Math. Maced.
Vol. 2 (2004)
27-50

SOME PROPERTIES AND RELATIONS CONCERNING BICENTRIC HEXAGONS AND OCTAGONS IN CONNECTION WITH PONCELET'S CLOSURE THEOREM

MIRKO RADIĆ

ABSTRACT. In this paper we restrict ourselves to the case when conics are circles one completely inside of the other. The great part is concerned with the condition which tangents of a bicentric hexagon or a bicentric octagon need to satisfy. Using very elementary mathematical facts, we have found some important relations (equalities and inequalities, Theorem 1-5).

1. INTRODUCTION

1.1. **A little of history.** A polygon which is both chordal and tangential is briefly called bicentric polygon. First they were concerned by the German mathematician Nicolaus Fuss (1755-1826), a friend of Leonhard Euler. He posed himself the following problem (known as Fuss' problem concerning bicentric quadrilateral):

Find the relation between the radii and the line-segment joining the centers of the circles of circumscription and inscription of a bicentric quadrilateral.

He found that

$$(r^2 - z^2)^2 = 2\rho^2(r^2 + z^2), \tag{1.1}$$

where r and ρ are radii and z is the distance between the centers of the circles of circumscription and inscription.

This problem is listed and considered in [4, pp. 188-192] as one of 100 great problems of elementary mathematics.

Fuss also found the corresponding formulas for the bicentric pentagon, hexagon, heptagon and octagon. These formulas are:

$$p^3q^3 + p^2q^2\rho(p+q) - pq\rho^2(p+q)^2 - \rho^3(p+q)(p-q)^2 = 0, \tag{1.2}$$

$$3p^4q^4 - 2p^2q^2\rho^2(p^2+q^2) = \rho^4(p^2-q^2)^2, \tag{1.3}$$

AMS Subject Classification. 51E12.
Key words and phrases. bicentric hexagon, bicentric octagon.

$$\begin{aligned}(pq-\rho(p-q)-2\rho^2)2pq\rho\sqrt{(p-\rho)(p+q)}+\\+(p^2q^2-\rho^2(p^2+q^2))2\rho\sqrt{(q-\rho)(p+q)}=\\=\pm(pq-\rho(p-q))(p^2q^2+\rho^2(p^2-q^2)),\end{aligned}\tag{1.4}$$

$$[\rho^2(p^2+q^2)-p^2q^2]^4=16p^4q^4\rho^4(p^2-\rho^2)(q^2-\rho^2),\tag{1.5}$$

where $p=r+z$, $q=r-z$.

About Fuss' results can be seen in [5] and [6]. Also very interesting informations about Fuss' results and, generally, about history of the results concerning bicentric polygons can be seen in [2] and [9].

A very remarkable theorem concerning bicentric polygon is given by the French mathematician Poncelet (1788-1867). In the formulation of this theorem will be used the so called Poncelet traverse. In short about this. First for the case when conics are circles one completely inside of the other.

Let C_1 and C_2 be two circles in a plane such that C_1 is completely inside of C_2. If from any point on C_2 we draw a tangent to C_1, extend the tangent so that it intersect C_2, and draw from the point of intersection a new tangent to C_1, extend this tangent similarly to intersect C_2, and continue in this way, we obtain the so-called Poncelet traverse which, when it consists of n chords of the circle C_2, it is called n-sided.

The Poncelet's closure theorem for circles when one is completely inside of the other is the following.

Let C_1 and C_2 be two circles in a plane such that C_1 is completely inside of C_2. If on C_2 there is a point of origin for which n-sided Poncelet traverse is closed, then the n-sided traverse will also be closed for any other point of origin on the circle C_2.

So, for example, the circles C_1 and C_2 shown in Figure 1 have the property that for every point A_1 on C_2 there are points A_2 and A_3 on C_2 such that $A_1A_2A_3$ is a triangle whose incircle is C_1 and circumcircle C_2. Thus, in this case every 3-sided Poncelet traverse is closed. As it is well known, the closing in this case will be iff

$$r^2-z^2=2r\rho,\tag{1.6}$$

where r is the radius of C_2, ρ is the radius of C_1 and z is the distance between the centers of C_1 and C_2.

In this connection let us remark that formula (1.6) has important role in the prehistory of Poncelet's closure theorem. (See [2, p. 291.])

Poncelet demonstrated that analogously holds a theorem for conic sections (general Poncelet's closure theorem):

If an n-sided Poncelet traverse constructed for two given conic sections is closed for one position of the point of origin, it is closed for any position of the point of origin.

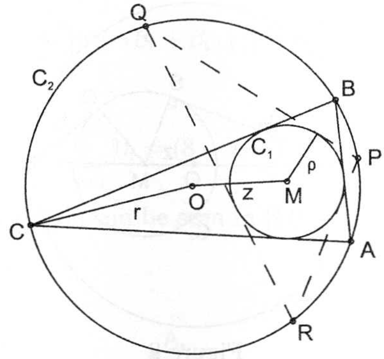

Figure 1

In this paper we restrict ourselves to the case when conics are circles one completely inside the other. In this case, as will be seen for bicentric hexagons and octagons, there are many interesting relations (equalities and inequalities) which need not be valid generally for conics.

1.2. **On notation which will be used.** Whenever $A_1 \dots A_n$ is a given or considered bicentric polygon in this paper, then the following notation is used:

C_1 is incircle of $A_1 \dots A_n$,
C_2 is circumcircle of $A_1 \dots A_n$,
ρ is radius of C_1,
r is radius of C_2,
$z = |MO|$, where M is center of C_1 and O is center of C_2,

$$\beta_i = \text{measure of } \sphericalangle MA_iA_{i+1}, i = 1, \dots, n \tag{1.7}$$

$$\gamma_i = \text{measure of } \sphericalangle OA_iA_{i+1}, i = 1, \dots, n \tag{1.8}$$

$$t_i + t_{i+1} = |A_iA_{i+1}|, i = 1, \dots, n \tag{1.9}$$

Of course, indices in (1.7)-(1.9) are calculated modulo n.

By t_m and t_M will be denoted the lengths of the least and the largest tangent that can be drawn from C_2 to C_1. See Figure 2, where $t_m = |PQ|$, $t_M = |RS|$. It is easy to see that

$$t_m = \sqrt{(r-z)^2 - \rho^2}, \qquad t_M = \sqrt{(r+z)^2 - \rho^2} \tag{1.10}$$

The following symbols will be used:

Symbol $S_j(x_1, \dots, x_n)$. Let $x_1, \dots, x_n$ be real numbers, and let j be an integer such that $1 \le j \le n$. Then $S_j(x_1, \dots, x_n)$ is the sum of all $\binom{n}{j}$ products of the

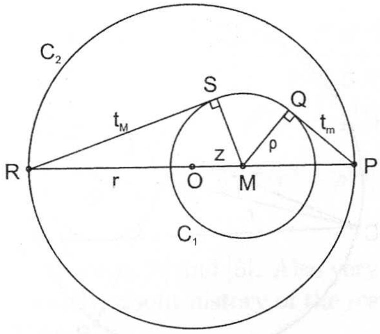

Figure 2

form $x_{i_1} \cdots x_{i_j}$, where $i_1, \ldots, i_j$ are different elements of the set $\{1, \ldots, n\}$, i.e.

$$S_j(x_1, \ldots, x_n) = \sum_{1 \le i_1 < \ldots < i_j \le n} x_{i_1} \ldots x_{i_j}. \tag{1.11}$$

Symbol C_j^n. The sum $S_j(\cot \beta_1, \ldots, \cot \beta_n)$ will be briefly written as C_j^n.

Remark 1. In the following where $t_1, \ldots, t_n$ will be lenghts of tangents we shall, for brevity, write S_j^n instead of $S_j(t_1, \ldots, t_n)$.

For example: $S_1^3 = t_1 + t_2 + t_3$, $S_2^3 = t_1t_2 + t_2t_3 + t_3t_1$, $S_3^3 = t_1t_2t_3$.

1.3. **About one kind of tangential polygons.** Let $A_1 \ldots A_n$ be a tangential polygon and let k be a positive integer such that

$$k \le \frac{n-1}{2} \text{ if } n \text{ is odd and } k \le \frac{n-2}{2} \text{ if } n \text{ is even.} \tag{1.12}$$

Then $A_1 \ldots A_n$ will be called a *k-tangential polygon* if any two of its consecutive tangents have only one common point and if

$$\sum_{i=1}^{n} \varphi_i = k \cdot 2\pi, \tag{1.13}$$

where φ_i = measure of $\sphericalangle A_iMA_{i+1}$, $i = 1, \ldots, n$, and M is the center of the inscribed circle into $A_1 \ldots A_n$. So, for example, the octagon $A_1 \ldots A_n$ shown in Figure 3 is a 3-tangential octagon.

It is easy to show that a k-tangential polygon $A_1 \ldots A_n$ has the property that

$$\sum_{i=1}^{n} \beta_i = (n - 2k)\frac{\pi}{2}, \tag{1.14}$$

where β_i = measure of $\sphericalangle MA_iA_{i+1}, i = 1, \ldots, n$. So, for example, for the 3-tangential octagon shown in Figure 3 we can write

$$\sum_{i=1}^{8} \varphi_i = 3 \cdot 2\pi, \quad \varphi_i = \pi - (\beta_i + \beta_{i+1}),\ i = 1, \ldots, 8$$

$$\sum_{i=1}^{8}[\pi - (\beta_i + \beta_{i+1})] = 3 \cdot 2\pi$$

$$\sum_{i=1}^{8}\beta_i = (8 - 2 \cdot 3)\frac{\pi}{2}.$$

(About a k-tangential polygon can be seen in [11].)

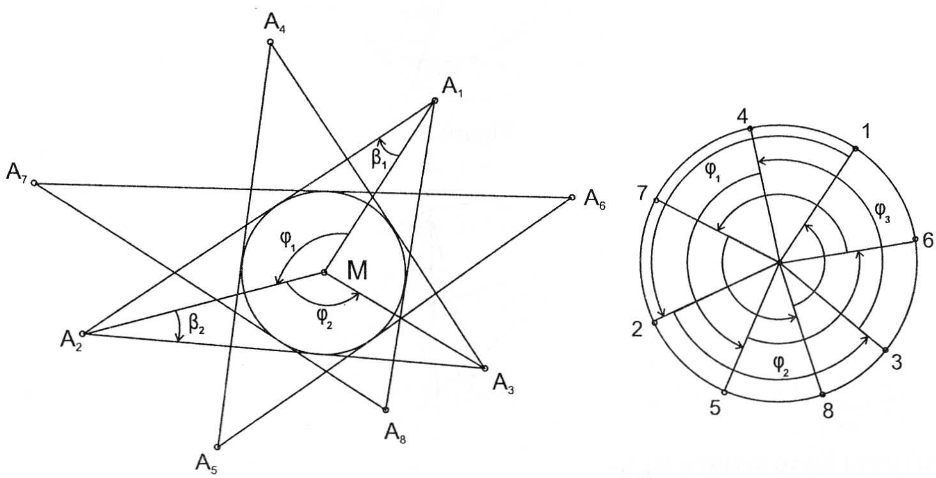

Figure 3

1.4. **Some kinds of bicentric polygon.** If one circle is completely inside of the other, then every bicentric polygon is a k-tangential polygon. So, for example, the bicentric octagon shown in Figure 5 is a 3-tangential polygon. When it is important to point out that a bicentric polygon is a k-tangential polygon, it will be said that this polygon has type k or, briefly, that this polygon is a k-bicentric polygon. For example, the octagon $A_1 \ldots A_8$ shown in Figure 5 is a 3-bicentric octagon.

1.5. **One property concerning bicentric polygons.** Here will be used oriented angles. First let us remark that the measure of an oriented angle will be taken positive or negative depending on whether this angle is positively or negatively oriented.

Concerning angles $\beta_1, \ldots, \beta_n$ given by (1.7), it is easy to see that the angles MA_iA_{i+1}, $i = 1, \ldots, n$, are either all positively oriented or all negatively oriented.

Concerning angles $\gamma_1, \ldots, \gamma_n$ given by (1.8) and the angles $\alpha_1, \ldots, \alpha_n$ given by

$$\alpha_i = \text{measure of } \sphericalangle A_{i+n-1}A_iA_{i+1},\ i = 1, \ldots, n$$

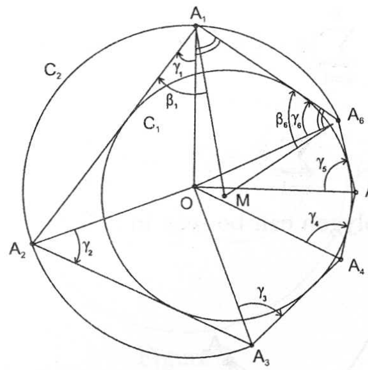

Figure 4

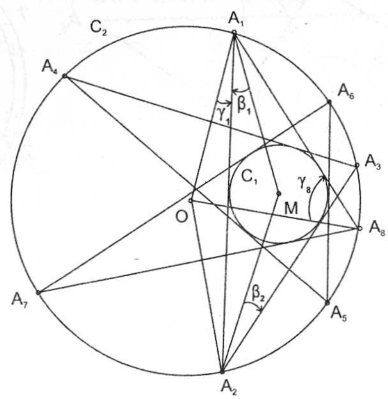

Figure 5

the following holds

$$\alpha_i = \gamma_i + \gamma_{i-1},\ i = 1, \ldots, n \tag{1.15}$$

where $\gamma_{i-1} = \gamma_n$ for $i = 1$.

So, for example, for bicentric hexagon $A_1 \ldots A_6$ shown in Figure 4 it holds

$$\begin{aligned}
\gamma_1 + \gamma_6 &= \alpha_1 = 2\beta_1, \\
\gamma_2 + \gamma_1 &= \alpha_2 = 2\beta_2, \\
\gamma_3 + \gamma_2 &= \alpha_3 = 2\beta_3, \\
\gamma_4 + \gamma_3 &= \alpha_4 = 2\beta_4, \\
\gamma_5 + \gamma_4 &= \alpha_5 = 2\beta_5, \\
\gamma_6 + \gamma_5 &= \alpha_6 = 2\beta_6.
\end{aligned}$$

In the case when O (center of C_2) is inside of C_1, then all γ_i have the same orientation. But if O is not inside of C_1, then all γ_i have not the same orientation.

See, for example, Figure 5, where

$$\begin{aligned} \alpha_1 &= \text{measure of } \sphericalangle A_8A_1A_2 < 0, \\ \gamma_1 &= \text{measure of } \sphericalangle OA_1A_2 > 0, \\ \gamma_8 &= \text{measure of } \sphericalangle OA_8A_1 < 0, \\ \alpha_1 &= \gamma_1 + \gamma_8 \text{ since } \sphericalangle OA_8A_1 = \sphericalangle A_8A_1O. \end{aligned}$$

In the case when n is even it holds

$$\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1} = \alpha_2 + \alpha_4 + \cdots + \alpha_n,$$

from which, since $\alpha_i = 2\beta_i$, we get

$$\beta_1 + \beta_3 + \beta_5 + \cdots + \beta_{n-1} = \beta_2 + \beta_4 + \beta_6 + \cdots + \beta_n \tag{1.16}$$

So, for example, for octagon shown in Figure 5 we have

$$\alpha_1 + \alpha_3 + \alpha_5 + \alpha_7 = (\gamma_1 + \gamma_8) + (\gamma_3 + \gamma_2) + (\gamma_5 + \gamma_4) + (\gamma_7 + \gamma_6) = \sum_{i=1}^{8} \gamma_i,$$

$$\alpha_2 + \alpha_4 + \alpha_6 + \alpha_8 = (\gamma_2 + \gamma_1) + (\gamma_4 + \gamma_3) + (\gamma_6 + \gamma_5) + (\gamma_8 + \gamma_7) = \sum_{i=1}^{8} \gamma_i.$$

Remark 2. As it is already said, $\beta_1, \ldots, \beta_n$ are either all positive or all negative. It only depends upon how vertices are numbered (clockwise or counterclockwise). So we can suppose that the angles $\beta_1, \ldots, \beta_n$ are positive.

2. Some properties and relations concerning bicentric hexagon

First about some relations concerning angles given by (1.7).

Let $A_1 \ldots A_6$ be a bicentric hexagon and let $\beta_1, \ldots, \beta_6$ be its angles given by (1.7). Then, for $k = 1$, according to (1.14) and (1.16), it holds

$$\beta_1 + \beta_3 + \beta_5 = \beta_2 + \beta_4 + \beta_6 = \pi.$$

From

$$\tan(\beta_1 + \beta_3 + \beta_5) = 0$$

or

$$\tan\beta_1 + \tan\beta_3 + \tan\beta_5 - \tan\beta_1 \tan\beta_3 \tan\beta_5 = 0,$$

using relations $\tan\beta_i = \frac{\rho}{t_i}$, $i = 1, 3, 5$ we get

$$t_1t_3 + t_3t_5 + t_5t_1 = \rho^2.$$

In the same way we get $t_2t_4 + t_4t_6 + t_6t_2 = \rho^2$.

Theorem 1. *Let t_m and t_M be given by*

$$t_m = \sqrt{(r-z)^2 - \rho^2}, \quad t_M = \sqrt{(r+z)^2 - \rho^2} \tag{2.1}$$

where r, ρ and z satisfy Fuss' relation for bicentric hexagon

$$3(r^2 - z^2)^4 = 4\rho^2(r^2 + z^2)(r^2 - z^2)^2 + 16r^2z^2\rho^4. \tag{2.2}$$

Then, for every t_1 such that $t_m \leq t_1 \leq t_M$ there are $t_2, \ldots, t_6$ such that

$$t_m \leq t_i \leq t_M, \ i = 2, \ldots, 6, \tag{2.3}$$

$$t_1t_3 + t_3t_5 + t_5t_1 = \rho^2, \tag{2.4}$$

$$t_2t_4 + t_4t_6 + t_6t_2 = \rho^2, \tag{2.5}$$

$$t_1t_4 = t_2t_5 = t_3t_6 = t_mt_M. \tag{2.6}$$

Proof. In order to prove this theorem, we have to examine the system of the above equations of $t_1, \ldots, t_6$ which satisfies the condition

$$t_m \leq t_i \leq t_M, \ i = 1, \ldots, 6.$$

For brevitiy in the following let t_mt_M be denoted by h, i.e.

$$h = t_mt_M. \tag{2.7}$$

Since from (2.6) we have

$$t_2 = \frac{h}{t_5}, \quad t_4 = \frac{h}{t_1}, \quad t_6 = \frac{h}{t_3}, \tag{2.8}$$

the relation (2.5) can be written as

$$t_1 + t_3 + t_5 = (\frac{\rho}{h})^2 t_1t_3t_5. \tag{2.9}$$

Using this relation and relation (2.4) we obtain

$$t_3 + t_5 = (\frac{\rho}{h})^2 t_1t_3t_5 - t_1,$$

$$t_1(t_3 + t_5) + t_3t_5 = \rho^2, \tag{2.10}$$

from which it follows that

$$t_3 + t_5 = a, \quad t_3t_5 = b \tag{2.11}$$

where

$$a = \frac{(\rho^4 - h^2)t_1}{\rho^2t_1^2 + h^2}, \quad b = \frac{h^2(\rho^2 + t_1^2)}{\rho^2t_1^2 + h^2}. \tag{2.12}$$

Hence we have the equations

$$t_3^2 - at_3 + b = 0, \quad t_5 = \frac{b}{t_3}. \tag{2.13}$$

The discriminant of the above square equation in t_3 is given by

$$D = (\rho^4 - h^2)^2 t_1^2 - 4h^2(\rho^2 + t_1^2)(\rho^2 t_1^2 + h^2). \tag{2.14}$$

We shall prove that $D \geq 0$ iff $t_m \leq t_1 \leq t_M$. For this purpose it is sufficient to prove that $D = 0$ for $t_1 = t_m$ and $t_1 = t_M$. It can be easily shown that

$$(\rho^4 - h^2)^2 t_m^2 - 4h^2(\rho^2 + t_m^2)(\rho^2 t_m^2 + h^2) = 0 \Longleftrightarrow (2.2)$$
$$(\rho^4 - h^2)^2 t_M^2 - 4h^2(\rho^2 + t_M^2)(\rho^2 t_M^2 + h^2) = 0 \Longleftrightarrow (2.2).$$

Thus, for every t_1 such that $t_m \leq t_1 \leq t_M$ there are t_3 and t_5 given by

$$t_3 = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b}, \quad t_5 = \frac{b}{t_3}. \tag{2.15}$$

Also, there are t_2, t_4, t_6 given by (2.8).

Obviously, such obtained $t_1, \ldots, t_6$ satisfy both (2.4) and (2.5).

Since t_1 can be replaced by any of $t_2, \ldots, t_6$, it is clear that $t_m \leq t_i \leq t_M$, $i = 1, \ldots, 6$.

Of course, instead of t_3 and t_5 given by (2.15), we can take $(t_3)_2$ and $(t_5)_2$ given by

$$(t_3)_2 = \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b}, \quad (t_5)_2 = \frac{b}{(t_3)_2}. \tag{2.16}$$

All essentially remain the same since we get the same tangents only with other notation. So instead of notations t_2 and t_3 we have notations t_6 and The difference is only in going round the incircle C_1. □

Corollary 1.1. It holds

$$h^2 = \rho^2 \frac{t_1 t_3 t_5}{t_1 + t_3 + t_5}. \tag{2.17}$$

Proof. It follows from (2.9). □

Corollary 1.2. It holds

$$h^2 = \rho^2 \frac{t_2 t_4 t_6}{t_2 + t_4 + t_6}. \tag{2.18}$$

Proof. From (2.17) ant (2.6) we obtain

$$\rho^2 \frac{t_2 t_4 t_6}{t_2 + t_4 + t_6} = \rho^2 (\frac{h}{t_5} \cdot \frac{h}{t_1} \cdot \frac{h}{t_3}) : (\frac{h}{t_5} + \frac{h}{t_1} + \frac{h}{t_3}) = \frac{\rho^2 h^2}{t_1 t_3 + t_3 t_5 + t_5 t_1} = h^2.$$

□

Corollary 1.3. The following inequality holds

$$\rho^2 \geq \frac{9 t_1 t_3 t_5}{t_1 + t_3 + t_5}. \tag{2.19}$$

Proof. Since $\frac{a}{b}+\frac{b}{a}\geq 2$ for every two positive numbers a and b, we can write

$$(t_1t_3+t_3t_5+t_5t_1)(t_1+t_3+t_5)=t_1t_3t_5(\frac{1}{t_1}+\frac{1}{t_3}+\frac{1}{t_5})(t_1+t_3+t_5)=$$
$$=t_1t_3t_5[3+(\frac{t_1}{t_5}+\frac{t_5}{t_1})+(\frac{t_3}{t_5}+\frac{t_5}{t_3})+(\frac{t_1}{t_3}+\frac{t_3}{t_1})]\geq 9t_1t_3t_5,$$

i.e.

$$\rho^2(t_1+t_3+t_5)\geq 9t_1t_3t_5.$$

□

Remark 3. It may be interesting that the inequality (2.19) follows from the well-known property concerning arithmetical and harmonical mean. Namely from

$$\frac{t_1+t_3+t_5}{3}\geq\frac{3}{\frac{1}{t_1}+\frac{1}{t_3}+\frac{1}{t_5}}$$

it follows

$$(t_1t_3+t_3t_5+t_5t_1)\frac{t_1+t_3+t_5}{t_1t_3t_5}\geq 9$$

or

$$\rho^2\geq 3\rho\sqrt{\frac{t_1t_3t_5}{t_1+t_3+t_5}}.$$

The equality holds only if $t_1=t_3=t_5$.

Corollary 1.4. It holds

$$\rho^2\geq 3h. \tag{2.20}$$

The equality $\rho^2=3h$ holds only if $t_1=t_2=\ldots=t_6$.

Proof. From (2.19) we obtain

$$\rho^2\geq\rho\sqrt{\frac{9t_1t_3t_5}{t_1+t_3+t_5}}\quad\text{or}\quad\rho^2\geq 3h.$$

From $t_1t_3+t_3t_5+t_5t_1=t_2t_4+t_4t_6+t_6t_2=\rho^2$, $t_1=t_3=t_5$, $t_2=t_4=t_6$ it follows $t_1=t_2$. □

Corollary 1.5. From

$$t_1t_3+t_3t_5+t_5t_1=\rho^2,$$
$$\frac{t_1t_3t_5}{t_1+t_3+t_5}=\frac{t_2t_4t_6}{t_2+t_4+t_6}$$

follows

$$t_2t_4+t_4t_6+t_6t_2=\rho^2,$$
$$t_2=\frac{h}{t_5},\quad t_4=\frac{h}{t_1},\quad t_6=\frac{h}{t_3},$$

where

$$h = \rho\sqrt{\frac{t_1t_3t_5}{t_1 + t_3 + t_5}}.$$

Proof. It is easy to see that

$$\frac{h}{t_5}\cdot\frac{h}{t_1} + \frac{h}{t_1}\cdot\frac{h}{t_3} + \frac{h}{t_3}\cdot\frac{h}{t_5} = \rho^2,$$

$$\frac{h}{t_5}\cdot\frac{h}{t_1}\cdot\frac{h}{t_3} : (\frac{h}{t_5} + \frac{h}{t_1} + \frac{h}{t_3}) = \frac{t_1t_3t_5}{t_1 + t_3 + t_5}.$$

□

The following theorem is in fact a corollary of Theorem 1.

Theorem 2. *Let $t_1, \dots, t_6$ have the same meaning described in Theorem 1. Then there is a bicentric hexagon $A_1 \dots A_6$ such that*

$$t_i + t_{i+1} = |A_iA_{i+1}|,\ i = 1, \dots, 6.$$

Proof. We have to prove that there are r and z such that

$$3(r^2 - z^2)^4 - 4\rho^2(r^2 + z^2)(r^2 - z^2)^2 = 16\rho^4r^2z^2, \tag{2.21}$$

$$h^2 = (r^2 - z^2)^2 - 2\rho^2(r^2 + z^2) + \rho^4, \tag{2.22}$$

where

$$\rho^2 = t_1t_3 + t_3t_5 + t_5t_1,$$

$$h^2 = \rho^2 \cdot \frac{t_1t_3t_5}{t_1 + t_3 + t_5}.$$

Let us remark that (2.21) is Fuss' relation for bicentric hexagon and that

$$(r^2 - z^2)^2 - 2\rho^2(r^2 + z^2) + \rho^4 = [(r - z)^2 - \rho^2][(r + z)^2 - \rho^2] = h^2.$$

It is easy to see that (2.21) and (2.22) are equivalent with

$$r^2 + z^2 = \frac{\rho^8 + 2h^2\rho^4 - 3h^4}{8h^2\rho^2}, \qquad r^2 - z^2 = \frac{\rho^4 - h^2}{2h} \tag{2.23}$$

since, by Corollary 1.4, holds $\rho^2 \geq 3h$, it follows

$$\rho^4 \geq 9h^2, \quad \rho^8 \geq 81h^4, \quad \rho^8 + 2h^2\rho^4 > 99h^4.$$

So, solving (2.21) and (2.22) for $r^2 + z^2$ and $r^2 - z^2$ we get (2.23). Reversely, from (2.23) follows (2.21) and (2.22).

Thus, not only ρ^2, but also r and z are completely determined by t_1, t_3, t_5 since ρ^2 and h are completely determined by t_1, t_3, t_5

It is easy to see that r, ρ, z in the above theorem are the same as in Theorem 1. □

In addition, as an appendix to the proof of Theorem 2, we shall prove the following assertion, where Figure 6 will be used. The circles C_1 and C_2 are such that r, ρ, z satisfy Fuss' releation (2.21).

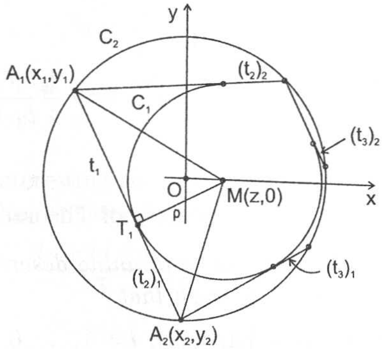

Figure 6

Let $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ be two different points on C_2 so that A_1A_2 is a tangent-line to C_1 and let t_1 and $(t_2)_1$ be given by

$$t_1 = |A_1T_1|,\ (t_2)_1 = |A_2T_1|$$

where T_1 is the tangential point of A_1A_2 and C_1. Then

$$t_1 = \sqrt{r^2 - \rho^2 + z^2 - 2zx_1}, \quad (t_2)_1 = \sqrt{r^2 - \rho^2 + z^2 - 2zx_2}$$

and it holds

$$(t_2)_1 = t_2,$$

where

$$t_2 = \frac{h}{t_5}$$

or

$$t_2 = \frac{ht_3}{b}$$

with t_3 and b given by (2.15) and (2.12).

Using computer it can be found that

$$(t_2)_1^2 = \frac{1}{(\rho^2+t_1^2)^2}\cdot\Big[-r^4\rho^2+2r^2z^2\rho^2-z^4\rho^2+2r^2\rho^4+2z^2\rho^4-\rho^6+r^4t_1^2$$
$$-2r^2z^2t_1^2+z^4t_1^2+2r^2\rho^2t_1^2+2z^2\rho^2t_1^2-2\rho^4t_1^2-\rho^2t_1^4$$
$$+2\sqrt{-(r-z)^2(r+z)^2\rho^2t_1^2((r-z)^2-\rho^2-t_1^2)((r+z-\rho)(r+z+\rho)-t_1^2)}\Big],$$

$$(t_2)_2^2 = \frac{1}{(\rho^2+t_1^2)^2}\cdot\Big[-r^4\rho^2+2r^2z^2\rho^2-z^4\rho^2+2r^2\rho^4+2z^2\rho^4-\rho^6+r^4t_1^2$$
$$-2r^2z^2t_1^2+z^4t_1^2+2r^2\rho^2t_1^2+2z^2\rho^2t_1^2-2\rho^4t_1^2-\rho^2t_1^4$$
$$-2\sqrt{-(r-z)^2(r+z)^2\rho^2t_1^2((r-z)^2-\rho^2-t_1^2)((r+z-\rho)(r+z+\rho)-t_1^2)}\Big]$$

and that holds

$$(t_2)_1^2-(\frac{h}{t_5})^2 = 256(r+z-\rho)^2(r-z+\rho)^2(-r+z+\rho)^2(r+z+\rho)^2$$
$$(-3(r^2-z^2)^4+4(r^2-z^2)^2(r^2+z^2)\rho^2+16r^2z^2\rho^4)^2$$
$$t_1^4(\rho^2+t_1^2)^2(r^4+(z^2-\rho^2)^2-2r^2(z^2+\rho^2)+\rho^2t_1^2)^2=0.$$

As can be easily seen the only factor which is equal to zero is

$$(-3(r^2-z^2)^4+4(r^2-z^2)^2(r^2+z^2)\rho^2+16r^2z^2\rho^4),$$

since holds Fuss' relation (2.21).

Concerning factor $(r^4+(z^2-\rho^2)^2-2r^2(z^2+\rho^2)+\rho^2t_1^2)^2$ it holds

$$r^4+(z^2-\rho^2)^2-2r^2(\rho^2+z^2)>0$$

since from $r^4-2r^2(z^2+\rho^2)+(z^2-\rho^2)^2=0$ follows

$$r=\rho-z \quad \text{or} \quad r=\rho+z,$$

which can not be.

Now we state the following corollary of the Theorem 2.

Corollary 2.1. Let $A_1\ldots A_6$ be a tangential hexagon such that

$$t_1t_3+t_3t_5+t_5t_1=\rho^2, \tag{2.24}$$

$$\frac{t_1t_3t_5}{t_1+t_3+t_5}=\frac{t_2t_4t_6}{t_2+t_4+t_6}, \tag{2.25}$$

where

$$t_i+t_{i+1}=|A_iA_{i+1}|,\ i=1,\ldots,6.$$

Then this hexagon is a bicentric one whose r and z are given by (2.23), where

$$h^2=\rho^2\cdot\frac{t_1t_3t_5}{t_1+t_3+t_5}. \tag{2.26}$$

Proof. See Corollary 1.5. □

Remark 4. In [11] it is shown that for a 1-tangential hexagon $A_1 \ldots A_6$ it holds $S_1^6\rho^4 - S_3^6\rho^2 + S_5^6 = 0$, where $t_i + t_{i+1} = |A_iA_{i+1}|$. (See Remark 1.) Here can be shown that from

$$S_1^6\rho^4 - S_3^6\rho^2 + S_5^6 = 0, \tag{2.27}$$

$$t_1t_3 + t_3t_5 + t_5t_1 = t_2t_4 + t_4t_6 + t_6t_2 = \rho^2, \tag{2.28}$$

$$t_1t_4 = t_2t_5 = t_3t_6 = h, \tag{2.29}$$

follows (2.26). The equation (2.27), using (2.28) and (2.29), in the end becomes

$$[h^2(t_1 + t_3 + t_5) - \rho^2 t_1t_3t_5][(t_1 + t_3 + t_5)\rho^2 - t_1t_3t_5] = 0. \tag{2.30}$$

Let us remark that $(t_1 + t_3 + t_5)^2 \neq t_1t_3t_5$ since ρ is the radius of the inscribed circle into hexagon $A_1 \ldots A_6$ and therefore it can not be the radius of the inscribed circle into a triangle whose sides have the lengths $t_1 + t_3$, $t_3 + t_5$, $t_5 + t_1$.

Example 1. Let $t_1 = 1$, $t_3 = 3$, $t_5 = 5$. Then

$$\rho^2 = t_1t_3 + t_3t_5 + t_5t_1 = 23.$$

$$h = \rho\sqrt{\frac{t_1t_3t_5}{t_1 + t_3 + t_5}} = 6.191391874,$$

$t_2 = \frac{h}{t_5} = 1.238278375$, $t_4 = \frac{h}{t_1} = 6.191391874$, $t_6 = \frac{h}{t_3} = 2.063797291$.

Using relations (2.23) we find that

$$r = 6.497111204, \quad z = 1.608585094.$$

For the angles $\beta_i = \arctan \frac{\rho}{t_i}$, $i = 1, \ldots, 6$, we have

$$\beta_1 = 78.22176785^\circ, \qquad \beta_2 = 75.52248781^\circ$$
$$\beta_3 = 57.97223989^\circ, \qquad \beta_4 = 37.76124391^\circ$$
$$\beta_5 = 43.80599227^\circ, \qquad \beta_6 = 66.71626828^\circ$$

$$\beta_1 + \beta_3 + \beta_5 = \beta_2 + \beta_4 + \beta_6 = 180^\circ.$$

Thus, a bicentric hexagon is completely determined by t_1, t_3, t_5.

Example 2. Here will be illustrated how using Theorem 1 can be relatively easily obtained $t_2, \ldots, t_6$ if t_1 is given.

Let $r = 4.696519101$, $\rho = 3$, $z = 1.675497439$. Then

$$t_m = 0.355769419, \quad t_M = 5.62161852, \quad t_mt_M = 2.$$

If $t_1 = 1$, then using relation (2.15) and (2.8), we get

$$t_3 = 5.34704211, \quad t_5 = 0.575372712$$
$$t_2 = 3.47600774, \quad t_4 = 2, \quad t_6 = 0.374038572.$$

Also, using the expression $\beta_i = \arctan \frac{\rho}{t_i}$, $i = 1, \ldots, 6$, we get

$$\beta_1 = 71.56505118^\circ, \qquad \beta_2 = 40.7961328^\circ, \qquad \beta_3 = 29.29191935^\circ$$
$$\beta_4 = 56.30993247^\circ, \qquad \beta_5 = 79.14302949^\circ, \qquad \beta_6 = 82.8939347^\circ$$

$$\beta_1 + \beta_3 + \beta_5 = \beta_2 + \beta_4 + \beta_6 = 180^\circ.$$

The corresponding hexagon is represented in Figure 7.

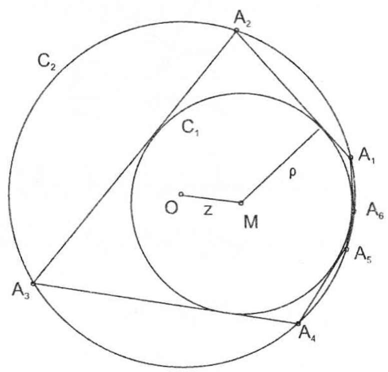

Figure 7

Let now be an other choice for t_1, say $t_1 = 5$. Then analogously as for $t_1 = 1$, we get

$$t_3 = 1.176380598, \quad t_5 = 0.504842108$$
$$t_2 = 3.961634674, \quad t_4 = 0.4, \quad t_6 = 1.700130046.$$

Also, for $\beta_i = \arctan \frac{\rho}{t_i}$, $i = 1, \ldots, 6$, we get

$$\beta_1 = 30.96375653^\circ, \qquad \beta_2 = 37.1353055^\circ, \qquad \beta_3 = 68.58852015^\circ$$
$$\beta_4 = 82.40535663^\circ, \qquad \beta_5 = 80.44772146^\circ, \qquad \beta_6 = 60.45933777^\circ$$

$$\beta_1 + \beta_3 + \beta_5 = \beta_2 + \beta_4 + \beta_6 = 180^\circ.$$

The corresponding hexagon is represented in Figure 8.

In connection with (2.15), let us remark that

$$\frac{a}{2} - \sqrt{\frac{a^2}{4} - b} = 0.504842108 = t_5.$$

Remark 5. It is easy to see that proving Theorem 1 and Theorem 2 we in fact give another proof of the Poncelet's closure theorem for bicentric hexagons, when conics are circles one inside of the other.

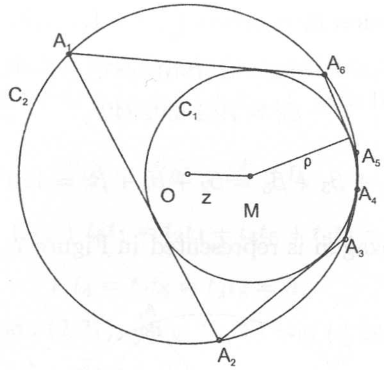

Figure 8

Theorem 3. *Let $A_1 \ldots A_6$ be a bicentric hexagon and let*

$$\rho^2 = t_1t_3 + t_3t_5 + t_5t_1, \quad h^2 = \rho^2 \cdot \frac{t_1t_3t_5}{t_1 + t_3 + t_5}$$

where $t_i + t_{i+1} = |A_iA_{i+1}|$, $i = 1, \ldots, 6$. Then

$$\sum_{i=1}^{6} t_it_{i+1} = \frac{\rho^4 - 3h^2}{h}, \tag{2.31}$$

$$\sum_{i=1}^{6} t_it_{i+1}t_{i+2}t_{i+3} = \rho^4 - 3h^2, \tag{2.32}$$

$$(t_1 + t_3 + t_5)(t_2 + t_4 + t_6) = \frac{\rho^4}{h}. \tag{2.33}$$

Proof. From $(t_1t_3 + t_3t_5 + t_5t_1)(t_2t_4 + t_4t_6 + t_6t_2) = \rho^4$, using equalities

$$t_1t_4 = t_2t_5 = t_3t_6 = h, \tag{2.34}$$

we get relation (2.32).

Now, using the relation (2.32) and relations (2.34), we get

$$\sum_{i=1}^{6} t_it_{i+1}t_{i+2}t_{i+3} = h(\sum_{i=1}^{6} t_it_{i+1}),$$

i.e.

$$\rho^4 - 3h^2 = h(\sum_{i=1}^{6} t_it_{i+1}),$$

which can be written as (2.31).

In order to prove (2.33), we can write

$$(t_1 + t_3 + t_5)(t_2 + t_4 + t_6) = \sum_{i=1}^{6} t_i t_{i+1} + 3h,$$

since $t_1t_4 + t_2t_5 + t_3t_6 = 3h$. The above equality, since holds (2.31), can be written as (2.33). □

Corollary 3.1. Let $A_1 \dots A_6$ and $B_1 \dots B_6$ be any two bicentric hexagons with the same r, ρ and z and let $u_i + u_{i+1} = |B_iB_{i+1}|$, $i = 1, \dots, 6$. Then

$$\sum_{i=1}^{6} t_i t_{i+1} = \sum_{i=1}^{6} u_i u_{i+1},$$

$$\sum_{i=1}^{6} t_i t_{i+1} t_{i+2} t_{i+3} = \sum_{i=1}^{6} u_i u_{i+1} u_{i+2} u_{i+3},$$

$$(t_1 + t_3 + t_5)(t_2 + t_4 + t_6) = (u_1 + u_3 + u_5)(u_2 + u_4 + u_6).$$

3. Some relations concerning bicentric octagons

First about some relations which will be used. Very important role in the following will play the Fuss' relation given by (1.5), here written as

$$[\rho^2(p^2 + q^2) - p^2q^2]^4 = 16p^4q^4\rho^4h^2,$$

where

$$h^2 = (p^2 - \rho^2)(q^2 - \rho^2).$$

This relation is equivalent to

$$[\rho^2(p^2 + q^2) - p^2q^2 - 2pq\rho\sqrt{h}][\rho^2(p^2 + q^2) - p^2q^2 + 2pq\rho\sqrt{h}] = 0,$$

where

$$\rho^2(p^2 + q^2) - p^2q^2 - 2pq\rho\sqrt{h} = 0 \tag{3.1}$$

is Fuss' relation for 1-bicentric octagon and

$$\rho^2(p^2 + q^2) - p^2q^2 + 2pq\rho\sqrt{h} = 0 \tag{3.2}$$

is Fuss' relation for 3-bicentric octagon.

From (3.1) we see that $\rho^2(p^2 + q^2) > p^2q^2$. Since $h^2 = (p^2 - \rho^2)(q^2 - \rho^2) = p^2q^2 - \rho^2(p^2 + q^2) + \rho^4$, we have the following inequality

$$\rho^2(p^2 + q^2) > \rho^2(p^2 + q^2) - \rho^4 + h^2,$$

from which follows that

$$\rho^2 > h.$$

In the same way we find that from (3.2) follows that

$$\rho^2 < h.$$

Now in short about some relations concerning the angles $\beta_1, \dots, \beta_8$.

Let $A_1 \dots A_8$ be a 1-bicentric octagon. Since in this case

$$\sum_{i=1}^{8} \alpha_i = 1080^\circ, \quad \sum_{i=1}^{8} \beta_i = 540^\circ$$

we have, according to (1.16), the following relatioms

$$\beta_1 + \beta_3 + \beta_5 + \beta_7 = \beta_2 + \beta_4 + \beta_6 + \beta_8 = 270^\circ. \tag{3.3}$$

Since

$$\cot(\beta_1 + \beta_3 + \beta_5 + \beta_7) = 0,$$

we have

$$C_4^4 - C_2^4 + 1 = 0, \tag{3.4}$$

where

$$C_j^4 = S_j(\cot\beta_1, \cot\beta_3, \cot\beta_5, \cot\beta_7), \ j = 2, 4.$$

Using relations $\frac{t_i}{\rho} = \cot\beta_i$, $i = 1, 3, 5, 7$, the equality (3.4) can be written as

$$\rho^4 - \rho^2(t_1t_3 + t_3t_5 + t_5t_7 + t_7t_1 + t_1t_5 + t_3t_7) + t_1t_3t_5t_7 = 0.$$

In the same way, starting from $\beta_2 + \beta_4 + \beta_6 + \beta_8 = 270^\circ$, we get

$$\rho^4 - \rho^2(t_2t_4 + t_4t_6 + t_6t_8 + t_8t_2 + t_2t_6 + t_4t_8) + t_2t_4t_6t_8 = 0.$$

In the case when $A_1 \dots A_8$ is 3-bicentric octagon, then

$$\sum_{i=1}^{8} \alpha_i = 360^\circ, \quad \sum_{i=1}^{8} \beta_i = 180^\circ.$$

Let us remark that

$$\sum_{i=!}^{8} \beta_i = (8 - 2k)90^\circ = 180^\circ \text{ for } k = 3.$$

According to (1.15) it holds

$$\beta_1 + \beta_3 + \beta_5 + \beta_7 = \beta_2 + \beta_4 + \beta_6 + \beta_8 = 90^\circ.$$

Since

$$\cot(\beta_1 + \beta_3 + \beta_5 + \beta_7) = \cot(\beta_2 + \beta_4 + \beta_6 + \beta_8) = 0,$$

we get the relations which have the same form as these obtained when (3.3) holds.

Now we can prove the following theorem.

Theorem 4. *Let t_m and t_M be given by*

$$t_m = \sqrt{(r-z)^2 - \rho^2}, \quad t_M = \sqrt{(r+z)^2 - \rho^2} \tag{3.5}$$

where r, ρ and z satisfy Fuss' relation for 1-bicentric octagon

$$\rho^2(p^2+q^2) - p^2q^2 = 2pq\rho\sqrt{h}, \tag{3.6}$$

where $p = r+z$, $q = r-z$, $h = t_m t_M$. Let t_1 and t_2 be any given lengths such that $t_1 = |A_1T_1|$, $t_2 = |A_2T_1|$, where A_1A_2 is a chord of C_2 which touches circle C_1 at point T_1. Then there are $t_3, \ldots, t_8$ so that

$$t_m \le t_i \le t_M, \; i = 3, \ldots, 8 \tag{3.7}$$

$$\rho^4 - \rho^2(t_1t_3 + t_3t_5 + t_5t_7 + t_7t_1 + t_1t_5 + t_3t_7) + t_1t_3t_5t_7 = 0, \tag{3.8}$$

$$\rho^4 - \rho^2(t_2t_4 + t_4t_6 + t_6t_8 + t_8t_2 + t_2t_6 + t_4t_8) + t_2t_4t_6t_8 = 0, \tag{3.9}$$

$$t_1t_5 = t_2t_6 = t_3t_7 = t_4t_8 = h. \tag{3.10}$$

Proof. In order to prove this theorem, we have to examine the system of the above equations in $t_3, \ldots, t_8$ which satisfies the conditions

$$t_m \le t_i \le t_M, i = 3, \ldots, 8. \tag{3.11}$$

First, using relations (3.10), let t_5, t_6, t_7, t_8 in (3.8) and (3.9) be replaced respectively by

$$\frac{h}{t_1}, \frac{h}{t_2}, \frac{h}{t_3}, \frac{h}{t_4}. \tag{3.12}$$

Then we have the relations which can be written as

$$\rho^2(h+t_1^2)t_3^2 - (\rho^4 - 2h\rho^2 + h^2)t_1t_3 + \rho^2(h+t_1^2)h = 0, \tag{3.13}$$

$$\rho^2(h+t_2^2)t_4^2 - (\rho^4 - 2h\rho^2 + h^2)t_2t_4 + \rho^2(h+t_2^2)h = 0. \tag{3.14}$$

From (3.13) we have

$$(t_3)_{1,2} = \frac{(\rho^2-h)^2t_1 \pm \sqrt{D}}{2\rho^2(h+t_1^2)}, \tag{3.15}$$

where

$$D = (\rho^2-h)^4t_1^2 - 4h\rho^4(h+t_1^2)^2. \tag{3.16}$$

It can be easily shown that $D = 0$ iff $t_1 = t_m$ or $t_1 = t_M$, i.e.

$$(\rho^2-h)^4t_m^2 - 4h\rho^4(h+t_m^2)^2 = 0 \Leftrightarrow (3.6)$$
$$(\rho^2-h)^4t_M^2 - 4h\rho^4(h+t_M^2)^2 = 0 \Leftrightarrow (3.6)$$

where (3.6) denotes Fuss' relation for 1-bicentric octagon.

Thus, t_1 is such that $t_m \le t_1 \le t_M$.

Of course, also holds $t_m \le t_3 \le t_M$, since from (3.13) we get

$$(t_1)_{1,2} = \frac{(\rho^2 - h)^2 t_3 \pm \sqrt{(\rho^2 - h)^4 t_3^2 - 4h\rho^4 (h + t_3^2)^2}}{2\rho^2 (h + t_3^2)}. \tag{3.17}$$

As can be seen from (3.17), it holds

$$(t_1)_1 \cdot (t_1)_2 = h. \tag{3.18}$$

Since by (3.10) holds $t_1 t_5 = h$, it follows that one of $(t_1)_1$ and $(t_1)_2$ must be t_5.

Also, from (3.15) it can be seen that

$$(t_3)_1 \cdot (t_3)_2 = h. \tag{3.19}$$

Since by (3.10) holds $t_3 t_7 = h$, it follows that one of $(t_3)_1$ and $(t_3)_2$ must be t_7.

It is easy to see that also must be $t_m \le t_5 \le t_M$, $t_m \le t_7 \le t_M$. So, from $t_1 t_5 = t_m t_M$ (since $h = t_m t_M$) it follows that can not be $t_5 > t_M$ since then

$$t_1 = t_m \cdot \frac{t_M}{t_5} < t_m.$$

In the same way, using (3.14) instead of (3.13), we find that analogously holds for t_2, t_4, t_6, t_8. □

Corollary 4.1. It holds

$$h = \frac{t_1 t_3 t_5 + t_3 t_5 t_7 + t_5 t_7 t_1 + t_7 t_1 t_3}{t_1 + t_3 + t_5 + t_7}, \tag{3.20}$$

$$h = \frac{t_2 t_4 t_6 + t_4 t_6 t_8 + t_6 t_8 t_2 + t_8 t_2 t_4}{t_2 + t_4 + t_6 + t_8}. \tag{3.21}$$

Proof. From (3.10) it follows

$$t_1 + t_3 + t_5 + t_7 = t_1 + t_3 + \frac{h}{t_1} + \frac{h}{t_3},$$

$$t_1 t_3 t_5 + t_3 t_5 t_7 + t_5 t_7 t_1 + t_7 t_1 t_3 = h(t_3 + \frac{h}{t_1} + \frac{h}{t_3} + t_1).$$

□

Corollary 4.2. Let (3.8) be written as

$$\rho^4 - a\rho^2 + b = 0, \tag{3.22}$$

where $a = t_1 t_3 + t_3 t_5 + t_5 t_7 + t_7 t_1 + t_1 t_5 + t_3 t_7$, $b = t_1 t_3 t_5 t_7$. If

$$(\rho^2)_{1,2} = \frac{a}{2} \pm \sqrt{(\frac{a}{2})^2 - b}, \tag{3.23}$$

then

$$(\rho^2)_1 > h, \tag{3.24}$$

$$(\rho^2)_2 < h. \tag{3.25}$$

(The proof is easy.)

Theorem 5. *Let $t_1, \ldots, t_8$ have the same meaning described in Theorem 4. Then there is a 1-bicentric octagon $A_1 \ldots A_8$ such that*

$$t_i + t_{i+1} = |A_i A_{i+1}|, \quad i = 1, \ldots 8. \tag{3.26}$$

Proof. First let us remark that Fuss' relation for 1-bicentric octagon is given by (3.6) and that in this relation we have

$$h = t_m t_M = \sqrt{(r-z)^2 - \rho^2}\sqrt{(r+z)^2 - \rho^2}. \tag{3.27}$$

Thus, here we have to prove that there are $p = r + z$, $q = r - z$ such that

$$(p^2 - \rho^2)(q^2 - \rho^2) = h^2, \tag{3.28}$$

$$\rho^2(p^2 + q^2) - p^2 q^2 = 2pq\rho\sqrt{h}. \tag{3.29}$$

It can be easily found that

$$p^2 q^2 = \frac{(\rho^4 - h^2)^2}{4\rho^2 h} \tag{3.30}$$

$$p^2 + q^2 = \frac{(\rho^4 - h^2)^2 + 4\rho^2 h(\rho^4 - h^2)}{4\rho^4 h}. \tag{3.31}$$

Since ρ^2 and h are completely determined by $t_1, \ldots, t_8$, the same holds for r and z.

It is easy to see that r, ρ, z in the above theorem are the same as in Theorem 4. □

Here let us remark that Theorem 5 is in some way analogous to the Theorem 2 and that we can give to it an appendix analogous to the given in Theorem 2. But for the brevity of the article we omit it.

Example 3. Let $\rho = 3$ and $h = 1$. Using (3.30) and (3.31) we get

$$r = 3.718489007, \quad z = 0.70272837.$$

Now we shall calculate the lenghts of two consecutive tangents and then to find all other using relations(3.15), (3.17) and (3.10). For this purpose we shall use the equations

$$(x - z)^2 + y^2 = \rho^2, \quad x^2 + y^2 = r^2.$$

Let us take, for example, $A_1(-2, 3.134830218)$ on the circle C_2. Then we have

$$t_1 = |A_1 T| = 2.851648846, \quad t_2 = |TA_2| = 2.990887753$$

where

$$T(-2.282513013, 0.29721017), \quad A_2(-2.578820413, -2.678963595).$$

Using relations (3.15) and (3.17) we find tha

$$t_3 = 1.592775033, \quad t_4 = 0.690680502.$$

Now, by (3.10) we get

$$t_5 = 0.350674313, \quad t_6 = 0.334348912$$
$$t_7 = 0.627835054, \quad t_8 = 1.447847445.$$

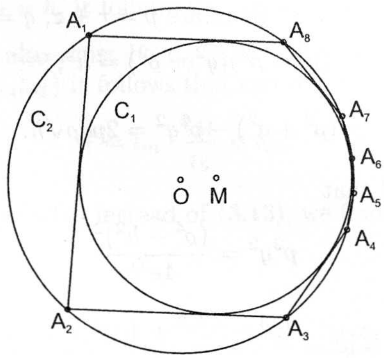

Figure 9

It can be checked that for the angles $\beta_i = \arctan \frac{\rho}{t_i}$, $i = 1, \ldots, 8$ it holds

$$\beta_1 = 46.452253301°, \qquad \beta_2 = 45.08714975°$$
$$\beta_3 = 62.03504957°, \qquad \beta_4 = 77.03489245°$$
$$\beta_5 = 83.33286979°, \qquad \beta_6 = 83.64064938°$$
$$\beta_7 = 78.17982764°, \qquad \beta_8 = 64.23730843°$$

$$\beta_1 + \beta_3 + \beta_5 + \beta_7 = \beta_2 + \beta_4 + \beta_6 + \beta_8 = 270°.$$

The corresponding 1-bicentric octagon is shown in Figure 9.

Before the following example, we shall consider in short 3-bicentric octagon.

As we have seen, there is a complete analogy between relations for a 1-bicentric octagon and these for 3-bicentric octagon. So, for example, for a 3-bicentric octagon hold the following relations

$$\rho^2(h + t_1^2)t_3^2 - (\rho^2 - h)^2 t_1 t_3 + \rho^2(h + t_1^2)h = 0, \tag{3.32}$$

$$\rho^2(h + t_2^2)t_4^2 - (\rho^2 - h)^2 t_2 t_4 + \rho^2(h + t_2^2)h = 0, \tag{3.33}$$

which have the same form as these given by (3.13) and (3.14) for a 1-bicentric octagon. Of course, here it holds $\rho^2 < h$, whereas in (3.13) and (3.14) holds $\rho^2 > h$. Therefore for a 3-bicentric octagon hold theorems analogous to the Theorem 4 and Theorem 5. Here is an example.

Example 4. Let $r = 3.718489007$, $z = 0.70272837$ be as in Exaqmple 3. Putting in Fuss' relation for bicentric octagon given by (1.5) instead of r and z the given values, we get the equation which can be written as

$$\rho^8 - 13.29936284\rho^6 + 392.4704253\rho^4 - 38447.315735\rho^2 + 5969.973647 = 0.$$

Its positive roots are

$$\rho_1 = 3, \quad \rho_2 = 1.378216239.$$

As we can see, ρ_1 is the same as ρ in Example 3. It is because $\rho_1 = 3$ is the positive root of the Fuss' relation for 1-bicentric octagon given by (3.1), whereas $\rho_2 = 1.378216239$ is the positive root of the Fuss' relation for 3-bicentric octagon given by (3.2).

In the case when $\rho = \rho_2$ we have

$$h = \sqrt{(r-z)^2 - \rho^2} \cdot \sqrt{(r+z)^2 - \rho^2} = 11.26858218.$$

If we take $A_1(-2, 3.134830218)$, as in Example 3, we find that

$$t_1 = |A_1T| = 3.902873425, \quad t_2 = |A_2T| = 3.391949986$$

where

$$T(-0.581194658, -0.501020765), \quad A_2(0.651875502, -3.660904099).$$

Using the equations (3.32) and (3.33) we find

$$t_3 = 2.844927483, \quad t_4 = 4.199807966.$$

Then we find

$$t_5 = \frac{h}{t_1} = 2.887252789, \quad t_6 = \frac{h}{t_2} = 3.32215458$$

$$t_7 = \frac{h}{t_3} = 3.960938283, \quad t_8 = \frac{h}{t_4} = 2.683118436.$$

It can be checked that

$$\beta_1 = 19.44958348^\circ, \qquad \beta_2 = 22.11287806^\circ$$
$$\beta_3 = 25.84772318^\circ, \qquad \beta_4 = 18.16785632^\circ$$
$$\beta_5 = 25.51725199^\circ, \qquad \beta_6 = 22.53141832^\circ$$
$$\beta_7 = 19.18544141^\circ, \qquad \beta_8 = 27.18784737^\circ$$

$$\beta_1 + \beta_3 + \beta_5 + \beta_7 = \beta_2 + \beta_4 + \beta_6 + \beta_8 = 90^\circ.$$

The corresponding 3-bicentric octagon is shown in Figure 10.

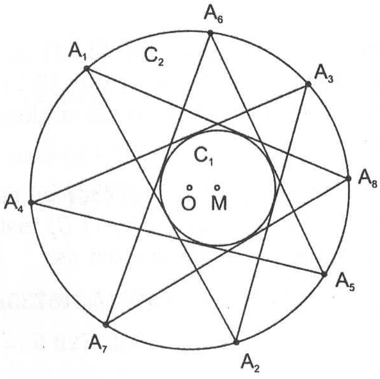

Figure 10

Remark 6. It can be easily shown that there is not a 2-bicentric octagon different from "double bicentric quadrilateral".

REFERENCES

[1] W. Barth, T. Bauer, Poncelét's Theorems, Expositiones Mathematicae 14 (1996) 125 - 144.

[2] H. J. M. Bos, C. Kers, F. Oort, D. W. Raven, Poncelét's closure theorem, Exposition. Math. 5 (1987) 289-364.

[3] H. J. M. Bos, The Closure theorem of Poncelét, Rend. Sem. Mat. Fis. Milano 54 (1984) 145-158.

[4] H. Dörrie, 100 Great Problems of Elementary Mathematics, Their History and Solution, Dover Publications, inc., 1965, originaly published in German under the title of Triumph der Mathematik.

[5] N. I. Fuss, De quadrilateris quibus circulum tam inscribere quam circumscribere licet, NAASP 1792 (Nova Acta), t.X (1797) 103-125 (14.VII.1794).

[6] N. I. Fuss, De polygonis simmetrice irregularibus calculo simul inscriptis et circumscriptis, NAASP 1792 (Nova Acta), t.XIII (1802) 166-189 (19.IV.1798).

[7] P.A. Griffiths, Harris, On Cayley's explicit solution of Poncelét's porism, L'Enseignement Math. 24 (1978) 31-40.

[8] A. Hrasko, Poncelét-type Problems, an Elementary Aproach, Elem. math. 55 (2000) 45-62.

[9] U. Herr, Über das Theorem von Poncelét, Staatsexamensarbeit vorgelegt dem Fachbereich Mathematik Johannes Gutenberg-Universität Mainz.

[10] J. V. Poncelét, Traité des propriétés projectives des figures, t.I, Paris, 1865, first ed. in 1822.

[11] M. Radić, Some relations and properties concerning tangential polygons, Mathematical Communications 4 (1999) 197-206.

[12] M. Radić, Some relations concerning triangles and bicentric quadrilaterals in connection with Poncelét's closure theorem, Math. Maced. 1 (2003) 35-58.

MIRKO RADIĆ, UNIVERSITY OF RIJEKA, FACULTY OF PHYLOSOPHY, DEPARTMENT OF MATHEMATICS, 51000 RIJEKA, OMLADINSKA 14, CROATIA