t_2, t_3, t_4$ and $u_1, u_2, u_3, u_4$ are the lengths of the (consecutive) tangents of$ABCD$ and $PQRS$ respectively.*

*Proof.* Using the expression for $t_3$ and $t_4$ given by (3.4) we find that

$$t_1 t_2 + t_2 t_3 + t_3 t_4 + t_4 t_1 = \frac{t_1^2 t_2^2 + \rho^2 (t_1^2 + t_2^2) + \rho^4}{t_1 t_2}. \tag{3.10}$$

Now let $r$ be the radius of the circumcircle to $ABCD$. We have to prove that $r$ is equal to the radius of the circumcircle to $PQRS$ iff holds (3.9).

In the proof we shall use the well-known relations which hold for bicentric quadrilateral:

$$r^2 = \frac{(ab + cd)(ac + bd)(ad + bc)}{16J^2}, \quad J^2 = abcd \tag{3.11}$$

where $a = t_1 + t_2$, $b = t_2 + t_3$, $c = t_3 + t_4$, $d = t_4 + t_1$, $J$ = area of $ABCD$.
Using (3.11) we find that

$$16r^2 = a^2 + b^2 + c^2 + d^2 + \frac{abc}{d} + \frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c},$$

which can be written as

$$16r^2 = (t_1+t_2)^2 + \left(\frac{t_1t_2+\rho^2}{t_1}\right)^2 + \left(\frac{(t_1+t_2)\rho^2}{t_1t_2}\right)^2 + \left(\frac{t_1t_2+\rho^2}{t_2}\right)^2 +$$
$$\left(\frac{(t_1+t_2)\rho}{t_1}\right)^2 + \left(\frac{(t_1t_2+\rho^2)\rho}{t_1t_2}\right)^2 + \left(\frac{(t_1+t_2)\rho}{t_2}\right)^2 + \left(\frac{t_1t_2+\rho^2}{\rho}\right)^2$$
$$= \left[\rho^2(t_1+t_2)^2 + (t_1t_2+\rho^2)^2\right]\left[\frac{1}{\rho^2} + \frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{\rho^2}{t_1^2t_2^2}\right]$$

or

$$16r^2\rho^2t_1^2t_2^2 = \left[t_1^2t_2^2 + \rho^2(t_1^2+t_2^2) + \rho^4 + 2\rho^2t_1t_2\right]^2 - 4\rho^4t_1^2t_2^2,$$

from which it follows that

$$16r^2\rho^2 + 4\rho^4 = \left[\frac{t_1^2t_2^2 + \rho^2(t_1^2+t_2^2) + \rho^4}{t_1t_2} + 2\rho^2\right]^2. \tag{3.12}$$

Analogously for the bicentric quadrilateral $PQRS$ we have

$$16r_1^2\rho^2 + 4\rho^4 = \left[\frac{u_1^2u_2^2 + \rho^2(u_1^2+u_2^2) + \rho^4}{u_1u_2} + 2\rho^2\right]^2, \tag{3.13}$$

where $r_1$ is the radius of the circumcircle of $PQRS$.

Thus, if (3.9) is valid, then $r_1 = r$. Theorem 3.2 is proved.

□

Now we shall prove that the left-hand side of (3.12) can be written as $4(r^2 + \rho^2 - z^2)^2$. For this purpose we shall prove that

$$16r^2\rho^2 + 4\rho^4 - 4(r^2+\rho^2-z^2)^2 = 0 \Leftrightarrow (1.1),$$

where (1.1) stands instead of Fuss' relation for bicentric quadrilaterals. Namely, starting from the Fuss' relation

$$2\rho^2(r^2+z^2) = (r^2-z^2)^2$$

or

$$2\rho^2r^2 + 2\rho^2z^2 = r^4 - 2r^2z^2 + z^4$$

and adding $\rho^4 + 2r^2\rho^2$ on both sides, we can write

$$(\rho^4 + 2r^2\rho^2) + (2r^2\rho^2 + 2\rho^2z^2) = (\rho^4 + 2r^2\rho^2) + (r^4 - 2r^2z^2 + z^4)$$

or

$$\rho^4 + 4r^2\rho^2 = (r^2+\rho^2-z^2)^2. \tag{3.14}$$

Thus the equality (3.12) can be written as

$$4(r^2+\rho^2-z^2)=\left[\frac{t_1^2t_2^2+\rho^2(t_1^2+t_2^2)+\rho^4}{t_1t_2}+2\rho^2\right]^2,$$

from which it follows that

$$\frac{t_1^2t_2^2+\rho^2(t_1^2+t_2^2)+\rho^4}{t_1t_2}=2(r^2-z^2). \tag{3.15}$$

Since hold (3.9) and (3.10) it follows that for every bicentric quadrilateral whose incircle is $C_1$ and circumcircle $C_2$ we have the equality

$$t_1t_2+t_2t_3+t_3t_4+t_4t_1=2(r^2-z^2). \tag{3.16}$$

Now we shall deduce some other relations.

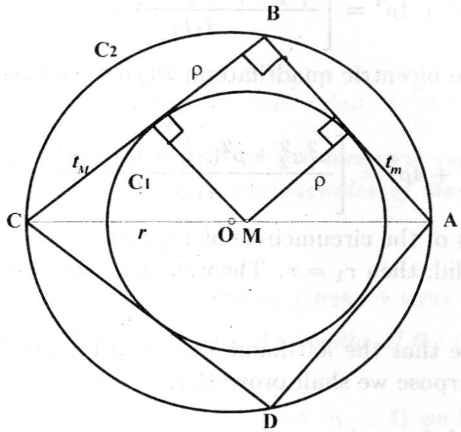


**Figure 7**

First in connection with Fuss' relation for bicentric quadrilaterals we shall prove that

$$\rho^2-t_mt_M=0\Leftrightarrow (r^2-z^2)^2-2\rho^2(r^2+z^2)=0, \tag{3.17}$$

where

$$t_m=\sqrt{(r-z)^2-\rho^2},\quad t_M=\sqrt{(r+z)^2-\rho^2}.$$

See Fig. 7. The quadrilateral $ABCD$ is a bicentric one, where

$$t_1=t_m,\quad t_2=\rho,\quad t_3=t_M,\quad t_4=\rho. \tag{3.18}$$

As can be seen, $t_m$ and $t_M$ are the lengths of the least and the largest tangent that can be drawn from $C_2$ to $C_1$. By (3.8) it holds

$$\rho^2 = t_m t_M. \tag{3.19}$$

Since $\rho^4 = t_m^2 t_M^2$, it can be written

$$\rho^4 = (r^2 - 2rz + z^2 - \rho^2)(r^2 + 2rz + z^2 - \rho^2)$$

or

$$0 = (r^2 - z^2)^2 - 2\rho^2(r^2 + z^2),$$

from which it is clear that holds (3.17).
Also it can be easily proved that

$$\rho(t_m + t_M) = r^2 - z^2, \tag{3.20}$$

$$\frac{t_m}{(r-z)^2} = \frac{t_M}{(r+z)^2}, \tag{3.21}$$

$$\frac{t_m}{\rho^2 + t_m^2} = \frac{t_M}{\rho^2 + t_M^2}. \tag{3.22}$$

These relations concern bicentric quadrilateral $ABCD$ shown in Fig.7. So the relation
(3.20) follows from (3.16) using the expressions given by (3.18). The relation (3.21) follows from (3.15) taking first $t_1 = t_m$, $t_2 = \rho$, and then taking $t_1 = \rho$, $t_2 = t_M$. The relation (3.22) follows from (3.21) using expressions for $t_m$ and $t_M$.

Now we can prove the following theorem (analogous to Theorem 2.2).

**Theorem 3.3.** *Let $r, z$ and $\rho$ be any given positive numbers such that (1.1) is satisfied and let $t_m$ and $t_M$ be given by*

$$t_m = \sqrt{(r-z)^2 - \rho^2}, \quad t_M = \sqrt{(r+z)^2 - \rho^2}. \tag{3.23}$$

*Then every positive solution $(t_1, t_2, t_3, t_4) \in \mathbb{R}_+^4$ of the equations*

$$t_1 t_2 + t_2 t_3 + t_3 t_4 + t_4 t_1 = 2(r^2 - z^2), \quad t_1 t_3 = \rho^2, \quad t_2 t_4 = \rho^2 \tag{3.24}$$

*is given by*

$$t_1 \text{ is a positive number such that } t_m \le t_1 \le t_M, \tag{3.25}$$

$$t_2 = \frac{(r^2 - z^2)t_1 + \sqrt{D}}{\rho^2 + t_1^2} \tag{3.26}$$

$$t_3 = \frac{\rho^2}{t_1}, \tag{3.27}$$

$$t_4 = \frac{\rho^2}{t_2}, \tag{3.28}$$

*where*

$$D = (r^2 - z^2)^2 t_1^2 - \rho^2(\rho^2 + t_1^2)^2. \tag{3.29}$$

*Proof.* The equation $t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = 2(r^2 - z^2)$, using equations $t_1t_3 = t_2t_4 = \rho^2$, can be written as

$$(\rho^2 + t_1^2)t_2^2 - 2(r^2 - z^2)t_1t_2 + \rho^2(\rho^2 + t_1)^2 = 0,$$

from which it follows that

$$(t_2)_{1,2} = \frac{(r^2 - z^2)t_1 \pm \sqrt{D}}{\rho^2 + t_1^2}.$$

It is unessential which of $(t_2)_1$ and $(t_2)_2$ will be taken for $t_2$ since

$$\frac{\rho^2}{(t_2)_1} = \frac{\rho^2(\rho^2 + t_1^2)}{(r^2 - z^2)t_1 + \sqrt{D}} = \frac{(r^2 - z^2)t_1 - \sqrt{D}}{\rho^2 + t_1^2} = (t_2)_2.$$

If we take $t_2 = (t_2)_1$ then $\frac{\rho^2}{t_2} = (t_2)_2$, that is, by (3.28), $(t_2)_2 = t_4$. But if we take $t_2 = (t_2)_2$ then $\frac{\rho^2}{t_2} = (t_2)_1$. Thus in this case $(t_2)_1 = t_4$.

Now, since in the expression of $t_2$ appears term $\sqrt{D}$, we have to prove that $D \geq 0$ for every $t_1$ such that $t_m \leq t_1 \leq t_M$. Of course, for this purpose it is sufficient to prove that $D = 0$ for $t_1 = t_m$ and $t_1 = t_M$.

It is easy to show that

$$(r^2 - z^2)^2 t_m^2 - \rho^2(\rho^2 + t_m^2)^2 = 0 \Leftrightarrow (1.1),$$
$$(r^2 - z^2)^2 t_M^2 - \rho^2(\rho^2 + t_M^2)^2 = 0 \Leftrightarrow (1.1),$$

where (1.1) stands instead of Fuss' relation given by (1.1). So for $t_1 = t_m$ we can write

$$\begin{aligned}&(r^2 - z^2)^2 t_m^2 - \rho^2(\rho^2 + t_m^2)^2\\ =&(r - z)^2\left[(r^2 - z^2)^2 - 2\rho^2(r^2 + z^2)\right] = 0.\end{aligned}$$

This completes the proof of Theorem 3.3. □

Although $t_1$ is not given explicitely but by condition $t_m \leq t_1 \leq t_M$, it is easy to check that for $t_1$, $t_2$, $t_3$, $t_4$ given by (3.25) – (3.28) in the end we get

$$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = \frac{(r^2 - z^2)t_1 + \sqrt{D}}{t_1} + \frac{(r^2 - z^2)t_1 - \sqrt{D}}{t_1} = 2(r^2 - z^2).$$

Before we state some of the corollaries of the Theorem 3.3 here is an example.

Example 2. Let $r = 3$ and $z = 1$. Using (1.1) we find that $\rho = 1.788854382$. Since $t_m = 0.894427191$, $t_M = 3.577708764$ we can take for $t_1$ any number from interval
$[0.894427191,\ 3.577708764]$. If we take $t_1 = 1.7$ then, by (3.26) – (3.28), we have

$$t_2 = 3.56997291, \quad t_3 = 1.882352941, \quad t_4 = 0.896365343.$$

The corresponding quadrilateral is shown in Figure 8.

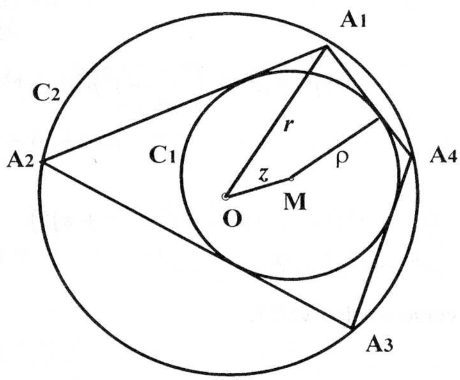


**Figure 8**

Notice 2. It is easy to see that proving the Theorem 3.3 we in fact give another proof of the Poncelet's closure theorem for bicentric quadrilaterals using very simple and elementary mathematical facts. Therefore this theorem may be interesting in itself.

The following corollaries of Theorem 3.3 may also be interesting.

**Corollary 3.3.1.** *The positive solutions of the equation*

$$\rho^2(t_1 + t_2 + t_3 + t_4)^2 = (t_1 + t_2)(t_2 + t_3)(t_3 + t_4)(t_4 + t_1)$$

*are given by* (3.25) – (3.28).

*(See* (3.3.)*)*

**Corollary 3.3.2.** *For every tangent drawn from $C_2$ to $C_1$ it holds*

$$(r^2 - z^2)t \geq (r^2 + t^2)\rho. \tag{3.30}$$

*Proof.* It follows from $D \geq 0$ for $t_m \leq t \leq t_M$.

□

**Corollary 3.3.3.** *Instead of* $t_m \leq t \leq t_M$ *it can be written*

$$\frac{r-z}{r+z} \cdot \rho \leq t \leq \frac{r+z}{r-z} \cdot \rho. \tag{3.31}$$

*Proof.* It holds

$$(r^2 - z^2)^2 = 2\rho^2(r^2 + z^2) \Leftrightarrow \sqrt{(r-z)^2 - \rho^2} = \frac{r-z}{r+z} \cdot \rho,$$
$$(r^2 - z^2)^2 = 2\rho^2(r^2 + z^2) \Leftrightarrow \sqrt{(r+z)^2 - \rho^2} = \frac{r+z}{r-z} \cdot \rho.$$

So from

$$(r-z)^2 - \rho^2 = \left(\frac{r-z}{r+z}\right)^2 \rho^2$$

it follows

$$(r^2 - z^2)^2 = \rho^2((r-z)^2 + (r+z)^2)$$
$$(r^2 - z^2)^2 = 2\rho^2(r^2 + z^2).$$

Obviously, the converse is also valid.

□

**Corollary 3.3.4.** *It holds*

$$r^2 \geq z^2 + 2\rho^2. \tag{3.32}$$

*Proof.* Using (3.30) it can be written

$$\rho t^2 - (r^2 - z^2)t + \rho^3 = 0,$$
$$t = \frac{(r^2 - z^2) \pm \sqrt{(r^2 - z^2)^2 - 4\rho^4}}{2},$$

from which it follows that $(r^2 - z^2)^2 - 4\rho^4 \geq 0$.

□

**Corollary 3.3.5.** *It holds*

$$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 \geq 4t_mt_M,$$

*where equality holds if* $z = 0$.

*Proof.* From (3.32) it follows that $2(r^2 - z^2) \geq 4\rho^2$. Since holds (3.16) and (3.19), we can write

$$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = 2(r^2 - z^2) \geq 4\rho^2 = 4t_mt_M.$$

If $z = 0$, then $2(r^2 - z^2) = 4\rho^2$ since $r^2 = 2\rho^2$ if $z = 0$.

□

The following two theorems are in fact some corollaries of the Theorem 3.1 and the Theorem 3.2.

**Theorem 3.4.** *For every bicentric quadrilateral it holds*

$$A(t_1, t_2, t_3, t_3) \cdot H(t_1, t_2, t_3, t_4) = \rho^2, \tag{3.33}$$

*where* $A(t_1, t_2, t_3, t_3)$ *and* $H(t_1, t_2, t_3, t_4)$ *are arithmetic and harmonic mean of* $t_1, t_2, t_3, t_4$ *respectively.*

*Proof.* From $t_1t_3 = t_2t_4 = \rho^2$ it follows that $t_1t_2t_3t_4 = \rho^4$. Thus the equality

$$(t_1 + t_2 + t_3 + t_4)\rho^2 = t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2$$

can be written as

$$(t_1 + t_2 + t_3 + t_4)\rho^2 = \frac{\rho^4}{t_1} + \frac{\rho^4}{t_2} + \frac{\rho^4}{t_3} + \frac{\rho^4}{t_4}$$

or

$$\frac{t_1 + t_2 + t_3 + t_4}{4} \cdot \frac{4}{\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}} = \rho^2.$$

Theorem 3.4 is proved.

□

**Theorem 3.5.** *Let* $ABCD$ *be any bicentric quadrilateral whose incircle is* $C_1$ *and circumcircle* $C_2$. *Then*

$$ef = 2(r^2 + 2\rho^2 - z^2), \tag{3.34}$$

*where* $e = |AC|$, $f = |BD|$. *In other words, for every bicentric quadrilateral whose incircle is* $C_1$ *and circumcircle* $C_2$ *the product of its diagonals is a constant, i.e., is* $2(r^2 + 2\rho^2 - z^2)$.

*Proof.* Let $a = t_1 + t_2$, $b = t_2 + t_3$, $c = t_3 + t_4$, $d = t_4 + t_1$ be the lengths of the sides of $ABCD$. Then by Ptolomy's theorem

$$ef = ac + bd,$$

and we can write

$$ac + bd = (t_1 + t_2)(t_3 + t_4) + (t_2 + t_3)(t_4 + t_1)$$

$$= (t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1) + 2(t_1t_3 + t_2t_4)$$
$$= 2(r^2 - z^2) + 2(\rho^2 + \rho^2)$$
$$= 2(r^2 + 2\rho^2 - z^2).$$

□

**Theorem 3.6.** *For every $t_1, t_2, t_3, t_4$ given by (3.25) – (3.28) it holds*

$$\sum_{i=1}^{4}(\rho^2 + t_i^2)(\rho^2 + t_{i+1}^2) = 4(r^2 - z^2)^2. \tag{3.35}$$

*Proof.* From (3.15) it follows

$$t_1^2t_2^2 + \rho^2(t_1^2 + t_2^2) + \rho^4 = 2(r^2 - z^2)t_1t_2$$

Analogously we have

$$t_2^2t_3^2 + \rho^2(t_2^2 + t_3^2) + \rho^4 = 2(r^2 - z^2)t_2t_3,$$
$$t_3^2t_4^2 + \rho^2(t_3^2 + t_4^2) + \rho^4 = 2(r^2 - z^2)t_3t_4,$$
$$t_4^2t_1^2 + \rho^2(t_4^2 + t_1^2) + \rho^4 = 2(r^2 - z^2)t_4t_1,$$

By adding, since $t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = 2(r^2 - z^2)$ we get (3.35).
Theorem 3.6 is proved.

□

**Theorem 3.7.** *For every bicentric quadrilateral whose incircle is $C_1$ and circumcircle $C_2$ it holds*

$$t_it_j + t_jt_k + t_kt_i > \rho^2, \qquad \rho^2 > \frac{t_it_jt_k}{t_i + t_j + t_k},$$

*where $t_i, t_j, t_k$ are the lengths of any three consecutive tangents.*

*Proof.* We shall prove more general assertion, namely, that the above inequalities hold for every tangential quadrilateral which have the same incircle.

First we shall prove the following lemma.

**Lemma 1.** *Let $t_1, t_2, t_3, t_4$ be any given lengths and let $\rho_Q$ be the length such that*

$$(t_1 + t_2 + t_3 + t_4)\rho_Q^2 = t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2.$$

*Then*

$$(t_1 + t_2 + t_3)\rho_Q^2 > t_1t_2t_3. \tag{3.36}$$

*Proof.* There is a length $\rho_T$ such that

$$(t_1 + t_2 + t_3)\rho_T^2 = t_1 t_2 t_3.$$

Let $A_1A_2A_3$ be corresponding triangle shown in Figure 9*a* and let $B_1B_2B_3B_4$ be corresponding tangential quadrilateral shown in Figure 9*b*. It is easy to see that holds (3.36), namely, from Figure 10*b* it is

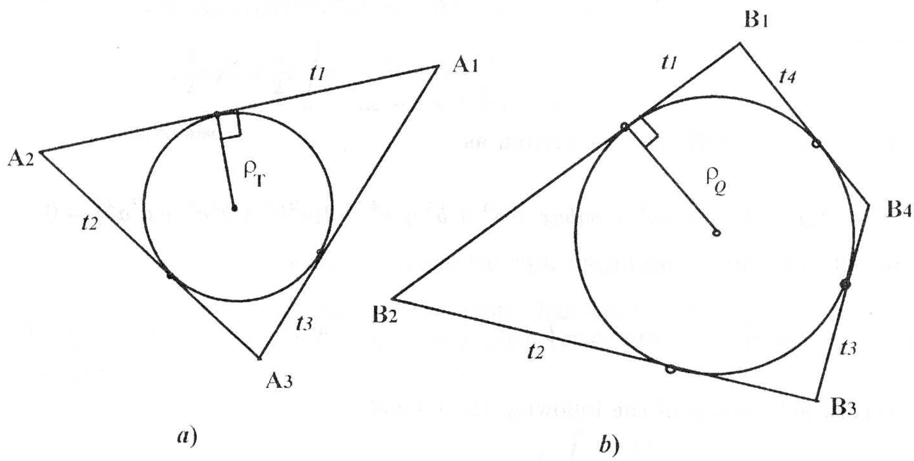


**Figure 9**

clear that only $t_1, t_2, t_3$ (without $t_4$) are not enough for closing. But it is enough for radius $\rho_T$. (More about tangential $n$-gons can be seen in [5].)

Now, Theorem 3.7 can be easily proved. Namely, from

$$(t_1 + t_2 + t_3 + t_4)\rho^2 = t_1 t_2 t_3 + t_2 t_3 t_4 + t_3 t_4 t_1 + t_4 t_1 t_2.$$

it follows that

$$t_4 = \frac{(t_1 + t_2 + t_3)\rho^2 - t_1 t_2 t_3}{t_1 t_2 + t_2 t_3 + t_3 t_1 - \rho^2}.$$

Since, according to above lemma,

$$(t_1 + t_2 + t_3)\rho^2 - t_1 t_2 t_3 > 0$$

and $t_4 > 0$, it follows that

$$t_1 t_2 + t_2 t_3 + t_3 t_1 - \rho^2 > 0.$$

Theorem 3.7 is proved. □

**Theorem 3.8.** *Let $a, b, c$ be any given lengths (in fact positive numbers) such that*

$$a - b + c > 0. \tag{3.37}$$

*Then there exists unique bicentric quadrilateral $ABCD$ such that*

$$a = |AB|, \quad b = |BC|, \quad c = |CD|.$$

*Proof.* Since the area of a bicentric quadrilateral whose sides have the lengths $a, b, c, d$ is given by $\sqrt{abcd}$, it can be written

$$(s-a)(s-b)(s-c)(s-x) = abcx, \tag{3.38}$$

where

$$2s = a + b + c + x.$$

The equation (3.38) can be written as

$$x^4 - 2(a^2+b^2+c^2)x^2 + 8abcx + a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 + c^2a^2) = 0.$$

Its solutions are

$$x_1 = -a + b + c, \qquad x_2 = a - b + c, \qquad x_3 = a + b - c, \qquad x_4 = -a - b - c.$$

Accordingly, if one of the following three conditions

$$-a + b + c > 0, \qquad a - b + c > 0, \qquad a + b - c > 0$$

is fulfilled, say the condition (3.37), then there is unique bicentric quadrilateral described in Theorem (3.8).

Theorem (3.8) is proved.

□

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