

Dokaz. Na osnovu 1. 2 Dfn imamo

$$(A) (B) [A \subseteq B. \equiv. (x) (x \in A. \rightarrow. x \in B)],$$

$$(A) (B) [B \subseteq A. \equiv. (x) (x \in B. \rightarrow. x \in A)].$$

Oдавде sleduje

$$(A) (B) [A \subseteq B. B \subseteq A : \equiv. (x) (x \in A. \equiv. x \in B)],$$

a zbog L 9. 3. 1 dobija se

$$(A) (B) [A \subseteq B. B \subseteq A : \equiv. A = B]$$

$$\text{L 9. 3. 6. } (A) (B) [A \subseteq B. \equiv. A \cap B = A].$$

Dokaz. Zbog 1. 2 Dfn imamo

$$A \subseteq B. \equiv. (x) [x \in A. \rightarrow. x \in B]$$

a zatim iz

$$x \in A. \rightarrow. x \in B : \equiv. : x \in A. \rightarrow : x \in A. x \in B,$$

zbog 1. 4 Dfn, sleduje $A \subseteq B. \equiv. A \subseteq A \cap B$, a zbog L 9. 3. 2

$$A \subseteq B. \equiv. A \cap B = A$$

što potvrđuje tačnost leme.

$$\text{L 9. 3 7. } (A) (B) (C) [A \subset B. B \subseteq C : \rightarrow. A \subset C]$$

Dokaz. Imamo najpre

$$(9. 3. 3) \quad A \subset B. \equiv. : A \subseteq B. A \neq B.$$

Dalje je (1. 2 Dfn)

$$A \subseteq B. \equiv. (x) [x \in A. \rightarrow. x \in B],$$

$$B \subseteq C. \equiv. (x) [x \in B. \rightarrow. x \in C].$$

Oдавde sleduje

$$A \subseteq B. B \subseteq C : \rightarrow. (x) [x \in A. \rightarrow. x \in C]$$

odnosno

$$A \subseteq B. B \subseteq C : \rightarrow. A \subseteq C.$$

Ako je $A = C$, iz relacija $A \subseteq B, B \subseteq C$ dobija se

$$A \subseteq B, B \subseteq A,$$

a na osnovu L 8. 3. 5 imali bi $A = B$, što je nemoguće zbog (9. 3. 3). Imamo dakle $A \neq C$ što sa relacijom $A \subseteq C$ daje $A \subset C$ (1. 2 Dfn).

Sada ćemo dokazati stav CCT 4. 1. 2 koji se može simbolički na sledeći način izraziti:

T 9. 3. 1. $(M) (N) \{(\exists A) [A \neq \Lambda. A \subseteq M. A \subseteq N].$

$(B) [B \neq \Lambda. B \subset M. B \subseteq N : \rightarrow. (\exists C) (B \subset C. C \subseteq M. C \subseteq N)] : \rightarrow M \subseteq N\}.$

Dokaz. Pretpostavimo da nije tako tj. da su uslovi

(9. 3. 4) $(\exists A) [A \neq \Lambda. A \subseteq M. A \subseteq N],$

(9. 3. 5) $(B) [B \neq \Lambda. B \subset M. B \subseteq N : \rightarrow. (\exists C) (B \subset C. C \subseteq M. C \subseteq N)]$

ispunjeni, ali da je ipak

(9. 3. 6) $\sim (M \subseteq N).$

Na osnovu B 2 i 1. 4 Dfn postoji klasa

(9. 3. 7) $D = M \cap N,$

a zbog L 9. 3. 2 imamo

(9. 3. 8) $D \subseteq M, D \subseteq N.$

Odavde, iz uslova (9. 3. 4), L 9. 3. 3 (9. 3. 7) sleduje

$$A \neq \Lambda, A \subseteq M \cap N = D,$$

odnosno zbog L 9. 3. 4

(9. 3. 9) $D \neq \Lambda.$

Takođe je $D \neq M$, jer bi inače iz $D = M$ sledovalo, s obzirom na (9. 3. 7), $M = M \cap N$, a zbog L 9. 3. 6 $M \subseteq N$, što protivreči pretpostavci (9. 3. 6). Dakle iz $D \neq M$ dobija se zbog (9. 3. 8) i (9. 3. 9)

$$D \neq \Lambda, D \subset M, D \subseteq N.$$

Međutim otuda i iz uslova (9. 3. 5) sleduje da postoji klasa E za koju je

(9. 3. 10) $D \subset E, E \subseteq M, E \subseteq N.$

Iz ovih relacija na osnovu L 9. 3. 3 imamo $E \subseteq M \cap N = D$, odnosno $E \subseteq D$, što sa (9. 3. 10) daje $D \subset E, E \subseteq D$, odakle (L 9. 3. 7) je $D \subset D$. Najzad imamo (1. 2 Dfn)

$$D \subset D. \equiv : D \subseteq D. D \neq D,$$

što je nemoguće. Dakle pretpostavka (9. 3. 6) je neodrživa, tj. teorema je dokazana. Kao što se vidi ona je posledica aksioma A 3 i B 2. S obzirom na aksiomu A 1 stav važi i za množine.

L I T E R A T U R A

1. G. LORIA, Storia delle Matematiche dall'alba civiltà al secolo XIX, Seconda edizione riveduta e aggiornata, Editore Ulrico Hoepli, Milano, 1950.
2. M. CANTOR, Vorlesungen über Geschichte der Mathematik, III Band, Zweite Auflage, Teubner, Leipzig, 1913.
3. F. HAUSDORFF, Grundzüge der Mengenlehre, Verlag von Veit und Comp., Leipzig, 1914.
4. G. KUREPA, Ensembles ordonnés et ramifiés, Publications mathématiques de l'Université de Belgrade, T. IV, 1935.
5. Ђ. КУРЕПА, О принципима индукције. Зборник радова Математичког института (Српска академија наука), бр. 1.
6. G. BIRKHOFF, Lattice Theory, Published by the American Mathematical Society, New York City, 1948.
7. H. KNESER, Eine direkte Ableitung des Zornschen Lemmas aus dem Auswahlaxiom. Mathematische Zeitschrift, B. 53, H. 2, 1950.
8. E. WITT, Beweisstudien zum Satz von M. Zorn. Mathematische Nachrichten, B. 4, 1951.
9. T. INAGAKI, Sur deux théorèmes concernant un ensemble partiellement ordonné. Mathematical Journal of Okayama University, V. 1, Nos. 1—2, 1952.
10. M. ZORN, A remark on method in transfinite algebra. Bulletin of American Mathematical Society V. 41, 1935.
11. O. FRINK, A proof of the maximal chain theorem. American journal of Mathematics, V. 74, N. 3, 1952.
12. Ђ. КУРЕПА, Teorija skupova, „Školska knjiga“, Zagreb, 1951.
13. H. LEBESGUE, Leçons sur l'intégration et la recherche des fonctions primitives, Gauthier—Villars et Cie, Paris, 1928.
14. A. KHINTCHINE, Das stetigkeitsaxiom des Linearkontinuums als Induktionsprinzip betrachtet. Fundamenta Mathematicae, T. 4, 1923.
15. А. ХИНЧИН, Простейший линейный континуум. Успехи математических наук, Т. 4, в. 2 (30), 1949.
16. М. ПОПАДИЋ, Jedno karakteristično svojstvo konačnih množina. Годишен зборник на Филозофскиот факултет на Универзитетот во Скопје, Природно-математички оддел, Кн. 4, № 6, 1951.
17. A. TARSKI, Sur les ensembles finis. Fundamenta Mathematicae, T. 6, 1925.
18. E. ZERMELO, Sur les ensembles finis et le principe de l'induction complète. Acta mathematica, T. 32, 1909.

19. Ђ. KUREPA, Dokaz principa totalne indukcije. Rad Jugoslovenske akademije znanosti i umjetnosti, Knj. 277, 1950.
20. G. KUREPA, Some principles of induktion. Rukopis predavanja održanog u Detroit-u, Mich., Wayne University, 30 oktobra 1950 i u Lafayette-u, Ind., Purdue University, 15 novembra 1950.
21. К. ГЕДЕЛЬ, Совместимость аксиомы выбора и обобщенной континуум-гипотезы с аксиомами теории множеств. Перевод с английского под редакцией А. А. Маркова. Успехи математических наук, Т. 3, в. 1 (23), 1948.
22. W. SIERPIŃSKI, Algebre des ensembles. Monografie matematyczne, T. XXIII. Warszawa — Wrocław 1951.
23. М ПОПАДИЋ, О уређеним множењима са коначним ланцима. Годинен зборник на Филозофскиот факултет на Универзитетот во Скопје, Природно-математички оддел, Кн. 5, № 1, 1952.

REGISTAR STVARI

Brojevi se odnose na strane na kojima je data definicija
ili objašnjenje navedenih pojmova

Aksioma izbora 8, 12, 25, 44

Atomizator 6

Baza, podmnožine date množine (gornja, donja, prava) 18
potsistema 14

Elementi uređene množine, ivični (donji i gornji) 7
krajnji (početni i završni) 7
neuporediv 6
unutrašnji 7

Granica, donja, vidi minoranta
gornja, vidi majoranta

Indukcija

formulacija principa indukcije 3, 31, 32, 34, 37, 41
princip totalne indukcije 3, 41, 44
princip transfinitne indukcije 3, 34, 41, 44

Induktor 41

Infimum 7

Interval (početni, završni) 17
redukovan 37

Inverzija uređene množine 17

Komad (početni, završni) 18
elementarni (početni, završni) 17
elementarni (u užem i širem smislu) 17

Lebesgue—Hinčin-ovo svojstvo 3, 16, 22
uopšteno 37

Lakuna (spoljašnja, unutrašnja) 18

Lanac 7

maksimalan 7

Majoranta 7

Minoranta 7

Množina, alakunarna (u užem i širem smislu) 19

dobro uređena 19
dvostruko dobro uređena 19
dvostruko razvrstana 19
ekstremalna (u širem i užem smislu) 19
infimalna (u širem i užem smislu) 19
izomorfna 17
koekstenzivna 18
koinicijalna 18
konfinalna 18
neuređena 7
otvorena 25

- partitivna 4, 9
- poludobro uređena 19
- polurazvrstana 19
- potpuno uređena 7
- razvrstana 19
- slična, vidi izomorfna
- supremalna (u širem i užem smislu) 19
- uređena 6
- u sebi gusta 25
- Podmnožina, prava 7
- Porodica množina, monotona 7
- Potsistem vezan za množinu 14
- Prekrivač množine 10
- Presek množine 18
- Prevoj 7
- Preslikavanje (donje, gornje) 30
- Relacija reda 6
 - dualna (ili inverzna) 17
- Rez množine, vidi presek množine
- Sistem množina, apsolutno zatvoren u odnosu na operator U 15
 - induktivan 4
 - induktivan u odnosu na induktor 42
 - karakterističan 17
 - neprekidan 15
 - potpuno aditivan 15
 - potencijalan 14
 - potencijalan u odnosu na induktor 42
- Spreg, induktivan 25, 32, 34
 - induktivan u odnosu na induktor 42
 - karakterističan 30
 - potencijalan 25, 31, 32, 34
 - potencijalan u odnosu na induktor 42
 - redukovani 26
 - saglasni 26
- Supremum 7

REGISTAR IMENA

Brojevi se odnose na strane u radu

Bernoulli, J. 3	Maurolico, F. 3
Dedekind, R. 25, 44	Pascal, B. 3
Euklid 3	Poincaré, H. 3
Gödel, K. 5, 44, 46	Sierpiński, W. 15
Hausdorff, F. 9	Tarski, A. 45
Hilbert, D. 3	Zermelo, E. 3, 25
Kurepa, Đ. 3, 21, 44, 45	Zorn, M. 8

S A D R Ź A J

PREDGOVOR	2
1. UVOD. Formulacija osnovnog problema. Induktivni sistemi	3
2. POMOĆNI STAVOVI. Definicije i pomoćni stavovi	6
3. OSNOVNI STAV. Kriterijum induktivnosti datog sistema. Potencijalni sistemi	10
4. INDUKTIVNI SISTEMI. Neke opšte vrste induktivnih sistema. Stavovi o induktivnim sistemima	15
5. NEKE PRIMENE. Primene teorije induktivnih sistema na uređene množine. Karakteristični sistemi	17
6. NOVE FORMULACIJE PRINCIPA INDUKCIJE. Uopštenje osnovnog problema i njegovo rešenje. Induktivni i potencijalni spregovi	25
7. JOŠ NEKE FORMULACIJE PRINCIPA INDUKCIJE. Nova formulacija osnovnog problema	30
8. OPŠTA FORMULACIJA PRINCIPA INDUKCIJE. Opšta formulacija osnovnog problema i njegovo rešenje. Neke primene	34
9. DODATAK. Jedna definicija konačnih množina. Dokaz jednog stava u okviru GÖDEL-ovog sistema aksioma za teoriju množina	44
LITERATURA	50
Registar imena	52
Registar stvari	53

Milan S. Popadić

ON INDUCTIVE SYSTEMS

(Summary)

1. Introduction

1.1. The object of this paper is, in fact, the content of the doctor dissertation which was defended at the University of Zagreb on 9th of February 1954. The article itself is in connection with certain works of Professor G. Kurepa. Professor G. Kurepa has suggested to me to try to „exhaust“ a plane by the elementary initial sections, what would represent a generalization of a his theorem [4, 23]¹⁾. The difficulties, which I have met during the work, conducted me to the introduction of the notion of the „inductive system“, as well to certain results connected with it, and, at last, it is found the solution of the mentioned problem.

On this occasion I consider obliging to thank sincerely to Professor G. Kurepa, for his suggestions and aid during my working. Thanks are particularly due for conversations about certain problems.

1.2. There are many kinds of „inductive conclusions“ in Mathematics. The best known are the statements: the principle of complete induction, the principle of transfinite induction and, according to Kurepa's term, Lebesgue—Khinchine property [4, 22]. The principle of complete induction is as follows (we suppose that the theory of natural numbers is founded in the frame of set theory):

T 1.2.1. For the set M of all natural numbers and a set N , from the conditions:

1. $1 \in M$;
 2. if $m \in M$, $m \in N$, it is also $m+1 \in N$
- it follows $M \subseteq N$.

The principle of transfinite induction is:

T 1.2.2. For a well-ordered set M and any set N , from the conditions:

1. the first element²⁾ of M belongs to the set N ;
 2. if $(-, x)_M \subseteq N$ for $x \in M$, then also $(-, x]_M \subseteq N$ ³⁾
- it follows $M \subseteq N$.

At last, as every simply ordered set without interior holes⁴⁾ possesses Lebesgue—Khinchine property, we have the following theorem [4, 28—24]:

T 1.2.3. For a simply ordered set M without interior holes and any set N , from the conditions:

1. there exists an elementary initial section A of M , satisfying the relation $\Delta \subset A \subseteq N$; ⁵⁾
 2. for every elementary initial section B of M , satisfying the relations $\Delta \subset B \subset M$ and $B \subseteq N$, there is an elementary initial section C of M , satisfying $B \subset C \subseteq N$
- it follows $M \subseteq N$.

1) Black printed numbers in brackets are referred to the ordinal numbers in the list of reference at the end of original text; other numbers represent the pages of the quoted paper.

2) See D 3.1.5.

3) See D 5.1.2.

4) See D 5.1.4.

5) Δ — empty set.

In all three cases one gets the result $M \subseteq N$, based on two hypothesis: the first hypothesis guaranties the existence at least one subset common to the sets M and N ; the second one guaranties the enlargement of the system of subsets of the same kind.

In this paper we have presented a general formulation of the inductive conclusion, which comprises, like special cases, all till now the known formulations. We determine also the necessary and sufficient conditions under which this proposition is true. Finally it is given the application of obtained results, as some theorems which we have considered of interest.

Let us mention that in this summary the basic problem is formally differently expressed than in original text. The notion of a system of transformations [in the original text (for instance in T. 8. 6. 1) denoted by F] is replaced by the notion of a binary relation (see T 3. 3. 1). Accordingly, we have made still some changes of formal character too.

1. 3. The numeration of statements and relations is positional, and, for certain terms, we use the following abbreviations: definition — D, proposition — Pr, lemma — L, theorem — T, consequence — C (with the mark of the statement from which it follows), problem — P.

2. The formulation of the basic problem

2. 1. Before we give a precise formulation of the main problem, we shall get some explanations and definitions.

D 2. 1. 1. Let $A_i, i=1, 2, \dots, n$, be n sets. By the *Cartesain product* $A_1 \times A_2 \times \dots \times A_n$ is meant the set of all n -tuples (x_1, x_2, \dots, x_n) , where $x_i \in A_i$. If $A_i = A, i=1, 2, \dots, n$, then this product we denote by A^n .

In a couple (x, y) , we shall call x and y the *left* and *right component*, respectively.

D 2. 1. 2. By a *binary relation* defined on a set A , we mean every non-empty aggregate¹⁾ $\varphi \subseteq A^2$. The symbols $(x, y) \in \varphi$ and $x \varphi y$ are equivalent.

D 2. 1. 3. By the *left (right)²⁾ domain* of a binary relation φ , defined on a set A , we mean the set of all left (right) components of its elements. The left domain we denote by $D\varphi$ the right one — by $W\varphi$. If $S \subseteq A$, $D_S\varphi (W_S\varphi)$ represents the set of the left (right) components of all elements of φ , whose right (left) components belong to S . For $x \in A$ we have $D_x\varphi = D_{\{x\}}\varphi$ and $W_x\varphi = W_{\{x\}}\varphi$.

D 2. 1. 4. Let φ be a binary relation. The set φ^{-1} of all couples (y, x) , for which $(x, y) \in \varphi$, is called the *converse* of the relation φ .

It is clear that $(\varphi^{-1})^{-1} = \varphi$.

D 2. 1. 5. A binary relation φ is *one-valued* if $W_x\varphi$ is an unit set for every $x \in D\varphi$. If φ^{-1} is one-valued too, then φ is a *one-to-one* relation.

It is obviously that φ and φ^{-1} are simultaneously one-to-one relations.

In the following $P(M)$ denotes the *partitive set* of M , i. e., the class of all subsets of M .

D 2. 1. 6. Let $S(A)$ be a system of subsets of a set A , and let B be any set. The subsystem $S_B(A) = S(A) \cap P(B)$ is said to be *bound* for B ; B is a *basis* of the subsystem.

2. 2. We now may state a *general scheme of the principle of induction*:

¹⁾ The terms „system“, „aggregate“, „class“ we understand as synonymous with „set“.

²⁾ By reading terms in brackets instead those before brackets, one obtains a dual statement.

Pr 2. 2. 1. Let M and N be any sets and let $S(M) \subseteq P(M)$. From the hypothesis that there is a couple — inductor — $(S_1(M), \varphi)$, where $S_1(M) \subseteq P(M)$ and φ is a binary relation with $D\varphi = S(M)$ and $W\varphi \subseteq P(M)$, and from the conditions:

1. $S_N(M) \cap S_1(M) \neq \Delta, \{\Delta\}$;

2. there exists $\psi \in P(\varphi)$ with $D\psi = S(M) \setminus \{\Delta, M\}$ ¹⁾ and with $W_{S_N(M) \setminus \{\Delta, M\}} \psi \subseteq P_N(M)$

— it follows $M \subseteq N$.

It is easily to show that the propositions expressed by the theorems 1. 2. 1, 1. 2. 2 and 1. 2. 3, are special cases of this proposition.

The following definition is of basic importance:

D 2. 2. 1. Let M and N be any sets. A system $S(M) \subseteq P(M)$ is *inductive for M with respect to an inductor $(S_1(M), \varphi)$* , where $S_1(M) \subseteq P(M)$ and φ is a binary relation with $D\varphi = S(M)$ and $W\varphi \subseteq P(M)$, if from the conditions:

1. $S_N(M) \cap S_1(M) \neq \Delta, \{\Delta\}$;

2. there exists $\psi \in P(\varphi)$ with $D\psi = S(M) \setminus \{\Delta, M\}$ and $W_{S_N(M) \setminus \{\Delta, M\}} \psi \subseteq P_N(M)$

— it follows $M \subseteq N$.

This statement may be proposed as follows: a system $S(M)$ is inductive for M with respect to an inductor $(S_1(M), \varphi)$, if the proposition 2. 2. 1 is true.

Finally, using the notation of the previous definition, the main problem of this paper is as follows:

P 2. 2. 1. Determine the necessary and sufficient conditions in order that $S(M)$ is inductive for M with respect to the inductor $(S_1(M), \varphi)$.

We shall solve this problem in the next section.

3. The fundamental theorem

3. 1. Previously we give still some definitions and statements necessary for our purpose.

If p is a statement, $\sim p$ is the *negation* of p .

D 3. 1. 1. A binary relation φ , defined on a set A , is an *ordering relation* if, for all $x, y, z \in A^2$, the following conditions are satisfied:

1. $(x, x) \in \varphi$;

2. if $(x, y), (y, z) \in \varphi$, then $(x, z) \in \varphi$;

3. if $(x, y), (y, x) \in \varphi$, then $x = y$.

φ being an ordering relation, $\varphi^* = \varphi \setminus \bigcup_{x \in D\varphi} \{(x, x)\}$ is a *strictly ordering relation*.

If we do not use some special symbols, we shall, instead $x \varphi y$, $x \varphi^* y$, usually write $x \leq y$, $x < y$, respectively. — The set A is *ordered (strictly ordered) by the relation φ (φ^*)* or, shorter, it is an *ordered (strictly ordered) set*. We say too: A is *with the ordering φ* .

We suppose that the properties of an ordering relation are known. A set with the ordering φ^{-1} is the *dual* of the same set with the ordering φ .

D 3. 1. 2. A set A is *simply ordered* by a relation φ , if $A^2 = \varphi \cup \varphi^{-1}$. If $\varphi = \varphi^{-1}$, A is an *unordered set*.

D 3. 1. 3. A simply ordered (unordered) subset of an ordered set A is a *chain (anti-chain)* of A .

1) $A \setminus B$ represents the set of all elements of A not in B .

2) If ρ is a binary relation, then the expression $x_1, x_2, \dots, x_n, \rho$ is equivalent to the system of expressions $x_1 \rho a, x_2 \rho a, \dots, x_n \rho a$.

D. 3. 1. 4. Let A be a set ordered by a relation φ . An element $a \in A$, for which $D_a \varphi = \{a\}$ ($W_a \varphi = \{a\}$), one calls the *minimal (maximal) element*. If $W_a \varphi = A$ ($D_a \varphi = A$), a is the *first (last) element*; first and last elements are called *extrem elements*.

D 3. 1. 5. Let A be a set with an ordering φ , and let $B \subseteq A$. An element $x \in A$ is a *lower (upper) bound* or *minorant (majorant)* of B , if $B \subseteq D_x \varphi$ ($B \subseteq W_x \varphi$); one says also that B is *bounded below (above)* in A . A *minimal (maximal) element* of all majorants (minorants) of B is called a *least upper (greatest lower) bound* or *supremum (infimum)* of B in A . If a supremum (infimum) is uniquely determined, then it is denoted by $\sup_A B$ ($\inf_A B$).

3. 2. To simplify the formulation and proof of the fundamental statement, we quote some definitions and theorems.

D. 3. 2. 1. A system S of sets *covers* an aggregate A , if $A \subseteq \bigcup X$. One says also „ S is a *covering* of A “.

T 3. 2. 1. In order that a system $S(M) \subseteq P(M)$ is *inductive* for the set M with respect to an inductor $(S_1(M), \varphi)$, where $S_1(M) \subseteq P(M)$ and φ is a binary relation with $D\varphi = S(M)$, $W\varphi \subseteq P(M)$, it is necessary that, for each $\psi \in P(\varphi)$ with $D\psi = S(M) \setminus \{\Delta, M\}$, $W\psi$ covers M .

Proof. Suppose that $S(M)$ is an inductive system, but yet there is $\psi \in P(\varphi)$ with $D\psi = S(M) \setminus \{\Delta, M\}$ so that the system $W\psi$ do not cover M . Since $W\psi \subseteq P(M)$, we have $\bigcup X = N \subseteq M$ and, because of the made hypothesis

(3. 2. 1) $N \subset M$

However, if M and N satisfy the conditions 1 and 2 of the definition 2. 2. 1, it follows $M \subseteq N$. This contradicts (3. 2. 1), whence one infers the truth of the theorem.

The following lemma is evident:

L 3. 2. 1. Let $S(A) \subseteq P(A)$, where A is a set. Then $\sup_{P(A)} S(A) = \bigcup X$.
 $X \in S(A)$

Thus the supremum of $S(A)$ is uniquely determined.

D 3. 2. 2 Let $S(M) \subseteq P(M)$, where M is a set, and let $(S_1(M), \varphi)$ be a couple, where $S_1(M) \subseteq P(M)$ and φ is a binary relation with $D\varphi = S(M)$, $W\varphi \subseteq P(M)$. The system $S(M)$ is *potential* for M with respect to the inductor $(S_1(M), \varphi)$ if the relation $\sim (W_{S_E(M)} \setminus \{\Delta\} \psi \in P(E))$ is satisfied for every $\psi \in P(\varphi)$ with $D\psi = S(M) \setminus \{\Delta, M\}$, provided $E \subset \sup_{P(M)} W\psi$ and $S_E(M) \cap S_1(M) \neq \Delta, \{\Delta\}$.

Remark that we might, with respect to L 3. 2. 1, propose the definition 3. 2. 2 without the notion of supremum.

T 3. 2. 2. In order that a system $S(M) \subseteq P(M)$ is *inductive* for the set M with respect to an inductor $(S_1(M), \varphi)$, where $S_1(M) \subseteq P(M)$ and φ is a binary relation with $D\varphi = S(M)$, $W\varphi \subseteq P(M)$, it is necessary that $S(M)$ is *potential* for M with respect to the given inductor.

Proof. Let $S(M)$ be an inductive system for M with respect to the inductor $(S_1(M), \varphi)$, but let us assume that it is not potential. Thus there is an element $\psi \in P(\varphi)$ with $D\psi = S(M) \setminus \{\Delta, M\}$, and there is a set $N \subset \sup_{P(M)} W\psi$, or, because of $W\psi \subseteq P(M)$ and L 3. 2. 1,

(3. 2. 2) $N \subset M$,

such that nevertheless, though it is

(3. 2. 3) $S_N(M) \cap S_1(M) \neq \Delta, \{\Delta\}$,

the following relation holds

$$(3.2.4) \quad W_{S_N(M) \setminus \{\Delta\}} \psi \subseteq P(N).$$

From the relation (3.2.3) it is obviously that condition 1 (D 2.2.1) is satisfied. Since, because of (8.2.2), $P_N(M) = P(M) \cap P(N) = P(N)$, it follows that the condition 2 (D 2.2.1) is satisfied too. Then, for the given system is inductive, we have $M \subseteq N$. This contradicts (8.2.2), and the theorem is proved.

3.3. We are, now, going to give a complete solution of proposed problem. The fundamental theorem of our exposition asserts:

T 3.3.1. *In order that a system $S(M) \subseteq P(M)$ is inductive for the set M with respect to an inductor $(S_1(M), \varphi)$, where $S_1(M) \subseteq P(M)$ and φ is a binary relation with $D\varphi = S(M)$, $W_\varphi \subseteq P(M)$, it is necessary and sufficient that $S(M)$ is potential for M with respect to the given inductor, and that the system W_ψ covers M for each $\psi \in P(\varphi)$ with $D\psi = S(M) \setminus \{\Delta, M\}$.*

Proof. The necessity of this conditions is established by T 3.2.1 and T 3.2.2. We shall now prove that they are sufficient too. Let $S(M)$ be potential for M with respect to the inductor $(S_1(M), \varphi)$, and assume that, for each $\psi \in P(\varphi)$ with $D\psi = S(M) \setminus \{\Delta, M\}$, the system W_ψ covers M . Thus we have (L 3.2.1)

$$(3.3.1) \quad \sup_{P(M)} W_\psi = M.$$

Suppose, also, the conditions 1 and 2 of D 2.2.1 are satisfied, but yet let

$$(3.3.2) \quad \sim (M \subseteq N),$$

i. e., the given system is not inductive. Hence we obtain

$$(3.3.3) \quad M \cap N = E \subset M, E \subset N$$

and, according to the condition 1 (D 2.2.1), $S_N(M) \cup S_1(M) \neq \Delta, \{\Delta\}$.

Since

$$(3.3.4) \quad S_N(M) = S(M) \cap P(N) = S(M) \cap P(M \cap N) = S_E(M),$$

it is also

$$(3.3.5) \quad S_E(M) \cap S_1(M) \neq \Delta, \{\Delta\}$$

By 2 (D 2.2.1) there is $\psi \in P(\varphi)$ with

$$(3.3.6) \quad D\psi = S(M) \setminus \{\Delta, M\}$$

and $W_{S_N(M) \setminus \{\Delta, M\}} \psi \subseteq P_N(M)$ or, because of (3.3.4) and $P_N(M) = P(E)$,

$$(3.3.7) \quad W_{S_E(M) \setminus \{\Delta, M\}} \psi \subseteq P(E).$$

However from (3.3.1), (3.3.3), (3.3.5), (3.3.6), and since $S(M)$ is potential, it follows $\sim (W_{S_E(M) \setminus \{\Delta\}} \psi \subseteq P(E))$ or $\sim (W_{S_E(M) \setminus \{\Delta, M\}} \psi \subseteq P(E))$, which is contrary to (3.3.7). Thus our assumption is not true, the theorem is proved.

3. SOME SPECIAL CASES

4.1. In what follows we shall quote one special case of Pr 2.2.1 and some general inductive system.

Pr 4.1.1. Let M and N be any sets. and let $S(M) \subseteq P(M)$. From the conditions:

1. $S_N(M) \neq \Delta, \{\Delta\}$;

2. for each set $B \in S_N(M) \setminus \{\Delta, M\}$ there is an aggregate $C \in S_N(M)$, satisfying $B \subset C$
— it follows $M \subseteq N$.

It is clear that Pr. 2. 2. 1 involves this proposition as a special case. Indeed, here $S_1(M) = S(M)$, and φ is a strict inclusion relation such that, for all $X, Y \in S(M)$, the relations $X \varphi Y$ and $X \subset Y$ are equivalent. Accordingly, we introduce the definition:

D 4. 1. 1. A system $S(M) \subseteq P(M)$ is *simply inductive (potential)* for M , if it is inductive (potential) for M with respect to the inductor $(S(M), \varphi)$ being a strict inclusion relation such that, for all $X, Y \in S(M)$, the relations $X \varphi Y$ and $X \subset Y$ are equivalent.

We have now the statement:

T 4. 1. 1. *In order that a system $S(M) \subseteq P(M)$ is simply potential for M , it is necessary and sufficient that, for each $E \subset \sup_{P(M)} S(M)$, the system $S_E(M) \neq \Delta$ has at least one maximal element.*

Proof. The condition is necessary. Let the system $S(M)$ be simply potential for M , i. e., potential for M with respect to the inductor $(S(M), \varphi)$, where $X \varphi Y$ is equivalent to $X \subset Y$ for all $X, Y \in S(M)$. Let

$$(4. 1. 1.) \quad E \subset \sup_{P(M)} S(M),$$

and suppose $S_E(M) \neq \Delta$ has no maximal elements. Then

$$(4. 1. 2.) \quad S_E(M) \neq \Delta, \{\Delta\},$$

and, for each $X \in S_E(M)$ there exists $Y \in S_E(M)$ such that $X \subset Y$.

Let us put

$$(4. 1. 3.) \quad A = S_E(M) \setminus \{\Delta\}, \quad B = S(M) \setminus \{\{\Delta\}, A\}$$

and $\psi = (A^2 \cup B^2) \cap \varphi$. Hence it follows $\psi \in P(\varphi)$ and, because of (4. 1. 3), $D\psi = S(M) \setminus \{\Delta\}$. Since $S(M) = D\psi$ and $\sim (M \in D\varphi)$, it is

$$(4. 1. 4.) \quad D\psi = S(M) \setminus \{\Delta, M\}.$$

By definition of ψ , we have

$$(4. 1. 5.) \quad W_{S_E(M) \setminus \{\Delta\}} \psi \subseteq S_E(M).$$

However, as by definition of φ

$$(4. 1. 6.) \quad \sup_{P(M)} W\varphi = \sup_{P(M)} S(M),$$

because $S(M)$ is potential, and since the relations (4. 1. 1.), (4. 1. 2.), (4. 1. 4) are satisfied, it follows $\sim (W_{S_E(M) \setminus \{\Delta\}} \psi \subseteq S_E(M))$. But this contradicts (4. 1. 5), and so the theorem is true.

The condition is sufficient. Let $E \subset \sup_{P(M)} S(M)$ or, because of (4. 1. 6), $E \subset \sup_{P(M)} W\varphi$, and let $S_E(M) \neq \Delta, \{\Delta\}$ have at least one maximal element C . Since $S_E(M) \setminus \{\Delta\} \subseteq S(M) \setminus \{\Delta, M\} = D\varphi \setminus \{\Delta, M\}$, then $C \in D\varphi \setminus \{\Delta, M\}$. Accordingly, we have $C \in D\psi$ for every $\psi \in P(\varphi)$ with $D\psi = S(M) \setminus \{\Delta, M\}$. Since C is a maximal element of $S_E(M)$, then $\sim (W_C \psi \subseteq S_E(M))$ and at last, because of $W_C \psi \subseteq W_{S_E(M) \setminus \{\Delta\}} \psi$, $\sim (W_{S_E(M) \setminus \{\Delta\}} \psi \subseteq S_E(M))$ too. This shows that the system $S(M)$ is really simply potential.

4. 2. In order to cite some very general simply inductive and simply potential systems, the following definitions are necessary.

D 4. 2. 1. A system $S(A) \subseteq P(A)$, where A is any set, is *supremal* if, for every chain F of $S(A)$, the relation $X' = \sup_{P(A)} F \neq \sup_{P(A)} S(A)$ implies $X' \in S(A)$.

D 4. 2. 2. A system S of sets is *absolutely closed with respect to the operator \cup* , if $\cup X \in S$ for each $F \subseteq S$.

$X \in F$

The following statements are evident.

T 4. 2. 1. Every *supremal system* $S(M) \subseteq P(M)$ is *simply potential for the set M* , and, as far $S(M)$, covers M , *simply inductive too*.

CT 4. 2. 1. 1. Every *finite system* $S(M) \subseteq P(M)$ is *simply potential for the set M* , and, as far $S(M)$ covers M , *simply inductive too*.

T 4. 2. 2. Every system $S(M) \subseteq P(M)$ *absolutely closed with respect to the operator \cup* is *simply potential for the set M* and, as far $S(M)$ covers M , *inductive too*.

CT 4. 2. 2. 1. The *partitive set* $P(M)$ of a set M is *simply inductive for M* [5, 110—111].

This is also a consequence of T 4. 2. 1.

4. 3. To simplify formulations of some statements, which will be later quoted, we introduce this definition:

D 4. 3. 1. Let C and C' be any classes of sets and let $M \in C$. A system $S(M) \subseteq P(M)$ is *characteristic for M* in the frame of C , if the propositions:

1. $M \in C'$;

2. $S(M)$ is a *simply inductive system for M*

— are equivalent.

5. Some applications

5. 1. We shall cite some applications of obtained results, in the main, to ordered sets. For that reason, we quote some definitions and statements.

D 5. 5. 1. Let A and B be two sets ordered by relations φ and ψ , respectively. A one-to-one relation ρ , with $D\rho = A$ and $W\rho = B$, is an *isomorphism* of A to B , if the relations $x\varphi y$ and $x'\psi y'$, x, x', y, y' satisfying $x\rho x', y\rho y'$, are equivalent. A is *isomorphic* to B , and one writes $A \approx B$.

The *isomorphism* is an *equivalence relation*.

D 5. 1. 2. Let A be a set ordered by a relation φ , and let $a, b \in A$. The aggregate $D_b\varphi = (-, b]_A$ ($W_a\varphi = [a, -)_A$) is an *initial (final) segment* of A ; $D_b\varphi^* = (-, b)_A$ ($W_a\varphi^* = (a, -)_A$) is an *initial (final) interval* of A . Initial (final) segments and intervals are called *elementary initial (final) sections* of A ; an elementary initial (final) section is denoted by $(-, b]_A$ ($[a, -)_A$). The non-empty intersection of $(-, b]_A$ and $[a, -)_A$ is an *elementary section* which we denote by $|a, b|_A$. All cited kinds of sets are called *elementary sections* in the larger sense. Especially, there are elementary sections: $(a, b)_A$ (*interval*), $[a, b)_A$, $(a, b]_A$, $[a, b]_A$ (*segment*). A set $B \subseteq A$ is an *initial (final) section* of A , if $D_B\varphi = B$ ($W_B\varphi = B$). Initial and final sections, as its nonempty intersections, are called *section* of A . Elementary sections are sections of A too.

D 5. 1. 3. The *cut* of an ordered set A is a couple (A_1, A_2) , where A_1 is an initial section of A , and A_2 is a final one, satisfying the relations $A_1 \cap A_2 = \Delta$, $A_1 \cup A_2 = A$.

D 5. 1. 4. By a *hole* of a non-empty simply ordered set A is meant every cut (A_1, A_2) of A , if there is not $\sup_A A_1$. If $A_1, A_2 \neq \Delta$, then the hole is *interior*, otherwise *exterior*.

A_1 and A_2 being non-empty sets, $\sup_A A_1$ and $\inf_A A_2$ exist only simultaneously.

D 5. 1. 5. Let A be a set ordered by a relation φ , and let $B, C \subseteq A$. B is *confinal* (coinitial) with C if $D_B \varphi = D_C \varphi$ ($W_B \varphi = W_C \varphi$). If B is confinal and cointial with C , then it is *coextensive* with C too.

Confinal, cointial and coextensive relations are sorts of the equivalence relation.

The following definitions determined some kinds of sets.

D 5. 1. 6. An ordered set A is called *well-ordered* if every non-void subset of A has a first element.

D 5. 1. 7. An ordered set A is called *ranged* if every nonvoid subset of A has at least one minimal element.

D 5. 1. 8. A simply ordered set A is called *semi-well-ordered* if every non-void subset of A , bounded below, has an first element.

D 5. 1. 9. An ordered set, whose each non-empty chain is semi-well-ordered, one calls a *semi-ranged* set.

D 5. 1. 10. By a *double-well-ordered* set is meant an ordered set if every its non-void subset has extrem elements.

D 5. 1. 11. By a *double-ranged* set is meant an ordered set if every its non-empty subset has minimal and maxime elements.

5. 2. It is easy to verify the following lemmas.

L. 5. 2. 1. If F is a system of initial (final) sections of an ordered set A , then the set $\cup X$ is also an initial (final) section of A .

$S \in F$

L 5. 2. 2. If F is a chain of the system of all sections of an ordered set A , then the set $\cup X$ is also a section of A .

$X \in F$

5. 3. At first, we cite a general theorem:

T 5. 3. 1. In order that a system $S(M)$ of initial (final) sections of an ordered set M is simply potential, it is necessary and sufficient that every non-void subsystem of $S(M)$, bound for an initial (final) section $D \subseteq \sup_{P(M)} S(M)$ of M , has at least one maximal element.

The necessity is evident, and it is easily to prove the sufficiency.

— As an immediate consequence of this theorem and of L 5. 2. 1, we have the statement formulated and proved by G. Kurepa [5, 11]:

T. 5. 3. 2. The system $S(M)$ of all initial (final) sections of an ordered set M is simply inductive for M .

Note that, in this case, $S(M)$ covers M .

T 5. 3. 3. The system of all sections of an ordered set M is simply inductive for M

It follows from T 4. 2. 1, L 3. 2. 2 and the fact that $S(M)$ covers M .

G. Kurepa has formulated Lebesgue-Khinchine property for simply ordered sets [4, 23—25; 13, 112¹]; 14, 164—166; 18, 186—191²]:

D 5. 3. 1. A simply ordered set A has *Lebesgue-Khinchine property* if the system of all initial (final) sections of A is simply inductive for A .

In the same paper the author proves this theorem:

T 5. 3. 4. In order that a simply ordered set has *Lebesgue-Khinchine property*, it is necessary and sufficient that it has no interior holes.

In connections with D 4. 3. 1, we have:

CT 5. 3. 4. 1. In the frame of the class of simply ordered sets, the system of all elementary initial (final) sections of an aggregate is characteristic for sets having no interior holes.

It is easy to establish the following theorems.

T 5. 3. 5. In the frame of the class of simply ordered sets, the system of all elementary sections of an aggregate is characteristic for sets having no holes.

¹) Here is used this property for proving a theorem.

²) In this paper the author has formulated this kind of inductive conclusion and given some applications.

T 5. 3. 6. In the frame of the class of ordered sets, the system $S(M)$ of all initial (final) segments of a set M is characteristic for M , if every chain, not being confinal (coinital) with M , has a last (first) element.

Note that the system $S(M)$ (ordered by inclusion relation) is isomorphic with M . The relation $\rho = U\{(x, (-, x]_M)\}$ is an isomorphism.

$x \in M$

CT 5. 3. 6. 1. In the frame of the class of simply ordered sets, the system of all initial (final) segments of a set is characteristic for duals of semi-well-ordered sets (for semi-well-ordered sets).

T 5. 3. 7. In the frame of the class of simply ordered sets, the system of all segments of a set is characteristic for double-ranged sets [23].¹⁾

CT 5. 3. 7. 1. In the frame of the class of simply ordered sets, the system of all segments of a set is characteristic for double-well-ordered sets [15]*.

In connection with this fact we have:

D 5. 3. 2. A set A is finite if it may be simply ordered such that the system of all its segments is simply inductive for A .

The equivalence of this and Dedekind's definition [12, 51] follows by CT 5. 3. 7. 1 and from a proof of E. Zermelo [18, 188], which assumes the axiom of choice.

5. 4. Supposing that the concepts of the open and dense-in-itself set are known, we shall cite still two theorems.

T 5. 4. 1. Every system $S(M)$ of open subsets of a set M is simply potential for M , and, if $S(M)$ covers M , simply inductive too.

T 5. 4. 2. Every system $S(M)$ of dense-in-itself subsets of a set M is simply potential for M , and, if $S(M)$ covers M , simply inductive too.

Both theorems follow from the fact that this systems are absolutely closed with respect to the operator U [12, 298, 307].

5. 5. We shall now expose some applications of general theorem.

T. 5. 5. 1. Let M be a ranged set, and let $R_0 M$ be the aggregate of all minimal elements of M . The system $U\{R_0 M U(-, x)_M\}$ is inductive

$X \in M$

for M with respect to the inductor $(R_0 M, \varphi)$, where $\varphi = U\{(A_x, B_x)\}$ $A_x = R_0 M U(-, x)_M$, $B_x = R_0 M U(-, x]_M$.

The proof may be found in [20]. If M is an ordered set, one obtains a special theorem from which it follows, also, the exactness of the principle of transfinite induction.

To generalize the definition 5. 3. 1 and the theorem 5. 3. 4, we introduce some new notions.

D 5. 5. 1. Let A be a simply ordered set. The symbol \bar{A}^n represents the set A^n ordered in the following manner: the relation $(x_1, x_2, \dots, x_n) \leq (y_1, y_2, \dots, y_n)$, where $x_i, y_i \in A$, $i=1, 2, \dots, n$, is equivalent to the system of relations $x_i \leq y_i$, $i=1, 2, \dots, n$.

D. 5. 5. 2. Let A be a simply ordered set, and let $u = (u_1, u_2, \dots, u_n)$, $v = (v_1, v_2, \dots, v_n)$ be elements of \bar{A}^n . The expression $u <^n v$ is equivalent to the system of symbols $u_i < v_i$, $i=1, 2, \dots, n$; $u \leq^n v$ signifies that $u <^n v$ or $u = v$.

D 5. 5. 3. Let A be a simply ordered set, and let $u \in A^n$. A set $(-, u)_{\bar{A}^n}^{(n)}$ of all elements $z \in \bar{A}^n$, satisfying $z <^n u$, is called a reduced initial interval of A^n . The symbol $(-, u)_{\bar{A}^n}'$ will denote either $(-, u)_{\bar{A}^n}$ or $(-, u)_{\bar{A}^n}^{(n)}$. Similarly we define the reduced final interval.

The following definition is a generalization of D 5. 3. 1.

¹⁾ In this paper is exposed a direct proof.

D 5. 5. 4. A simply ordered set M possesses *generalized Lebesgue-Khinchine property* if, for M and any set N , from conditions:

1. there exists an element $z \in \bar{M}^n$ satisfying the relation $\Delta \subset (-, z|_{\bar{M}^n} \subset N$;

2. for every element $u \in \bar{M}^n$, satisfying $\Delta \subset (-, u|_{\bar{M}^n} \subset \bar{M}^n, (-, u|_{\bar{M}^n} \subset N$, there exist an element $v \in M^n$, and the set $(-, v|_{M^n}$, which satisfy relations $u \leq^n v$ and $(-, v|_{M^n} \subset (-, v|_{M^n} \subset N$

— it follows $\bar{M}^n \subset N$.

This may be proposed as follows:

D 5. 5. 5. A simply ordered set M possesses *generalized Lebesgue-Khinchine property* if the system $S(\bar{M}^n)$ of all initial sections $(-, z|_{\bar{M}^n}$, $z \in \bar{M}^n$, is inductive for \bar{M}^n with respect to the inductor $(S(\bar{M}^n), \varphi)$, where $\varphi = \cap \{((-, u|_{\bar{M}^n}, (-, v|_{\bar{M}^n})\}, u, v$ satisfying $u \leq^n v$ and $(-, u|_{\bar{M}^n} \subset (-, v|_{\bar{M}^n}$. $u, v \in \bar{M}^n$

A generalization of the theorem 5. 3. 4 is as follows:

T 5. 5. 2. In order that a simply ordered set M possesses *generalized Lebesgue-Khinchine property*, it is necessary and sufficient that M has no interior holes.

The proof may be derived directly or from T 3. 3. 1, but in both cases depends on the lemma:

L 5. 5. 1. Let F be a chain of the system of all initial sections $(-, z|_{\bar{M}^n}$, $z \in \bar{M}^n$, where M is a simply ordered set. If F is bounded above in the mentioned system, then there is an initial section $(-, u|_{\bar{M}^n} \subset \bigcup_{X \in F} X$, $u \in M^n$, such that $\sim ((-, v|_{\bar{M}^n} \subset \bigcup_{X \in F} X$ for each $v \in \bar{M}^n$ satisfying $u \leq^n v$ and $(-, u|_{\bar{M}^n} \subset (-, v|_{\bar{M}^n}$.

5. 6. We are going to expose theorems I. 2. 1 and I. 2. 2, using our terminology. Then the principle of complete induction is as follows:

T 5. 6. 1. Let N be the set of all natural numbers. The system $S(N) = \bigcup \{ \{n\} \}$ is inductive for N with respect to the inductor $(\{ \{n\} \}, \varphi)$, where $\varphi = \bigcup \{ \{n\}, \{n+1\} \}$, $n \in N$

The proof of this theorem is simple, if one knows that is a well-ordered set. Note that G. Kurepa has proved this statement, not depending on the axiom of choice [19, 238–248]. In his paper the natural numbers are defined as cardinal numbers of finite sets.

The principle of transfinite induction is as follows:

T 5. 6. 2. The system $S(M) = \bigcup \{ (-, x)_M \}$, M being a well-ordered set, is inductive for M with respect to the inductor $(R_0 M, \varphi)$, where $R_0 M$ is the set containing only the first element of M , and $\varphi = \bigcup \{ ((-, x)_M, (-, x)_M) \}$, $x \in M$

6. Appendix

6. 1. Among diverse definitions of the finite set we shall accept that of A. Tarski [17, 46]. G. Kurepa has formulated this definition as follows [12, 51]:

D 6. 1. 1. A set A is finite if the system $P(A)$ is \bar{r} -ranged.

In the same paper A. Tarski has shown, depending on the axiom of choice, that this definition is equivalent to that of Dedekind.

We have now the following theorem:

T 6.1.1. *In order that a set A is finite, it is necessary and sufficient that every system $S(A) \subseteq P(A)$, which covers A , is simply inductive for A .*

The proof depends on some properties of partitive sets, and on the fact that $P(A)$, in this case, is a double-ranged set.

6.2. At last, accepting Gödel's system of axioms for set theory [21, 91–108], one proves that the theorem CT 4.2.2.1 follows from the axioms $A1, A3, B2$. Symbolically this statement may be expressed as follows:

T 6.2.1. $(M)(N) \{(\exists A)(A \neq \Delta, A \subseteq M, A \subseteq N). (B)[B \neq \Delta, B \subseteq M, B \subseteq N: \rightarrow (\exists C)(B \subseteq C \subseteq M, C \subseteq N): \rightarrow M \subseteq N]\}$.

Here we use for the implication the symbol \rightarrow instead \supset . Also the intersection of sets A and B we denote by $A \cup B$. The proof is indirect and it follows from the mentioned axioms and lemmas L 9.3.2, L 9.3.3, L 9.3.4, L 9.3.7, L 9.3.10, quoted in the original text.

I N D E X

- | | |
|---|---|
| Absolutely closed system 16 | Initial segment 61 |
| Anti-chain 57 | Interval 61 |
| Axiom of choice 64 | Interior hole 61 |
| Basis of a bound system 56 | Kurepa, G. 55, 64 |
| Bound system 56 | Last element 58 |
| Bounded above 58 | Least upper bound, see supremum |
| Bounded below 58 | Lebesgue – Khinchine property 55, 62 |
| Binary relation 56 | Left component of a couple 56 |
| Cartesian product 56 | Left domain of a binary relation 56 |
| Chain 57 | Majorant 58 |
| Characteristic system 61 | Maximal element 58 |
| Coextensive 62 | Minimal element 58 |
| Coinitial 62 | Minorant 58 |
| Confinal 62 | One-to-one relation 56 |
| Converse of a binary relation 56 | One-valued relation 56 |
| Covering 58 | Ordered set 57 |
| Cut 61 | Ordering relation 57 |
| Dedekind, R. 64 | Partitive set 56 |
| Double-ranged set 62 | Potential 58 |
| Double-well-ordered set 62 | Principle of complete induct. 55, 64 |
| Dual of an ordered set 57 | Principle of transfinite induct. 55, 64 |
| Elementary final section 61 | Ranged set 62 |
| Elementary initial section 61 | Reduced final interval 63 |
| Elementary section 61 | Reduced initial interval 63 |
| Exterior hole 61 | Right component of a couple 56 |
| Extrem element 58 | Right domain of a binary relation 56 |
| Final interval 61 | Section 61 |
| Final section 61 | Segment 61 |
| Final segment 61 | Semi-ranged set 62 |
| First element 58 | Semi-well-ordered set 62 |
| General scheme of the principle of induction 56 | Simply inductive system 60 |
| Generalized Lebesgue – Khinchine property 64 | Simply ordered set 57 |
| Gödel, K. 65 | Simply potential set 60 |
| Greatest lower bound, see infimum | Strictly ordered set 57 |
| Hole 61 | Strictly ordering relation 57 |
| Inductive system 57 | Supremum 58 |
| Inductor 57 | Supremal system 61 |
| Infimum 58 | Tarski, A. 64 |
| Initial interval 61 | Unordered set 57 |
| Initial section 61 | Well-ordered set 62 |
| | Zermelo, E. 68 |