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ABOUT SOME KINDS OF BICENTRIC POLYGONS AND CONCERNING RELATIONS

MIRKO RADIĆ AND ALEKSANDAR ZATEZALO

Abstract. In the article we present some new properties and relations concerning bicentric polygons with common circumcircle and center of their incircles. A new generalized definition of Fuss' relation for k-bicentric n-gons is introduced. A general conjecture about k-bicentric n-gons is stated and it is proved for some special cases.

1. Introduction

A polygon which is both chordal and tangential is briefly called bicentric polygon. The first that was concerned with bicentric polygons is German mathematician Nicolaus Fuss (1755-1826), a friend of Leonhard Euler. He posed himself the following problem (known as Fuss' problem of the bicentric quadrilaterals):

To find the relation between the radii and the line segment joining the centers of the circles of circumscription and inscription of a bicentric quadrilateral.

He found that

$$\left(r^2 - z^2\right)^2 = 2\rho^2\left(r^2 + z^2\right), \tag{1.1}$$

where r and ρ are radii and z is the distance between the centers of the circles of circumscription and inscription. (See [5].) This problem is listed and considered in [4, pp. 188-192], as one of the 100 great problems of elementary mathematics.

Fuss also found corresponding formulas for the bicentric pentagon, hexagon, heptagon and octagon, see [6]. These formulas may be stated as follows:

$$p^3q^3 + p^2q^2\rho(p+q) - pq\rho^2(p+q)^2 - \rho^3(p+q)(p-q)^2 \doteq 0, \tag{1.2}$$

$$3p^4q^4 - 2p^2q^2\rho^2\left(p^2+q^2\right) = \rho^4\left(p^2-q^2\right)^2, \tag{1.3}$$

$$\left(pq - \rho(p-q) - 2\rho^2\right)2pq\rho\sqrt{(p-\rho)(p+q)} + \left(p^2q^2 - \rho^2\left(p^2+ q^2\right)\right)2\rho\sqrt{(q-\rho)(p+q)} = \pm\left(pq - \rho(p-q)\right)\left(p^2q^2 + \rho^2\left(p^2-q^2\right)\right), \tag{1.4}$$

$$\left(\rho^2\left(p^2+q^2\right) - p^2q^2\right)^4 = 16p^4q^4\rho^4\left(p^2-\rho^2\right)\left(q^2-\rho^2\right), \tag{1.5}$$

where $p = r + z$, $q = r - z$.

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The corresponding formula for triangles is

$$r^2 - z^2 = 2r\rho, \tag{1.6}$$

and had already been given by Euler.

The very remarkable theorem concerning bicentric polygons is given by French mathematician Poncelet (1788 - 1867). In the formulation of this theorem will be used the so-called Poncelet traverse. In short about this.

Let C_1 and C_2 be two circles in a plane. If from any point on C_2 we draw a tangent to C_1, extend the tangent line so that it intersects C_2, and draw from the point of intersection a new tangent to C_1, extend this tangent similarly to intersect C_2, and continue in this way, we obtain the so-called Poncelet traverse which, when it consists of n chords of circle C_2 (circle of circumscription), it is called n-sided.

The Poncelet's theorem (for circles) can be expressed as follows.

If on the circle of circumscription there is one point of origin for which n-sided Poncelet traverse is closed, then the n-sided traverse will also be closed for any other point of origin on the circle.

Poncelet demonstrated that analogously holds for conic sections so that the general theorem reads:

Poncelet's closure theorem. *If an n-sided Poncelet traverse constructed for two given conic section is closed for one position of the point of origin, it is closed for any position of the point of origin.*

Although the Poncelet's closure theorem date from nineteenth century, many mathematicians have been working on number of problems in connection with this theorem. Many interesting and useful information about it we have found in quoted articles concerning Poncelet's closure theorem, especially in [2], [8] and[9].

In this paper we shall restrict ourselves to the case when conics are circles, one inside the other.

In Section 2, we give notation and some known results from [11] used in the article.

In Section 3, we derive rational relations which give us Fuss' equations for 1-bicentric and 2-bicentric pentagons.

In Sections 4, 5, and 6, in similar way as in Section 3, we derive rational relations for all possible k-gons for hexagons, heptagons, and octagons, respectively.

In Section 7, we give in Theorem 7.2 and Corollary 7.3, how radius of incircle of a k-bicentric polygon depends on k. Then we give general definition of the Fuss' equation for k-bicentric n-gon, see Definition 7.7, which allowed us in Theorem 7.8, to give dependence of Fuss' equations of k-bicentric n-gons on k. In the end, using general definition of Fuss' equations, we present conjecture about Fuss' equations of k-bicentric n-gons. It is proved for some special cases given as examples. The open problem is formal proof of the conjecture.

Results in Sections 2, 3, 4, 5, and 6 motivate general results in Section 7. This article also explains factors in relations obtained in [9] using definition of k-bicentric n-gon.

2. Notation and some known results

If $A_1 \dots A_n$ is a bicentric polygon under consideration, then by C_1 will be denoted its incircle and by C_2 circumcircle. By ρ will be denoted the radius of C_1 and that of C_2 by r. The center of C_1 will be denoted by M and that of C_2 by O. The distance between M and O will be denoted by z.

Very important role will have the angles β_i given by

$$\beta_i = \text{ measure of } \angle MA_iA_{i+1}, i = 1, \dots, n. \tag{2.1}$$

Of course, indices in (2.1) are calculated modulo n.

Some symbols, which we have used in [11], will also be used in this paper.

Let us define symbol $S_j(x_1, \dots, x_n)$. Let $x_1, \dots, x_n$ be real numbers, and let j be an integer such that $1 \le j \le n$. Then $S_j(x_1, \dots, x_n)$ is the sum of all $\binom{n}{j}$ products of the form $x_{i_1} \dots x_{i_j}$, where $i_1, \dots, i_j$ are different elements of the set $\{1, \dots, n\}$, that is

$$S_j(x_1, \dots, x_n) = \sum_{1 \le i_1 < \dots < i_j \le n} x_{i_1} \dots x_{i_j}. \tag{2.2}$$

Let $T_j^n := S_j(\tan\beta_1, \dots, \tan\beta_n)$ and $C_j^n := S_j(\cot\beta_1, \dots, \cot\beta_n)$.

Remark 2.1. In the following we shall, for brevity, write S_j^n instead of $S_j(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ will be lengths of tangents.

For example we have $S_1^3 = t_1 + t_2 + t_3$, $S_2^3 = t_1t_2 + t_2t_3 + t_3t_1$, and $S_3^3 = t_1t_2t_3$. Let

$$\left[\frac{n-1}{2}\right] := \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd,} \\ \frac{n-2}{2}, & \text{if } n \text{ is even.} \end{cases}$$

In [11, pp. 200], we have the following definition (Definition 1).

Let $A_1, \dots, A_n$ be tangential polygon and let k be a positive integer such that $k \le \left[\frac{n-1}{2}\right]$. Then the polygon $A_1, \dots, A_n$ will be called k-tangential polygon if any two of its consecutive sides have only one point in common and if holds

$$\sum_{i=1}^{n} \beta_i = (n - 2k)\frac{\pi}{2}, \tag{2.3}$$

where $2\beta_i =$ measure of $\angle A_{n-1+i}A_iA_{i+1}$, $i = 1, \dots, n$.

Of course, if M is the center of the incircle of $A_1 \dots A_n$, then

$$\beta_i = \text{measure of } \angle MA_iA_{i+1}, i = 1, \dots, n.$$

It is easy to see that a tangential polygon $A_1 \dots A_n$ is k-tangential if

$$\sum_{i=1}^{n} \varphi_i = 2k\pi, \tag{2.4}$$

where $\varphi_i =$measure of $\angle A_iMA_{i+1}$, $i = 1, \dots, n$. Namely, from (2.3) and

$$2\beta_i = \text{measure of } \angle A_{n-1+i}A_iA_{i+1}, i = 1, \dots, n,$$

it follows that

$$\beta_i + \beta_{i+1} = \pi - \varphi_i, i = 1, \ldots, n.$$

Hence

$$\sum_{i=1}^{n} (\beta_i + \beta_{i+1}) = n\pi - 2k\pi$$

or

$$2\sum_{i=1}^{n} \beta_i = (n - 2k)\pi.$$

For example, in Figure 1, we have drawn one 2 tangential hexagon. More about this will be said in connection with the equation (2.6).

In [11] the following theorem is proved.

Theorem 2.2. *Let $t_1, \ldots, t_n$ be any given lengths (in fact positive numbers) and let k be a positive integer such that $k \le \left[\frac{n-1}{2}\right]$. Then there exists a k-tangential polygon whose (consecutive) tangents have the lengths $t_1, \ldots, t_n$.*

The connection between $t_1, \ldots, t_n$ and ρ is given by the following corollary.

Corollary 2.3 (Slightly modified). *Let ρ_k be the radius of a k-tangential polygon whose tangents have the lengths $t_1, \ldots, t_n$. Then, if n is odd, every ρ_k, $k = 1, \ldots, \frac{n-1}{2}$, is a solution of the equation*

$$S_1^n x^{n-1} - S_3^n x^{n-3} + S_5^n x^{n-5} - \cdots + (-1)^s S_n^n = 0, \tag{2.5}$$

where $s = (1 + 3 + 5 + \ldots n) + 1$.

If n is even, then every ρ_k, $k = 1, \ldots, \frac{n-2}{2}$, is a solution of the equation

$$S_1^n x^{n-2} - S_3^n x^{n-4} + S_5^n x^{n-6} - \cdots + (-1)^s S_{n-1}^n = 0, \tag{2.6}$$

where $s = (1 + 3 + 5 + \cdots + n - 1) + 1$.

For example, if $n = 3, 4, 5, 6$, then

$$\begin{aligned} S_1^3 x^2 - S_3^3 &= 0, && \text{(triangle)} \\ S_1^4 x^2 - S_3^4 &= 0, && \text{(quadrangle)} \\ S_1^5 x^4 - S_3^5 x^2 + S_5^5 &= 0, && \text{(pentagon)} \\ S_1^6 x^4 - S_3^6 x^2 + S_5^6 &= 0, && \text{(hexagon)} \end{aligned}$$

and

$$\begin{aligned} S_1^3 \rho_k^2 - S_3^3 &= 0, \quad k = 1, \\ S_1^4 \rho_k^2 - S_3^4 &= 0, \quad k = 1, \\ S_1^5 \rho_k^4 - S_3^5 \rho_k^2 + S_5^5 &= 0, \quad k = 1, 2, \\ S_1^6 \rho_k^4 - S_3^6 \rho_k^2 + S_5^6 &= 0, k = 1, 2. \end{aligned} \tag{2.7}$$

So, if $n = 6$ and $t_j = j$, $j = 1, \ldots, 6$, then the equation (2.6) can be written as

$$21x^4 - 735x^2 + 1764 = 0,$$

from which we get $\rho_1 \approx 5.69280$, $\rho_2 \approx 1.60995$. See Figure 1. (The other hexagon, which is 1-tangential, is not drawn since ρ_1 is rather large for construction in the text.)

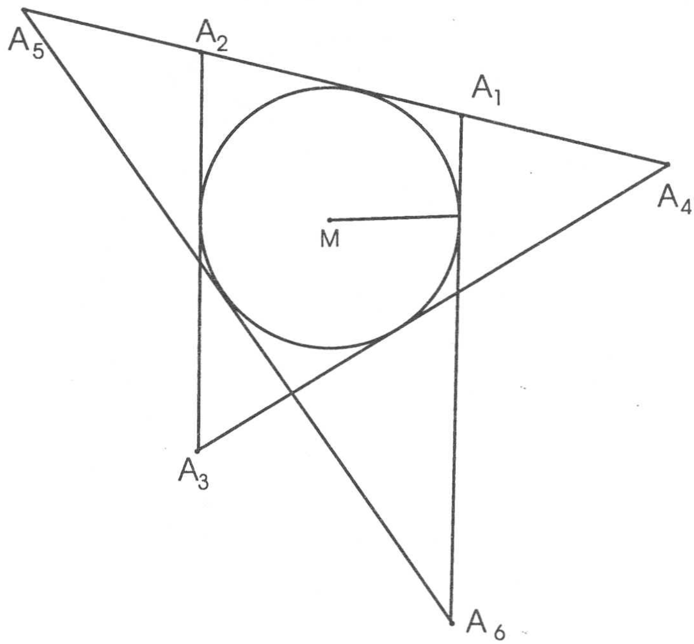


FIGURE 1. 2-tangential hexagon with $\rho_2 \approx 1.60995$

Remark 2.4. There are bicentric polygons with property that, as tangential polygons, are k-tangential. When it is important to point out, it will be said that these polygons have type k or, briefly, that these polygons are k-bicentric polygons.

3. Some relations concerning Bicentric Pentagons

First we prove the following theorem.

Theorem 3.1. *The relation between r, ρ, and z for 1-bicentric pentagon can be written as*

$$\rho = \rho\sqrt{\frac{r-z-\rho}{2r}} + (r-z)\sqrt{1-\left(\frac{\rho}{r+z}\right)^2}\sqrt{\frac{r+z+\rho}{2r}}. \qquad (3.1)$$

Proof. We can take (by Poncelet's closure theorem) a pentagon which is symmetric with respect to x-axis and consider situation shown in Figure 2, where $A_1\,(r\cos\varphi_1, r\sin\varphi_1)$, $A_2\,(r\cos\varphi_2, r\sin\varphi_2)$, $\varphi_i = \angle POA_i$, $i = 1, 2$ and $0 < \varphi_1 < \varphi_2 \le \frac{3\pi}{5}$.

It is easy to find that distances of the point $M(z,0)$ from the lines A_1A_5, A_1A_2, and A_2A_3 are given by

$$\rho + z = r\cos\varphi_1, \qquad (3.2)$$

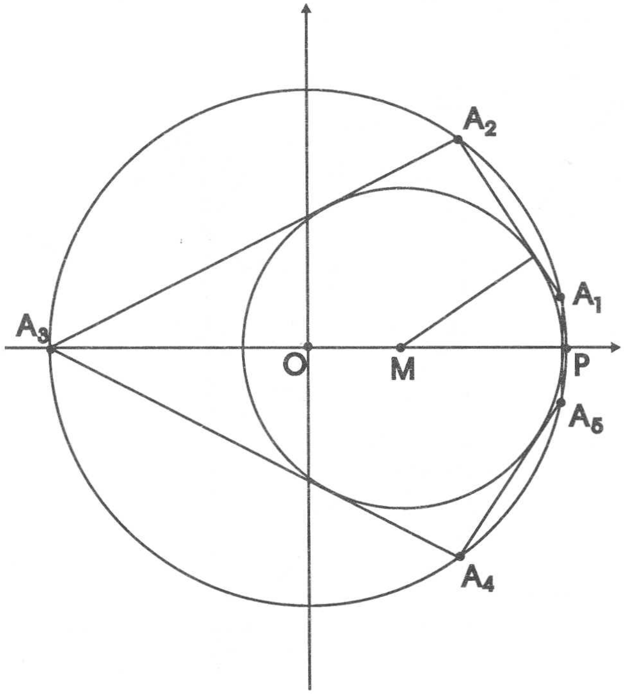


FIGURE 2. 1-bicentric pentagon symmetric with respect to x-axis

$$\rho = -z\cos\frac{\varphi_1+\varphi_2}{2} + r\cos\frac{\varphi_1-\varphi_2}{2}, \tag{3.3}$$

$$\rho = z\sin\frac{\varphi_2}{2} + r\sin\frac{\varphi_2}{2}. \tag{3.4}$$

Since from (3.2) and (3.4) we have

$$\sin\frac{\varphi_1}{2} = \sqrt{\frac{r-z-\rho}{2r}}, \quad \cos\frac{\varphi_1}{2} = \sqrt{\frac{r+z+\rho}{2r}},$$

$$\sin\frac{\varphi_2}{2} = \frac{\rho}{r+z}, \quad \cos\frac{\varphi_2}{2} = \sqrt{1-\left(\frac{\rho}{r+z}\right)^2},$$

and (3.3) can be written as

$$\rho = (r-z)\cos\frac{\varphi_1}{2}\cos\frac{\varphi_2}{2} + (r+z)\sin\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2},$$

it is easy to see that above equality can be written as (3.1).

The theorem is proved. □

After rationalization and factorization of (3.1), we get

$$(r+z+\rho)^2\left(z^2-r^2+2r\rho\right)\left(3r^4z^2-r^6-3r^2z^4+z^6-2r^5\rho+4r^3z^2\rho-\right.$$
$$\left.2rz^4\rho+4r^4\rho^2-4r^2z^2\rho^2+8rz^2\rho^3\right)=0. \tag{3.5}$$

The third factor on the right hand side of (3.5) can be written as Fuss' relation (1.2).

Corollary 3.2. *It holds*

$$\left(\frac{\rho}{r-z}\right)^2 > \frac{r+z-\rho}{2r}. \tag{3.6}$$

Proof. From (3.1) we have

$$\left(\frac{\rho}{r-z}\right)^2 > \frac{(r+z)^2-\rho^2}{(r+z)^2}\cdot\frac{r+z+\rho}{2r}$$

or

$$\left(\frac{\rho}{r-z}\right)^2 > \left(\frac{r+z+\rho}{r+z}\right)^2\cdot\frac{r+z-\rho}{2r},$$

from which and $\frac{r+z+\rho}{r+z} > 1$, we get (3.6). □

Theorem 3.3. *The relation between r, ρ, and z for 2-bicentric pentagon can be written as*

$$\rho = (r-z)\sqrt{1-\left(\frac{\rho}{r+z}\right)^2}\sqrt{\frac{r+z-\rho}{2r}} - \rho\sqrt{\frac{r-z+\rho}{2r}}. \tag{3.7}$$

Proof. Here we may take a symmetric pentagon as shown in Figure 3, where

$$A_1\,(r\cos\varphi_1, r\sin\varphi_1)\ ,\ A_2\,(-r,0)\ ,\ A_3\,(r\cos\varphi_1, -r\sin\varphi_1)\,,$$

$$A_4\,(r\cos\varphi_2, r\sin\varphi_2)\ ,\ A_5\,(r\cos\varphi_2, -r\sin\varphi_2)\ ,\ 0<\varphi_1<\varphi_2\le\frac{3\pi}{5},$$

where $\varphi_1 = \angle POA_1$, $\varphi_2 = \angle POA_4$.

Let us remark that for every $0 \le z < r$ there are unique angles φ_1 and φ_2 such that $A_1A_2A_3A_4A_5$ is symmetric with respect to x-axis and that $0 < \varphi_1 < \varphi_2 \le \frac{3\pi}{5}$.

The distances of the point $M(z,0)$ from lines A_1A_2, A_3A_4 and A_4A_5 are given by

$$\rho = (r+z)\sin\frac{\varphi_1}{2}, \tag{3.8}$$

$$\rho = r\cos\frac{\varphi_1+\varphi_2}{2} - z\cos\frac{\varphi_1-\varphi_2}{2}, \tag{3.9}$$

$$\rho = z - r\cos\varphi_2. \tag{3.10}$$

From (3.8) and (3.10) we have

$$\sin\frac{\varphi_1}{2} = \frac{\rho}{r+z}\ ,\ \cos\frac{\varphi_1}{2} = \sqrt{1-\left(\frac{\rho}{r+z}\right)^2}, \tag{3.11}$$

$$\sin\frac{\varphi_2}{2} = \sqrt{\frac{r-z+\rho}{2r}}\ ,\ \cos\frac{\varphi_2}{2} = \sqrt{\frac{r+z-\rho}{2r}}. \tag{3.12}$$

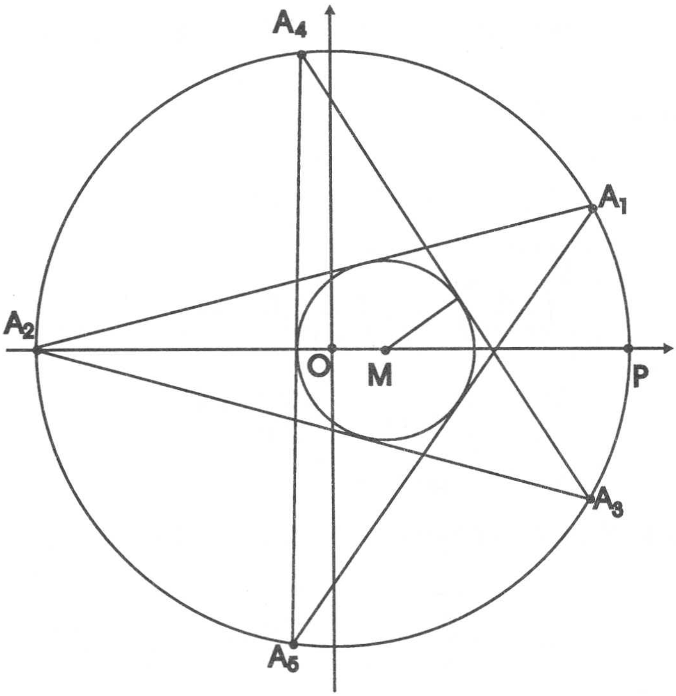


FIGURE 3. 2-bicentric pentagon symmetric with respect to x-axis

Equation (3.9) can be written as

$$\rho = (r - z)\cos\frac{\varphi_1}{2}\cos\frac{\varphi_2}{2} - (r + z)\sin\frac{\varphi_1}{2}\sin\frac{\varphi_2}{2}. \tag{3.13}$$

From (3.11), (3.12), and (3.13), we get (3.7).

The theorem is proved. □

After rationalization and factorization of Equation (3.7) we get

$$(\rho - r - z)^2\left(r^2 - z^2 + 2r\rho\right)\left(r^6 - 3r^4z^2 + 3r^2z^4 - z^6 - 2r^5\rho + 4r^3z^2\rho - 2rz^4\rho - 4r^4\rho^2 + 4r^2z^2\rho^2 + 8rz^2\rho^3\right) = 0. \tag{3.14}$$

Thus, for 2-bicentric pentagons instead of Fuss' relation (1.2), we have relation

$$r^6 - 3r^4z^2 + 3r^2z^4 - z^6 - 2r^5\rho + 4r^3z^2\rho - 2rz^4\rho - 4r^4\rho^2 + 4r^2z^2\rho^2 + 8rz^2\rho^3 = 0. \tag{3.15}$$

As can be seen ,these relations have exactly the same terms but differ only in some signs + and − in front of the terms.

These relations can be obtained in other ways, see [9, pp. 73].

Corollary 3.4. *It holds*

$$\left(\frac{\rho}{r-z}\right)^2 < \frac{r+z+\rho}{2r}. \tag{3.16}$$

Proof. From (3.7) we have

$$\left(\frac{r+z-\rho}{r+z}\right)^2 \cdot \frac{r+z+\rho}{2r} > \left(\frac{\rho}{r-z}\right)^2,$$

which gives (3.16). □

Corollary 3.5. *It holds*

$$\left(\frac{\rho}{r-z}\right)^2 < \frac{r+z+\rho}{r-z+\rho}. \tag{3.17}$$

Proof. From (3.7) we have

$$(r-z)^2 \cdot \frac{(r+z)^2-\rho^2}{(r+z)^2} \cdot \frac{r+z-\rho}{2r} > \rho^2 \cdot \frac{r-z+\rho}{2r}$$

or

$$\left(\frac{r+z-\rho}{r+z}\right)^2 (r+z+\rho) > \left(\frac{\rho}{r-z}\right)^2 (r-z+\rho),$$

from which we have (3.17). □

4. Some relations concerning Bicentric Hexagons

First we prove the following theorem.

Theorem 4.1. *The relation between r, ρ, and z for 1-bicentric hexagon can be written as*

$$1 = \sqrt{1-\left(\frac{\rho}{r+z}\right)^2} + \sqrt{1-\left(\frac{\rho}{r-z}\right)^2}. \tag{4.1}$$

Proof. We can take a hexagon which is symmetric in relation to axis x and consider situation shown in Figure 4, where

$$A_1(r,0)\ ,\ A_2(r\cos\varphi_1, r\sin\varphi_1)\ ,\ A_3(r\cos\varphi_2, r\sin\varphi_2)\ ,\ A_4(-r,0).$$

It is easy to see that distances of the point $M(z,0)$ from the lines A_1A_2, A_2A_3, and A_3A_4 are given by

$$\rho = (r-z)\sin\frac{\varphi_1}{2},$$

$$\rho = -z\cos\frac{\varphi_1+\varphi_2}{2} + r\cos\frac{\varphi_1-\varphi_2}{2},$$

$$\rho = (r+z)\sin\frac{\varphi_2}{2},$$

from which we easily get (4.1). The theorem is proved. □

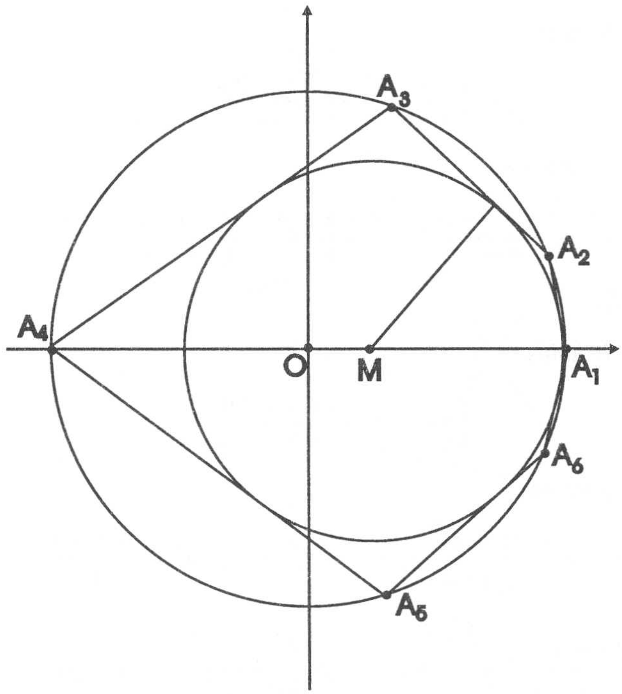


FIGURE 4. 1-bicentric hexagon symmetric with respect to x-axis

From rationalization of (4.1) we get

$$\rho^4\left(\frac{1}{(r+z)^2}-\frac{1}{(r-z)^2}\right)^2+2\rho^2\left(\frac{1}{(r+z)^2}+\frac{1}{(r-z)^2}\right)-3=0, \qquad (4.2)$$

which can be written as Fuss' relation (1.3).

Remark 4.2. Using Poncelet's closure theorem it can be easily proved that a 2-bicentric hexagon is in fact a "double triangle" (Figure 5). The proof is as follows.

Since $n=6$, starting form a point A_1 on C_2, the point A_7 must be A_1. But if the radius of incircle is less or greater than the radius of the incircle shown in Figure 5, then we have the situation shown in Figure 6.a. or that shown in Figure 6.b., respectively. Thus, in each of these two cases the corresponding traverse can not be closed. Accordingly, the relation between r, ρ, and z for 2-bicentric hexagon is given by Euler's relation $2r\rho = r^2 - z^2$.

Corollary 4.3. *It holds*

$$3\left(r^2-z^2\right)^2 > 4\rho^2\left(r^2+z^2\right), \qquad (4.3)$$

$$\sqrt{3}\left(r^2-z^2\right)^2 > 4rz\rho^2. \qquad (4.4)$$

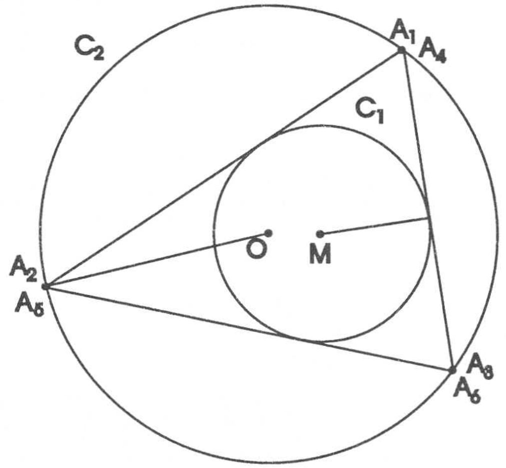
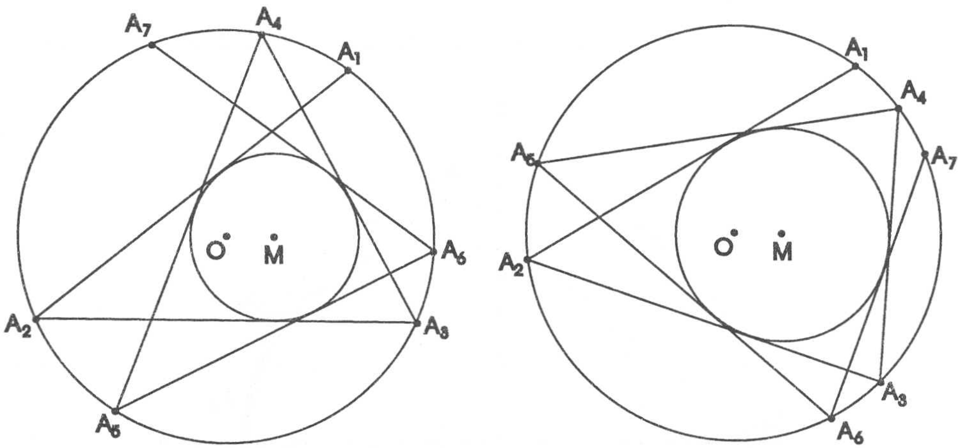


FIGURE 5. 2-bicentric hexagon, i.e. "double triangle"



a. smaller radius b. greater radius

FIGURE 6. two possible cases depending on the radius ρ of the incircle

Proof. From (4.2) we have

$$3 > 2\rho^2 \left(\left(\frac{1}{r+z} \right)^2 + \left(\frac{1}{r-z} \right)^2 \right),$$

which can be written as (4.3).

From (4.2) we have

$$3 \left(r^2 - z^2 \right)^4 > 16 r^2 z^2 \rho^4,$$

which can be written as (4.4). □

5. Some relations concerning Bicentric Heptagons

Since $\frac{7-1}{2} = 3$, there is k-bicentric heptagon for $k = 1, 2, 3$.

Theorem 5.1. *The relation between r, ρ, and z for 1-bicentric heptagon can be written as*

$$\left(\sqrt{1-\left(\frac{\rho}{r+z}\right)^2}\sqrt{\frac{r-z-\rho}{2r}} - \frac{\rho}{r+z}\sqrt{\frac{r+z+\rho}{2r}}\right)^2 =$$

$$\left(\frac{\rho}{r-z}\right)^2\left(\sqrt{\frac{r-z-\rho}{2r}} - \frac{\rho}{r+z}\right)^2 +$$

$$\left(\frac{\rho}{r+z}\right)^2\left(\sqrt{1-\left(\frac{\rho}{r+z}\right)^2} - \sqrt{\frac{r+z+\rho}{2r}}\right)^2. \quad (5.1)$$

Proof. Let $A_1 \ldots A_7$ be a 1-bicentric heptagon which is symmetric in relation to axis x as shown in Figure 7, and let

$$A_1(-r, 0) \quad , \quad A_{i+1}(r\cos\varphi_i, r\sin\varphi_i) \quad , \; i = 1, 2, 3.$$

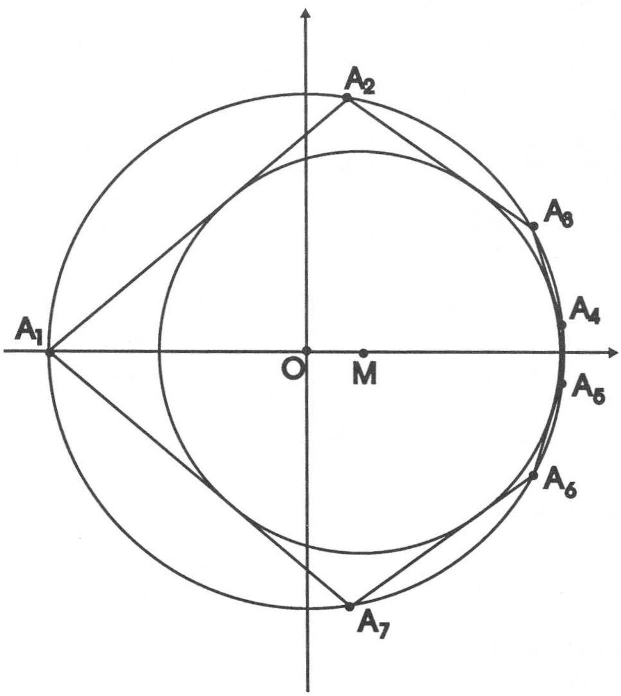


Figure 7. 1-bicentric heptagon

Then distances of the point $M(z,0)$ from the lines A_1A_2, A_2A_3, A_3A_4, and A_4A_5 are given by

$$\rho = (r+z)\sin\frac{\varphi_1}{2}, \tag{5.2}$$

$$\rho = r\cos\frac{\varphi_1-\varphi_2}{2} + z\cos\frac{\varphi_1+\varphi_2}{2}, \tag{5.3}$$

$$\rho = r\cos\frac{\varphi_2-\varphi_3}{2} + z\cos\frac{\varphi_2+\varphi_3}{2}, \tag{5.4}$$

$$\rho = -z + r\cos\varphi_3. \tag{5.5}$$

The relations (5.3) and (5.4) can be written as

$$\rho = \cos\frac{\varphi_2}{2}\left((r+z)\cos\frac{\varphi_1}{2}\right) + \sin\frac{\varphi_2}{2}\left((r-z)\sin\frac{\varphi_1}{2}\right),$$

$$\rho = \cos\frac{\varphi_2}{2}\left((r+z)\cos\frac{\varphi_3}{2}\right) + \sin\frac{\varphi_2}{2}\left((r-z)\sin\frac{\varphi_3}{2}\right),$$

from which it follows that $\cos\frac{\varphi_2}{2} = \frac{D_1}{D}$, $\sin\frac{\varphi_2}{2} = \frac{D_2}{D}$, where

$$D = \begin{vmatrix} (r+z)\cos\frac{\varphi_1}{2} & (r-z)\sin\frac{\varphi_1}{2} \\ (r+z)\cos\frac{\varphi_3}{2} & (r-z)\sin\frac{\varphi_3}{2} \end{vmatrix},$$

$$D_1 = \begin{vmatrix} \rho & (r-z)\sin\frac{\varphi_1}{2} \\ \rho & (r-z)\sin\frac{\varphi_3}{2} \end{vmatrix}, \quad D_2 = \begin{vmatrix} (r+z)\cos\frac{\varphi_1}{2} & \rho \\ (r+z)\cos\frac{\varphi_3}{2} & \rho \end{vmatrix}.$$

Using the equality $\cos^2\frac{\varphi_2}{2} + \sin^2\frac{\varphi_2}{2} = 1$ and the relations (5.2) and (5.5), we get (5.1). The theorem is proved. □

After rationalization and factorization of (5.1), we get

$$\begin{aligned}(r^6 - 3r^4z^2 + 3r^2z^4 - z^6 + 2r^5\rho - 4r^3z^2\rho + 2rz^4\rho - 4r^4\rho^2 + 4r^2z^2\rho^2 - \\ 8rz^2\rho^3)\,(r^{12} - 6r^{10}z^2 + 15r^8z^4 - 20r^6z^6 + 15r^4z^8 - 6r^2z^{10} + z^{12} - \\ 4r^{11}\rho + 20r^9z^2\rho - 40r^7z^4\rho + 40r^5z^6\rho - 20r^3z^8\rho + 4rz^{10}\rho - 4r^{10}\rho^2 + \\ 16r^8z^2\rho^2 - 24r^6z^4\rho^2 + 16r^4z^6\rho^2 - 4r^2z^8\rho^2 + 8r^9\rho^3 - 48r^5z^4\rho^3 + \\ 64r^3z^6\rho^3 - 24rz^8\rho^3 - 16r^6z^2\rho^4 + 32r^4z^4\rho^4 - 16r^2z^6\rho^4 - 32r^5z^2\rho^5 + \\ 32rz^6\rho^5 + 64r^4z^2\rho^6) = 0.\end{aligned}$$

Theorem 5.2. *The relation between r, ρ, and z for 2-bicentric heptagon can be written as*

$$\left(\sqrt{1-\left(\frac{\rho}{r+z}\right)^2}\sqrt{\frac{r-z+\rho}{2r}} + \frac{\rho}{r+z}\sqrt{\frac{r+z-\rho}{2r}}\right)^2 =$$

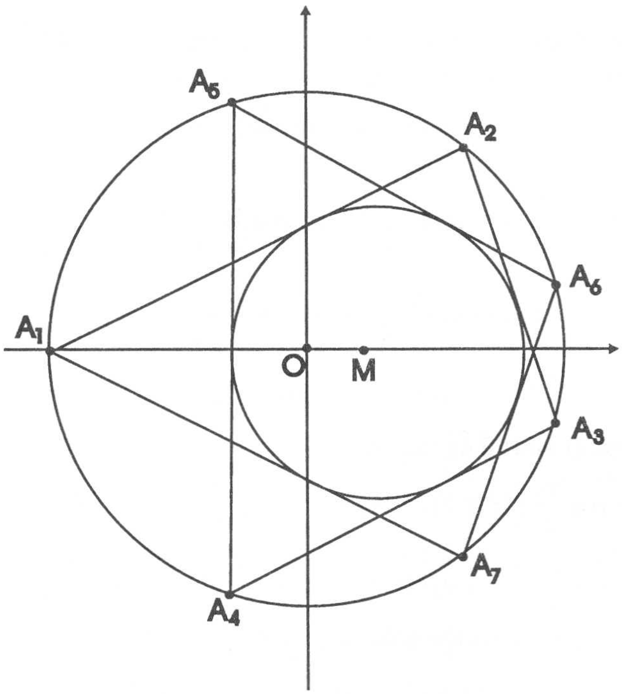


FIGURE 8. 2-bicentric heptagon symmetric with respect to x-axis

$$\left(\frac{\rho}{r-z}\right)^2\left(\sqrt{\frac{r-z+\rho}{2r}}+\frac{\rho}{r+z}\right)^2+$$
$$\left(\frac{\rho}{r+z}\right)^2\left(\sqrt{1-\left(\frac{\rho}{r+z}\right)^2}-\sqrt{\frac{r+z-\rho}{2r}}\right)^2. \quad (5.6)$$

Proof. Let $A_1\ldots A_7$ be a 2-bicentric heptagon which is symmetric in relation to axis x as shown in Figure 8, and let

$$A_1\,(-r,0)\;,\;A_{i+1}\,(r\cos\varphi_i, r\sin\varphi_i)\;,\;i=1,2,3.$$

Then

$$\rho=(r+z)\sin\frac{\varphi_1}{2},$$
$$\rho=-r\cos\frac{\varphi_1-\varphi_2}{2}+z\cos\frac{\varphi_1+\varphi_2}{2},$$
$$\rho=r\cos\frac{\varphi_2-\varphi_3}{2}-z\cos\frac{\varphi_2+\varphi_3}{2},$$
$$\rho=z-r\cos\varphi_3,$$

from which, in the same way as in Theorem 5.1, we get (5.6). The theorem is proved. □

After rationalization and factorization of (5.6) we get

$$(r+z-\rho)^4\left(r^2-z^2+2r\rho\right)^3\left(r^6-3r^4z^2+3r^2z^4-z^6-2r^5\rho+4r^3z^2\rho-\right.$$
$$2rz^4\rho-4r^4\rho^2+4r^2z^2\rho^2+8rz^2\rho^3\Big)\left(r^{12}-6r^{10}z^2+15r^8z^4-20r^6z^6+\right.$$
$$15r^4z^8-6r^2z^{10}+z^{12}+4r^{11}\rho-20r^9z^2\rho+40r^7z^4\rho-40r^5z^6\rho+20r^3z^8\rho-$$
$$4rz^{10}\rho-4r^{10}\rho^2+16r^8z^2\rho^2-24r^6z^4\rho^2+16r^4z^6\rho^2-4r^2z^8\rho^2-8r^9\rho^3+$$
$$48r^5z^4\rho^3-64r^3z^6\rho^3+24rz^8\rho^3-16r^6z^2\rho^4+32r^4z^4\rho^4-16r^2z^6\rho^4+$$
$$\left.32r^5z^2\rho^5-32rz^6\rho^5+64r^4z^2\rho^6\right)=0.$$

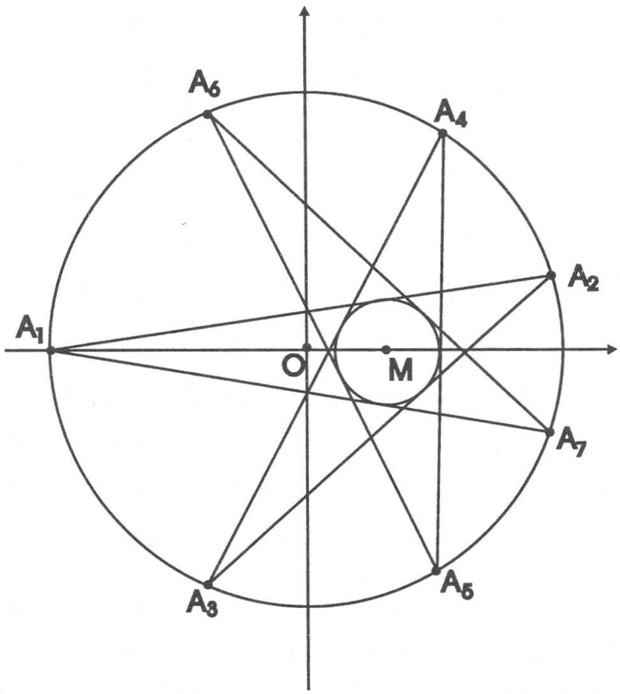


FIGURE 9. 3-bicentric heptagon symmetric with respect to x-axis

Theorem 5.3. *The relation between r, ρ, and z for 3-bicentric heptagon can be written as*

$$\left(\sqrt{1-\left(\frac{\rho}{r+z}\right)^2}\sqrt{\frac{r-z-\rho}{2r}}-\frac{\rho}{r+z}\sqrt{\frac{r+z+\rho}{2r}}\right)^2=$$

$$\left(\frac{\rho}{r-z}\right)^2\left(\sqrt{\frac{r-z-\rho}{2r}}+\frac{\rho}{r+z}\right)^2+$$
$$\left(\frac{\rho}{r+z}\right)^2\left(\sqrt{1-\left(\frac{\rho}{r+z}\right)^2}+\sqrt{\frac{r+z+\rho}{2r}}\right)^2. \quad (5.7)$$

Proof. Let $A_1\dots A_7$ be a 3-bicentric heptagon which is symmetric with respect to x-axis as shown in Figure 9, and let

$$A_1(-r,0)\ ,\ A_{i+1}(r\cos\varphi_i, r\sin\varphi_i)\ ,\ i=1,2,3.$$

Then

$$\rho=(r+z)\sin\frac{\varphi_1}{2},$$
$$\rho=-r\cos\frac{\varphi_1-\varphi_2}{2}+z\cos\frac{\varphi_1+\varphi_2}{2},$$
$$\rho=r\cos\frac{\varphi_2-\varphi_3}{2}-z\cos\frac{\varphi_2+\varphi_3}{2},$$
$$\rho=-z+r\cos\varphi_3,$$

from which we get (5.7). The theorem is proved. □

After rationalization and factorization of (5.7), we get

$$(r+z+\rho)^4\left(r^2-z^2-2r\rho\right)^3\left(r^6-3r^4z^2+3r^2z^4-z^6+2r^5\rho-4r^3z^2\rho+\right.$$
$$\left.2rz^4\rho-4r^4\rho^2+4r^2z^2\rho^2-8rz^2\rho^3\right)\left(r^{12}-6r^{10}z^2+15r^8z^4-20r^6z^6+\right.$$
$$15r^4z^8-6r^2z^{10}+z^{12}-4r^{11}\rho+20r^9z^2\rho-40r^7z^4\rho+40r^5z^6\rho-20r^3z^8\rho+$$
$$4rz^{10}\rho-4r^{10}\rho^2+16r^8z^2\rho^2-24r^6z^4\rho^2+16r^4z^6\rho^2-4r^2z^8\rho^2+8r^9\rho^3-$$
$$48r^5z^4\rho^3+64r^3z^6\rho^3-24rz^8\rho^3-16r^6z^2\rho^4+32r^4z^4\rho^4-16r^2z^6\rho^4-$$
$$\left.32r^5z^2\rho^5+32rz^6\rho^5+64r^4z^2\rho^6\right)=0.$$

6. Some relations concerning bicentric octagons

Here will be in short about bicentric octagons since there is a great analogy with what we said about bicentric heptagons. Namely, in the same way as for 1-bicentric heptagon and 3-bicentric heptagon we find that for 1-bicentric octagon and 3-bicnetric octagon hold the following relations

$$\left(\rho^2-t_mt_M\right)^2=\left(\frac{\rho}{p}\right)^2(p\rho-qt_M)^2+\left(\frac{\rho}{q}\right)^2(q\rho-pt_m)^2, \quad (6.1)$$

$$\left(\rho^2-t_mt_M\right)^2=\left(\frac{\rho}{p}\right)^2(p\rho+qt_M)^2+\left(\frac{\rho}{q}\right)^2(q\rho+pt_m)^2, \quad (6.2)$$

where $p=r+z$, $q=r-z$, $t_m=\sqrt{q^2-\rho^2}$, $t_M=\sqrt{p^2-\rho^2}$.

It may be interesting that each of the above two relations has the property that after rationalization and factorization we get the same relation which can be expressed as

$$\left[\left(\rho^2\left(p^2+q^2\right)-p^2q^2\right)^4-16p^4q^4\rho^4\left(p^2-\rho^2\right)\left(q^2-\rho^2\right)\right]\left[p^2q^2-\rho^2\left(p^2+q^2\right)\right]=0.$$

Thus for 1-bicentric octagon and 3-bicentric octagon holds the following relation

$$\left(\rho^2\left(p^2+q^2\right)-p^2q^2\right)^4-16p^4q^4\rho^4\left(p^2-\rho^2\right)\left(q^2-\rho^2\right)=0,$$

which can be written as

$$r^{16}-8z^2r^{14}-8\rho^2r^{14}+8\rho^4r^{12}+28z^4r^{12}+40\rho^2z^2r^{12}-56z^6r^{10}+ 48z^2\rho^4r^{10}-72\rho^2z^4r^{10}-264\rho^4z^4r^8+40\rho^2z^6r^8+70z^8r^8-128z^2\rho^6r^8+ 128\rho^6z^4r^6-56z^{10}r^6+40\rho^2z^8r^6+416\rho^4z^6r^6+128z^2\rho^8r^6+128\rho^6z^6r^4- 72\rho^2z^{10}r^4-264\rho^4z^8r^4+28z^{12}r^4+48\rho^4z^{10}r^2+128\rho^8z^6r^2-8z^{14}r^2- 128\rho^6z^8r^2+40\rho^2z^{12}r^2+8\rho^4z^{12}+z^{16}-8\rho^2z^{14}=0. \quad (6.3)$$

In accordance with what we said in Remark 4.2 in connection with 2-bicentric hexagon, it is clear that Fuss' relation for 2-bicentric octagon is the same as the relation for bicentric quadrilateral.

7. Some properties concerning k-bicentric polygons and one conjecture

Let $n \geq 3$ and r and z, $z<r$, such that for every $k \in \left\{1,2,\ldots,\left[\frac{n-1}{2}\right]\right\}$ there exist a circle $C_1^{(k)}=M\left(\rho_k\right)$ inside of C_2 and k-bicentric n-gon whose incircle is $C_1^{(k)}$ and circumcircle C_2, see Figure 10.

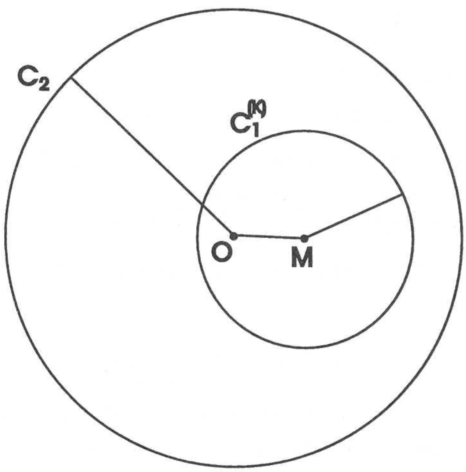


FIGURE 10

Remark 7.1. For any r and z such that $z < r$, in general it is not true that for bicentric n-gons there exist all ρ_k, $k = 1, \dots, \left[\frac{n-1}{2}\right]$ as it is the case for k-tangential n-gons (in which case we do not have r and z since tangential n-gon does not have to be chordal), see Corollary 2.3. For example if $n = 10$, $r = 5$, and $z = 4$, there are no ρ_1 and ρ_3.

For convenience, in the following, we shall by $\rho(n,k)$ denote the length of radius of incircle of a k-bicentric n-gon. The case when $k|n$ and $k \neq 1$ may be particularly interesting. For example, according to what we said in the Remark 2.1 it is clear that

$$\rho(6,2) = \rho(3,1).$$

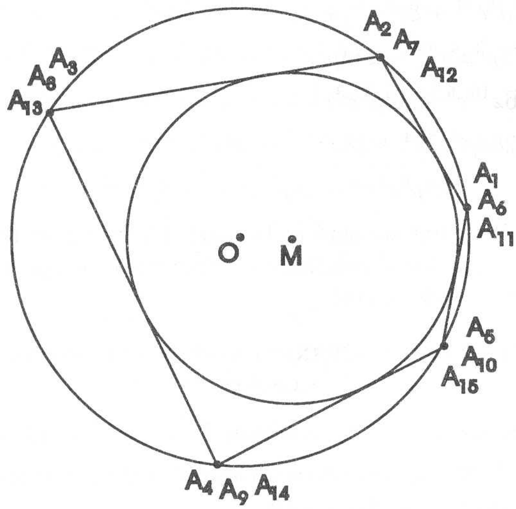


FIGURE 11

In the same way can be seen

$$\rho(15,3) = \rho(5,1) \ , \ \rho(15,6) = \rho(5,2).$$

See Figures 11 and 12. Thus 3-bicentric 15-gon is in fact 3-fold 1-bicentric 5-gon, but 6-bicentric 15-gon is in fact 3-fold 2-bicentric 5-gon.

Generally, the following theorem can be easily proved.

Theorem 7.2. *Let $k = pq$, where p is the greatest common divisor of k and n. Then*

$$\rho(n,k) = \rho\left(\frac{n}{p}, q\right), \tag{7.1}$$

Proof. By definition of k-bicentric n-gon it is clear that p-fold q-bicentric $\frac{n}{p}$-gon is pq-bicentric n-gon, and by Poncelet's closure theorem there is no a pq-bicentric n-gon such that $\rho(n,k)$ be different from $\rho\left(\frac{n}{p}, q\right)$. The argumentation is the same as that at Remark 2.1 for 2-bicentric hexagon. □

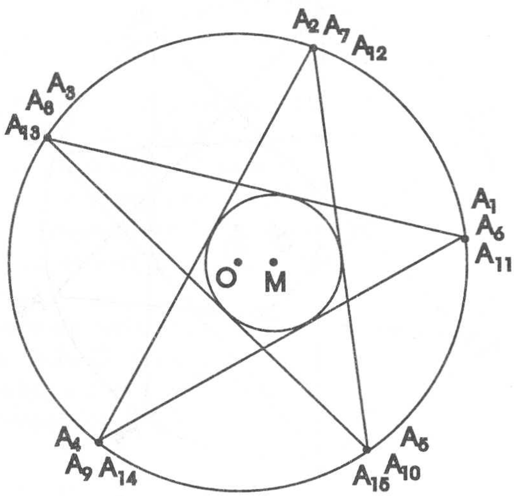


FIGURE 12

Let $GCD(n,k)$ denotes greatest common divisor of a natural numbers n and k.

Corollary 7.3. *It holds*

$$\rho(n,k) = \rho\left(\frac{n}{d}, \frac{k}{d}\right), \tag{7.2}$$

where $d = GCD(n,k)$.

For example, let $n = 20$. Then $k \in \{1,\dots,9\}$ since $\frac{20-2}{2} = 9$. If $k = 4$ then $\rho(20,4) = \rho(5,1)$. In this case the corresponding 20-gon is 4-fold 1-bicentric 5-gon.

If $k = 8$ then $\rho(20,8) = \rho(5,2)$. In this case the corresponding 20-gon is 4-fold 2-bicentric 5-gon.

If $k = 6$ then $\rho(20,6) = \rho(10,3)$. In this case the corresponding 20-gon is 2-fold 3-bicentric 10-gon.

If $GCD(n,k) = 1$ then the corresponding n-gon is 1-fold k-bicentric n-gon. For example, the octagon shown in Figure 13 is a 1-fold 3-bicentric octagon.

In connection with $k \in \{1,\dots,\left[\frac{n-1}{2}\right]\}$ we shall prove the following theorem (which may be interesting in itself).

Theorem 7.4. *The number of elements in the set* $\{1,\dots,\left[\frac{n-1}{2}\right]\}$ *relatively prime to* n *is* $\frac{\varphi(n)}{2}$, *where* $\varphi(n)$ *is Euler's* φ*-function.*

Proof. We have to prove that for every positive integer m the following two assertions are true:

In each of the sets

$$\{1,2,\dots,m\}\ ,\ \{m+1,m+2,\dots,2m\} \tag{7.3}$$

there are equal number of elements which are relatively prime to $2m+1$.

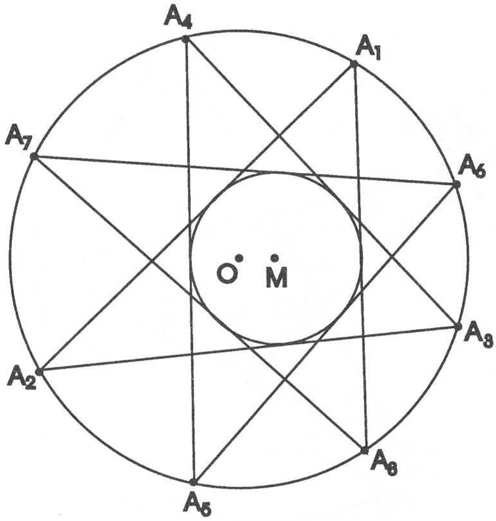


FIGURE 13

In each of the sets

$$\{1,2,\ldots,m\}\ ,\ \{m+1,m+2,\ldots,2m,2m+1\} \tag{7.4}$$

there are equal number of elements which are relatively prime to $2m+2$.

For the first assertion it is enough to prove that

$$GCD(k,2m+1)=d \Leftrightarrow GCD(2m+1-k,2m+1)=d.$$

The proof is easy, namely, from

$$k=pd\ ,\ 2m+1=qd$$

it follows that $2m+1-k=(q+p)d$. Thus, each divisor of k and $2m+1$ is also a divisor of $2m+1-k$. The converse is also valid since from

$$k=ad\ ,\ 2m+1-k=bd,$$

it follows that $2m+1=(a+b)d$.

In the same way can be proven the second assertion. □

Corollary 7.5. *If n is odd and $\frac{n-1}{2}$ is even, then in the set $\{1,\ldots,\frac{n-1}{2}\}$ there are $\frac{\varphi(n)}{4}$ even integers which are relatively prime to n. If n is even and $\frac{n-2}{2}$ is even, then in the set $\{1,\ldots,\frac{n-2}{2}\}$ there are $\frac{\varphi(n)}{4}$ odd integers which are relatively prime to n.*

Let us say something about Fuss' relation for k-bicentric n-gon.

Although Fuss' has found relations between r, z, and ρ only for bicentric n-gon, $4\le n\le 8$, it is in his honor to call such relations Fuss' relations also in the case when $n>8$.

Thus, intuitively, Fuss' relation for a k-bicentric n-gon would be an equation between r, z, and ρ such that $\rho_{k,n} = \rho(n,k)$ is its root, where r and z are given (fixed) such that $r > z$.

The important property of any relation called Fuss' relation is that it has the property that $\rho_{k,n}$ is its root for $k = 1$ and $n = 3,4,5,6,7,8$. We get Fuss' equations from rational equations which have $\rho_{k,n}$ as one of its positive roots. For example, such rational equations are (5.1), (5.6), and (5.7) for k-bicentric heptagon, $k = 1,2,3$, respectively. This gives us motivation for general definition of Fuss' relation. Let us first introduce some notation and well-known notions from general algebra.

Let $Q(r,z)$ denotes the smallest field obtained by adjunction of the real numbers r and z to the field Q of rational numbers, and $F(n,k)$ be the irreducible polynomial in variable ρ over $Q(r,z)$ which has $\rho_{k,n}$ as one of its roots and all its coefficients are multiples of whole numbers and potentials of r and z without common divisors.

Remark 7.6. Since we get polynomial $F(n,k)$ in ρ from the minimal polynomial in $Q(r,z)$ with root $\rho_{k,n}$ by multiplying it by the common denominator of all rational numbers in its coefficients, it is obvious that such polynomial is unique.

Remark 7.6 and previous considerations make the following general definition of Fuss' relation natural.

Definition 7.7. Relation $F(n,k)(\rho) = 0$ is Fuss' relation for k-bicentric n-gon, shortly written as $F(n,k) = 0$.

Here are some examples.

$$F(5,1) = r^6 - 3r^4z^2 + 3r^2z^4 - z^6 + 2r^5\rho - 4r^3z^2 + \\ 2rz^4\rho - 4r^4\rho^2 + 4r^2z^2\rho^2 - 8rz^2\rho^3,$$

$$F(5,2) = r^6 - 3r^4z^2 + 3r^2z^4 - z^6 - 2r^5\rho + 4r^3z^2\rho - \\ 2rz^4\rho - 4r^4\rho^2 + 4r^2z^2\rho^2 + 8rz^2\rho^3,$$

$$F(6,1) = 3r^8 - 4r^6\rho^2 - 12r^6z^2 + 18r^4z^4 + 4r^4z^2\rho^2 - 12r^2z^6 - \\ 16r^2z^2\rho^4 + 4r^2z^4\rho^2 + 3z^8 - 4z^6\rho^2,$$

$$F(6,2) = r^2 - 2r\rho - z^2.$$

Thus, $F(6,2) = F(3,1)$. (See (3.5), (3.14), (4.2), and (1.6).)

The following theorem is a corollary of Theorem 7.2.

Theorem 7.8. *If $k = pq$ and p is the greatest common divisor of k and n, then*

$$F(n,k) = F\left(\frac{n}{p}, q\right). \tag{7.5}$$

For example:

$$F(20,2) = F(10,1) \ , \ F(20,4) = F(5,1),$$

$$F(20,6) = F(10,3) \ , \ F(20,8) = F(5,2).$$

The equations $F(n,k_1) = 0$ and $F(n,k_2) = 0$ may have two or more roots in common, and it is often so. For example, $F(7,1) = 0$ and $F(7,3) = 0$ have the same roots since

$$F(7,1) = F(7,3) = r^{12} - 6r^{10}z^2 + 15r^8z^4 - 20r^6z^6 + 15r^4z^8 - 6r^2z^{10} + z^{12} - 4r^{11}\rho + 20r^9z^2\rho - 40r^7z^4\rho + 40r^5z^6\rho - 20r^3z^8\rho + 4rz^{10}\rho - 4r^{10}\rho^2 + 16r^8z^2\rho^2 - 24r^6z^4\rho^2 + 16r^4z^6\rho^2 - 4r^2z^8\rho^2 + 8r^9\rho^3 - 48r^5z^4\rho^3 + 64r^3z^6\rho^3 - 24rz^8\rho^3 - 16r^6z^2\rho^4 + 32r^4z^4\rho^4 - 16r^2z^6\rho^4 - 32r^5z^2\rho^5 + 32rz^6\rho^5 + 64r^4z^2\rho^6. \quad (7.6)$$

See the corresponding rational expressions for (5.1) and (5.7).

Now we can state the following conjecture.

Conjecture. *Let's take i and j to be odd integers in set $\{1,\ldots[\frac{n-1}{2}]\}$ such that $GCD(i,n) = GCD(j,n) = 1$. Then $F(n,i) = F(n,j)$.*

Let's take u and v to be even integers in set $\{1,\ldots[\frac{n-1}{2}]\}$ such that $GCD(u,n) = GCD(v,n) = 1$. Then $F(n,u) = F(n,v)$.

Here are some examples which strongly suggest the above conjecture.

Example 1. By (7.6), we have $F(7,1) = F(7,3)$.

Example 2.

$$F(8,1) = F(8,3) = r^{16} - 8z^2r^{14} - 8\rho^2r^{14} + 8\rho^4r^{12} + 28z^4r^{12} + 40\rho^2z^2r^{12} - 56z^6r^{10} + 48z^2\rho^4r^{10} - 72\rho^2z^4r^{10} - 264\rho^4z^4r^8 + 40\rho^2z^6r^8 + 70z^8r^8 - 128z^2\rho^6r^8 + 128\rho^6z^4r^6 - 56z^{10}r^6 + 40\rho^2z^8r^6 + 416\rho^4z^6r^6 + 128z^2\rho^8r^6 + 128\rho^6z^6r^4 - 72\rho^2z^{10}r^4 - 264\rho^4z^8r^4 + 28z^{12}r^4 + 48\rho^4z^{10}r^2 + 128\rho^8z^6r^2 - 8z^{14}r^2 - 128\rho^6z^8r^2 + 40\rho^2z^{12}r^2 + 8\rho^4z^{12} + z^{16} - 8\rho^2z^{14}.$$

Example 3. By checking the equation given in [9, pp. 77] and using relations for 2-bicentric 9-gon and 4-bicentric 9-gon, both of them symmetric with respect to

x-axis (as pentagons in Figures 2 and 3), we get

$$F(9,2) = F(9,4) = \\ r^{18} - 6\rho r^{17} - 9z^2r^{16} + 8\rho^3r^{15} + 48z^2\rho r^{15} + 36z^4r^{14} - 168z^4\rho r^{13} + 8z^2\rho^3r^{13} - 84z^6r^{12} - 96z^2\rho^4r^{12} + 336z^6\rho r^{11} - 32z^2\rho^5r^{11} - 216z^4\rho^3r^{11} + 256z^2\rho^6r^{10} + 126z^8r^{10} + 480\rho^4z^4r^{10} - 32z^4\rho^5r^9 - 420z^8\rho r^9 + 680z^6\rho^3r^9 - 960\rho^4z^6r^8 - 512\rho^6z^4r^8 - 126z^{10}r^8 - 256z^2\rho^8r^8 - 1000z^8\rho^3r^7 - 128z^4\rho^7r^7 + 336z^{10}\rho r^7 + 448z^6\rho^5r^7 + 84z^{12}r^6 + 960\rho^4z^8r^6 - 832z^8\rho^5r^5 + 384z^6\rho^7r^5 - 168z^{12}\rho r^5 + 792z^{10}\rho^3r^5 - 480\rho^4z^{10}r^4 - 36z^{14}r^4 + 512\rho^6z^8r^4 + 512z^6\rho^9r^3 + 608z^{10}\rho^5r^3 - 384z^8\rho^7r^3 - 328z^{12}\rho^3r^3 + 48z^{14}\rho r^3 - 256\rho^6z^{10}r^2 + 256\rho^8z^8r^2 + 9z^{16}r^2 + 96\rho^4z^{12}r^2 - 6z^{16}\rho r + 56z^{14}\rho^3r + 128z^{10}\rho^7r - 160z^{12}\rho^5r - z^{18}.$$

Example 4. By checking the equation given in [9, pp. 78] and using relations for 1-bicentric 10-gon and 3-bicentric 10-gon, both of them symmetric with respect to x-axis, we get

$$F(10,1) = F(10,3) = 5r^{24} - 20\rho^2r^{22} - 60z^2r^{22} + 330z^4r^{20} + 180\rho^2z^2r^{20} + 16\rho^4r^{20} - 1100z^6r^{18} - 304z^2\rho^4r^{18} - 700\rho^2z^4r^{18} + 1872\rho^4z^4r^{16} + 1152z^2\rho^6r^{16} + 2475z^8r^{16} + 1500\rho^2z^6r^{16} - 5760\rho^6z^4r^{14} - 5952\rho^4z^6r^{14} - 3960z^{10}r^{14} - 1800\rho^2z^8r^{14} - 1792z^2\rho^8r^{14} + 4620z^{12}r^{12} + 840\rho^2z^{10}r^{12} + 10368\rho^6z^6r^{12} + 11424\rho^4z^8r^{12} + 1024z^2\rho^{10}r^{12} + 3328\rho^8z^4r^{12} + 840\rho^2z^{12}r^{10} - 5760\rho^6z^8r^{10} + 2816\rho^8z^6r^{10} - 14112\rho^4z^{10}r^{10} - 3960z^{14}r^{10} + 11424\rho^4z^{12}r^8 - 1800\rho^2z^{14}r^8 - 8704\rho^8z^8r^8 - 5760\rho^6z^{10}r^8 + 2475z^{16}r^8 - 1024\rho^{10}z^6r^8 - 1100z^{18}r^6 + 4096\rho^{12}z^6r^6 + 10368\rho^6z^{12}r^6 - 1024\rho^{10}z^8r^6 + 2816\rho^8z^{10}r^6 - 5952\rho^4z^{14}r^6 + 1500\rho^2z^{16}r^6 + 3328\rho^8z^{12}r^4 + 330z^{20}r^4 + 1872\rho^4z^{16}r^4 - 700\rho^2z^{18}r^4 - 5760\rho^6z^{14}r^4 + 180\rho^2z^{20}r^2 - 60z^{22}r^2 - 1792\rho^8z^{14}r^2 + 1024\rho^{10}z^{12}r^2 + 1152\rho^6z^{16}r^2 - 304\rho^4z^{18}r^2 + 16\rho^4z^{20} - 20\rho^2z^{22} + 5z^{24}.$$

Example 5. By checking the equation given in [9, pp. 97] and using relations for k-bicentric 11-gon, $k = 1,2,3,4,5$, symmetric with respect to x-axis, we get

$$F(11,1) = F(11,3) = F(11,5) = -r^{30} + 6\rho r^{29} + 15z^2r^{28} + 12\rho^2r^{28} - 32\rho^3r^{27} - 84z^2\rho r^{27} - 105z^4r^{26} - 156\rho^2z^2r^{26} - 16\rho^4r^{26} + 32\rho^5r^{25} + 280z^2\rho^3r^{25} + 546z^4\rho r^{25} + 455z^6r^{24} + 240z^2\rho^4r^{24} + 936\rho^2z^4r^{24} - 2184z^6\rho r^{23} - 864z^4\rho^3r^{23} + 384z^2\rho^5r^{23} - 3432\rho^2z^6r^{22} - 1408z^2\rho^6r^{22} - 1584\rho^4z^4r^{22} - 1365z^8r^{22} - 5088z^4\rho^5r^{21} - 1792z^2\rho^7r^{21} + 6006z^8\rho r^{21} + 176z^6\rho^3r^{21} + 12480z^4\rho^6r^{20} + 3003z^{10}r^{20} + 6160z^6\rho^4r^{20} + 8580z^8\rho^2r^{20} + 4864z^2\rho^8r^{20} + 7040z^8\rho^3r^{19} + 3072z^2\rho^9r^{19} + 22720z^6\rho^5r^{19} + 11264z^4\rho^7r^{19} - 12012z^{10}\rho r^{19} - 48960z^6\rho^6r^{18} - 15444z^{10}\rho^2r^{18} - 5005z^{12}r^{18} - 27648z^4\rho^8r^{18} -$$

$$7168z^2\rho^{10}r^{18} - 15840z^8\rho^4r^{18} - 54720z^8\rho^5r^{17} + 18018z^{12}\rho r^{17} - 2048z^2\rho^{11}r^{17} -$$
$$26136z^{10}\rho^3r^{17} - 26496z^6\rho^7r^{17} - 12288z^4\rho^9r^{17} + 28512z^{10}\rho^4r^{16} + 6435z^{14}r^{16} +$$
$$111360z^8\rho^6r^{16} + 4096z^2\rho^{12}r^{16} + 57600z^6\rho^8r^{16} + 20592z^{12}\rho^2r^{16} +$$
$$20480z^4\rho^{10}r^{16} + 52800z^{12}\rho^3r^{15} + 12288z^6\rho^9r^{15} - 20592z^{14}\rho r^{15} +$$
$$21504z^8\rho^7r^{15} + 78336z^{10}\rho^5r^{15} - 36960z^{12}\rho^4r^{14} - 161280z^{10}\rho^6r^{14} -$$
$$6435z^{16}r^{14} - 12288z^6\rho^{10}r^{14} - 20592z^{14}\rho^2r^{14} - 37632z^8\rho^8r^{14} +$$
$$24576z^6\rho^{11}r^{13} + 18018z^{16}\rho r^{13} - 63168z^{12}\rho^5r^{13} - 70752z^{14}\rho^3r^{13} +$$
$$21504z^{10}\rho^7r^{13} + 12800z^8\rho^9r^{13} - 48384z^{10}\rho^8r^{12} + 153216z^{12}\rho^6r^{12} +$$
$$5005z^{18}r^{12} + 4096z^6\rho^{12}r^{12} + 5120z^8\rho^{10}r^{12} + 15444z^{16}\rho^2r^{12} +$$
$$34848z^{14}\rho^4r^{12} - 33792z^{10}\rho^9r^{11} + 14976z^{14}\rho^5r^{11} + 66528z^{16}\rho^3r^{11} -$$
$$16384z^6\rho^{13}r^{11} - 12012z^{18}\rho r^{11} - 64512z^{12}\rho^7r^{11} - 57344z^8\rho^{11}r^{11} -$$
$$23760z^{16}\rho^4r^{10} - 8580z^{18}\rho^2r^{10} - 56320z^{10}\rho^{10}r^{10} + 112896z^{12}\rho^8r^{10} -$$
$$16384z^6\rho^{14}r^{10} - 3003z^{20}r^{10} - 94080z^{14}\rho^6r^{10} - 28672z^8\rho^{12}r^{10} + 32768z^6\rho^{15}r^9 +$$
$$6006z^{20}\rho r^9 + 59136z^{14}\rho^7r^9 + 19968z^{12}\rho^9r^9 - 44440z^{18}\rho^3r^9 + 24480z^{16}\rho^5r^9 +$$
$$24576z^8\rho^{13}r^9 + 55296z^{10}\rho^{11}r^9 + 34560z^{16}\rho^6r^8 + 3432z^{20}\rho^2r^8 + 11440z^{18}\rho^4r^8 +$$
$$1365z^{22}r^8 + 104448z^{12}\rho^{10}r^8 - 91392z^{14}\rho^8r^8 - 24576z^{12}\rho^{11}r^7 - 21504z^{16}\rho^7r^7 -$$
$$30080z^{18}\rho^5r^7 + 2048z^{14}\rho^9r^7 - 2184z^{22}\rho r^7 + 20768z^{20}\rho^3r^7 + 45056z^{12}\rho^{12}r^6 +$$
$$34560z^{16}\rho^8r^6 - 71680z^{14}\rho^{10}r^6 - 3696z^{20}\rho^4r^6 - 936z^{22}\rho^2r^6 - 5760z^{18}\rho^6r^6 -$$
$$455z^{24}r^6 - 6480z^{22}\rho^3r^5 + 4096z^{14}\rho^{11}r^5 - 2304z^{18}\rho^7r^5 - 8192z^{12}\rho^{13}r^5 +$$
$$16032z^{20}\rho^5r^5 - 4608z^{16}\rho^9r^5 + 546z^{24}\rho r^5 - 4608z^{18}\rho^8r^4 - 320z^{20}\rho^6r^4 +$$
$$105z^{26}r^4 + 720z^{22}\rho^4r^4 + 17408z^{16}\rho^{10}r^4 + 16384z^{12}\rho^{14}r^4 + 156z^{24}\rho^2r^4 -$$
$$24576z^{14}\rho^{12}r^4 + 1216z^{24}\rho^3r^3 - 84z^{26}\rho r^3 + 4096z^{20}\rho^7r^3 - 4416z^{22}\rho^5r^3 -$$
$$64z^{24}\rho^4r^2 + 192z^{22}\rho^6r^2 - 12z^{26}\rho^2r^2 - 256z^{20}\rho^8r^2 - 15z^{28}r^2 - 896z^{22}\rho^7r +$$
$$512z^{20}\rho^9r + 512z^{24}\rho^5r + 6z^{28}\rho r - 104z^{26}\rho^3r + z^{30},$$
$$F(11,2) = F(11,4) = r^{30} + 6\rho r^{29} - 15z^2r^{28} - 12\rho^2r^{28} - 32\rho^3r^{27} - 84z^2\rho r^{27} +$$
$$105z^4r^{26} + 156\rho^2z^2r^{26} + 16\rho^4r^{26} + 32\rho^5r^{25} + 280z^2\rho^3r^{25} + 546z^4\rho r^{25} -$$
$$455z^6r^{24} - 240z^2\rho^4r^{24} - 936\rho^2z^4r^{24} - 2184z^6\rho r^{23} - 864z^4\rho^3r^{23} + 384z^2\rho^5r^{23} +$$
$$3432\rho^2z^6r^{22} + 1408z^2\rho^6r^{22} + 1584\rho^4z^4r^{22} + 1365z^8r^{22} - 5088z^4\rho^5r^{21} -$$
$$1792z^2\rho^7r^{21} + 6006z^8\rho r^{21} + 176z^6\rho^3r^{21} - 12480z^4\rho^6r^{20} - 3003z^{10}r^{20} -$$
$$6160z^6\rho^4r^{20} - 8580z^8\rho^2r^{20} - 4864z^2\rho^8r^{20} + 7040z^8\rho^3r^{19} + 3072z^2\rho^9r^{19} +$$
$$22720z^6\rho^5r^{19} + 11264z^4\rho^7r^{19} - 12012z^{10}\rho r^{19} + 48960z^6\rho^6r^{18} +$$
$$15444z^{10}\rho^2r^{18} + 5005z^{12}r^{18} + 27648z^4\rho^8r^{18} + 7168z^2\rho^{10}r^{18} +$$
$$15840z^8\rho^4r^{18} - 54720z^8\rho^5r^{17} + 18018z^{12}\rho r^{17} - 2048z^2\rho^{11}r^{17} -$$
$$26136z^{10}\rho^3r^{17} - 26496z^6\rho^7r^{17} - 12288z^4\rho^9r^{17} - 28512z^{10}\rho^4r^{16} -$$
$$6435z^{14}r^{16} - 111360z^8\rho^6r^{16} - 4096z^2\rho^{12}r^{16} - 57600z^6\rho^8r^{16} - 20592z^{12}\rho^2r^{16} -$$
$$20480z^4\rho^{10}r^{16} + 52800z^{12}\rho^3r^{15} + 12288z^6\rho^9r^{15} - 20592z^{14}\rho r^{15} +$$
$$21504z^8\rho^7r^{15} + 78336z^{10}\rho^5r^{15} + 36960z^{12}\rho^4r^{14} + 161280z^{10}\rho^6r^{14} +$$
$$6435z^{16}r^{14} + 12288z^6\rho^{10}r^{14} + 20592z^{14}\rho^2r^{14} + 37632z^8\rho^8r^{14} +$$

$$24576z^6\rho^{11}r^{13} + 18018z^{16}\rho r^{13} - 63168z^{12}\rho^5 r^{13} - 70752z^{14}\rho^3 r^{13} + 21504z^{10}\rho^7 r^{13} + 12800z^8\rho^9 r^{13} + 48384z^{10}\rho^8 r^{12} - 153216z^{12}\rho^6 r^{12} - 5005z^{18}r^{12} - 4096z^6\rho^{12}r^{12} - 5120z^8\rho^{10}r^{12} - 15444z^{16}\rho^2 r^{12} - 34848z^{14}\rho^4 r^{12} - 33792z^{10}\rho^9 r^{11} + 14976z^{14}\rho^5 r^{11} + 66528z^{16}\rho^3 r^{11} - 16384z^6\rho^{13}r^{11} - 12012z^{18}\rho r^{11} - 64512z^{12}\rho^7 r^{11} - 57344z^8\rho^{11}r^{11} + 23760z^{16}\rho^4 r^{10} + 8580z^{18}\rho^2 r^{10} + 56320z^{10}\rho^{10}r^{10} - 112896z^{12}\rho^8 r^{10} + 16384z^6\rho^{14}r^{10} + 3003z^{20}r^{10} + 94080z^{14}\rho^6 r^{10} + 28672z^8\rho^{12}r^{10} + 32768z^6\rho^{15}r^9 + 6006z^{20}\rho r^9 + 59136z^{14}\rho^7 r^9 + 19968z^{12}\rho^9 r^9 - 44440z^{18}\rho^3 r^9 + 24480z^{16}\rho^5 r^9 + 24576z^8\rho^{13}r^9 + 55296z^{10}\rho^{11}r^9 - 34560z^{16}\rho^6 r^8 - 3432z^{20}\rho^2 r^8 - 11440z^{18}\rho^4 r^8 - 1365z^{22}r^8 - 104448z^{12}\rho^{10}r^8 + 91392z^{14}\rho^8 r^8 - 24576z^{12}\rho^{11}r^7 - 21504z^{16}\rho^7 r^7 - 30080z^{18}\rho^5 r^7 + 2048z^{14}\rho^9 r^7 - 2184z^{22}\rho r^7 + 20768z^{20}\rho^3 r^7 - 45056z^{12}\rho^{12}r^6 - 34560z^{16}\rho^8 r^6 + 71680z^{14}\rho^{10}r^6 + 3696z^{20}\rho^4 r^6 + 936z^{22}\rho^2 r^6 + 5760z^{18}\rho^6 r^6 + 455z^{24}r^6 - 6480z^{22}\rho^3 r^5 + 4096z^{14}\rho^{11}r^5 - 2304z^{18}\rho^7 r^5 - 8192z^{12}\rho^{13}r^5 + 16032z^{20}\rho^5 r^5 - 4608z^{16}\rho^9 r^5 + 546z^{24}\rho r^5 + 4608z^{18}\rho^8 r^4 + 320z^{20}\rho^6 r^4 - 105z^{26}r^4 - 720z^{22}\rho^4 r^4 - 17408z^{16}\rho^{10}r^4 - 16384z^{12}\rho^{14}r^4 - 156z^{24}\rho^2 r^4 + 24576z^{14}\rho^{12}r^4 + 1216z^{24}\rho^3 r^3 - 84z^{26}\rho r^3 + 4096z^{20}\rho^7 r^3 - 4416z^{22}\rho^5 r^3 + 64z^{24}\rho^4 r^2 - 192z^{22}\rho^6 r^2 + 12z^{26}\rho^2 r^2 + 256z^{20}\rho^8 r^2 + 15z^{28}r^2 - 896z^{22}\rho^7 r + 512z^{20}\rho^9 r + 512z^{24}\rho^5 r + 6z^{28}\rho r - 104z^{26}\rho^3 r - z^{30}.$$

Example 6. By checking the equation given in [9, pp. 99] and using relations for 1-bicentric 12-gon and 5-bicentric 12-gon, both of them symmetric with respect to x-axis, we get

$$F(12,1) = F(12,5) = r^{32} - 16\rho^2 r^{30} - 16z^2 r^{30} + 208z^2\rho^2 r^{28} + 120z^4 r^{28} + 16\rho^4 r^{28} - 1232z^4\rho^2 r^{26} + 224z^2\rho^4 r^{26} - 560z^6 r^{26} + 1820z^8 r^{24} - 1792z^2\rho^6 r^{24} - 3920z^4\rho^4 r^{24} + 4368z^6\rho^2 r^{24} + 16128z^4\rho^6 r^{22} - 4368z^{10}r^{22} + 5888z^2\rho^8 r^{22} - 10192z^8\rho^2 r^{22} + 23744z^6\rho^4 r^{22} - 82544z^8\rho^4 r^{20} - 8192z^2\rho^{10}r^{20} + 16016z^{10}\rho^2 r^{20} - 40960z^4\rho^8 r^{20} - 62720z^6\rho^6 r^{20} + 8008z^{12}r^{20} - 16016z^{12}\rho^2 r^{18} + 189728z^{10}\rho^4 r^{18} - 11440z^{14}r^{18} + 121600z^6\rho^8 r^{18} + 4096z^2\rho^{12}r^{18} + 134400z^8\rho^6 r^{18} + 28672z^4\rho^{10}r^{18} - 306768z^{12}\rho^4 r^{16} - 161280z^{10}\rho^6 r^{16} - 204800z^8\rho^8 r^{16} + 12870z^{16}r^{16} - 20480z^6\rho^{10}r^{16} + 6864z^{14}\rho^2 r^{16} - 4096z^4\rho^{12}r^{16} + 75264z^{12}\rho^6 r^{14} + 232960z^{10}\rho^8 r^{14} - 28672z^8\rho^{10}r^{14} + 57344z^6\rho^{12}r^{14} - 11440z^{18}r^{14} + 6864z^{16}\rho^2 r^{14} + 359040z^{14}\rho^4 r^{14} + 28672z^{10}\rho^{10}r^{12} + 75264z^{14}\rho^6 r^{12} - 16016z^{18}\rho^2 r^{12} + 8008z^{20}r^{12} - 229376z^{12}\rho^8 r^{12} - 306768z^{16}\rho^4 r^{12} - 65536z^6\rho^{14}r^{12} - 258048z^8\rho^{12}r^{12} - 161280z^{16}\rho^6 r^{10} + 401408z^{10}\rho^{12}r^{10} + 16016z^{20}\rho^2 r^{10} + 65536z^8\rho^{14}r^{10} - 4368z^{22}r^{10} + 28672z^{12}\rho^{10}r^{10} + 232960z^{14}\rho^8 r^{10} + 189728z^{18}\rho^4 r^{10} + 65536z^6\rho^{16}r^{10} - 65536z^8\rho^{16}r^8 + 1820z^{24}r^8 + 65536z^{10}\rho^{14}r^8 - 28672z^{14}\rho^{10}r^8 + 134400z^{18}\rho^6 r^8 - 82544z^{20}\rho^4 r^8 - 258048z^{12}\rho^{12}r^8 - 10192z^{22}\rho^2 r^8 - 204800z^{16}\rho^8 r^8 + 121600z^{18}\rho^8 r^6 - 65536z^{12}\rho^{14}r^6 - 20480z^{16}\rho^{10}r^6 - 62720z^{20}\rho^6 r^6 -$$

$$560z^{26}r^6 + 57344z^{14}\rho^{12}r^6 + 65536z^{10}\rho^{16}r^6 + 23744z^{22}\rho^4r^6 + 4368z^{24}\rho^2r^6 - 40960z^{20}\rho^8r^4 + 28672z^{18}\rho^{10}r^4 + 16128z^{22}\rho^6r^4 + 120z^{28}r^4 - 4096z^{16}\rho^{12}r^4 - 1232z^{26}\rho^2r^4 - 3920z^{24}\rho^4r^4 + 224z^{26}\rho^4r^2 + 4096z^{18}\rho^{12}r^2 - 8192z^{20}\rho^{10}r^2 - 16z^{30}r^2 - 1792z^{24}\rho^6r^2 + 5888z^{22}\rho^8r^2 + 208z^{28}\rho^2r^2 + 16z^{28}\rho^4 - 16z^{30}\rho^2 + z^{32}.$$

If our conjecture is true, it would further hold

$$F(13,1) = F(13,3) = F(13,5)\ ,\ F(13,2) = F(13,4) = F(13,6),$$
$$F(14,1) = F(14,3) = F(14,5)\ ,\ F(15,1) = F(15,7),$$
$$F(16,1) = F(16,3) = F(16,5) = F(16,7),$$

and so on.

We consider that the reason why it is so lie in the following fact concerning relation (2.3), that is, $\beta_1 + \cdots + \beta_n = (n-2k)\frac{\pi}{2}$. So, for example, we have that

$$\sin(13-2k)\frac{\pi}{2} = -1 \text{ for } k = 1,3,5,$$
$$\sin(13-2k)\frac{\pi}{2} = +1 \text{ for } k = 2,4,6,$$
$$\cos(14-2k)\frac{\pi}{2} = +1 \text{ for } k = 1,3,5,$$
$$\cos(14-2k)\frac{\pi}{2} = -1 \text{ for } k = 2,4,6.$$

The situation in the case when $GCD(k,n) = 1$ is a quite different from that when $GCD(k,n) > 1$ since then $(n-2k)\frac{\pi}{2}$ can be written as $k\left(\frac{n}{k}-2\right)\frac{\pi}{2}$.

We have tried to prove this conjecture, but (at least for the time being) we have not achieved success.

Remark 7.9. In this paper we restrict ourselves to the case when one circle is inside of the other. The case when one circle is not inside the other may also be interesting.

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Faculty of Philosophy in Rijeka, Department of Mathematics, Omladinska 14, Rijeka 51 000, Croatia
E-mail address: mradic@ffri.hr