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ON A LAME'S POLYNOMIALS

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Abstract. This paper examines the Lame differential equation in algebraic form. For this class of differential equations necessary and sufficient conditions are obtained for existence of the polynomial solution. The algebraic equation is obtained which roots are the appropriate parameter values for which the differential equation has a polynomial solution which can be interpreted as eigenvalues in their Sturm-Liouville problem.

1. Introduction

In the theory of partial differential equations and the Calculus of Variations classical result for solving the internal bounday value of Dirichlet's problem for the Laplace's partial differential equation in a sphere is known. With the transformation in spherical coordinates and applying Fourier's method of separation of variables, differential equations whose solutions are classical orthogonal Legendre's polynomials are obtained which are custom functions for appropriate Sturm-Liouville problem. Therefore, solutions of the Laplace's partial differential equation are obtained in the form of homogeneous polynomials of appropriate degree and are called spherical harmonic functions.

This approach of solving the same problem, but for ellipsoid was used by Lame with the introduction of confocal coordinates in Laplase's partial differential equation. Using the elliptic functions and using the Fourier's method of separation of the variables, a differential equation was obtained of the following kind:

$$-\frac{d^2y}{du^2} + [a\wp(u) + b]y = 0,$$

where $\wp(u)$ is the Weierstrass' function of the parameters $e_1, e_2, e_3, e_1+e_2+e_3 = 0$, and where a and b are parameters to be determined by the conditions of the problem.

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The requirement of Dirichlet's problem of seeking a solution homogeneous polynomial of degree n ie harmonious analytic function in the relevant field, we get a parameter a that can be of type $a = -n(n+1)$, n-natural number.

The same condition is obtained by a different approach in Lame's classical theory of differential equation in algebraic form of the Fuchsian class four regular singularities

$$\varphi(x)y" + \frac{1}{2}\varphi'(x)y' + (ax+b)y = 0,$$

where $\varphi(x) = 4(x-e_1)(x-e_2)(x-e_3)$, to satisfy the conditions this equation has general solution univocal analytical function.

The relationship between the two types is given by $\wp(u) = x$.

It shows that this conditions is necessary and sufficient.

The Lame's equation is a special case of Shredinger's equation and Hill's equation. This paper examines Lame's differential equation in algebraic form

$$2\varphi(x)y" + \varphi'(x)y' - 2[n(n+1)x - \lambda]y = 0, \qquad (1')$$

where n-natural number and λ-parameter.

2. Main result

In [2] necessary and sufficient conditions are obtained for the of class differential equations with polynomial coefficients of the following kind

$$(\alpha_3x^3 + \alpha_2x^2 + \alpha_1x + \alpha_0)y" + (\beta_2x^2 + \beta_1x + \beta_0)y' + (\gamma_1x + \gamma_0)y = 0, \qquad (1)$$

to have a polynomial solution.

Equation (1) has a polynomial solution of degree m if and only if there is a natural number m which satisfies the conditions

$$\alpha_3m^2 + (\beta_2 - \alpha_3)m + \gamma_1 = 0$$
$$\Delta_m = 0, \qquad (2)$$

where

$$\Delta_m = \begin{vmatrix} Q_0 & R_0 & S_0 & 0 & ... & 0 & 0 & 0 & 0 \\ P_1 & Q_1 & R_1 & S_1 & 0 & ... & 0 & 0 & 0 \\ 0 & P_2 & Q_2 & R_2 & S_2 & 0 & ... & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & ... & 0 & P_{m-1} & Q_{m-1} & R_{m-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_m & Q_m \end{vmatrix}, \qquad (3)$$

$$P_k = (k-1)(k-2)\alpha_3 + (k-1)\beta_2 + \gamma_1,$$
$$Q_k = k(k-1)\alpha_2 + k\beta_1 + \gamma_0,$$
$$R_k = k(k+1)\alpha_1 + (k+1)\beta_0,$$
$$S_k = (k+1)(k+2)\alpha_0,$$
$$k = 0, 1, 2, ..., m.$$

The m is the smallest natural number if the characteristic quadratic equation

$$\alpha_3 t^2 + (\beta_2 - \alpha_3)t + \gamma_1 = 0$$

has two roots of natural numbers.

Lame equation (1') in algebraic form

$$4(x^3 + E_1 x - E_2)y" + 2(3x^2 + E_1)y' - [n(n+1)x - \lambda]y = 0, \tag{4}$$

where

$$E_1 = e_1e_2 + e_1e_3 + e_2e_3, E_2 = e_1e_2e_3,$$

is a special case of equation (1) with

$$\alpha_3 = 4, \alpha_2 = 0, \alpha_1 = 4E_1, \alpha_0 = -4E_2,$$
$$\beta_2 = 6, \beta_1 = 0, \beta_0 = 2E_1, \gamma_1 = -n(n+1), \gamma_0 = \lambda.$$

In accordance with the conditions (2) Lame's equation (4) has a polynomial solution of degree m called the Lame' polynomial or Lame' function of the first kind if and only if satisfies the conditions

$$4m^2 + 2m - n(n+1) = 0,$$
$$\Delta_m = 0, \tag{5}$$

where in the determinant (3) is substituted

$$P_k = 2(k-1)(2k-1) - n(n+1),$$
$$Q_k = \lambda,$$
$$R_k = 2(k+1)(2k+1)E_1,$$
$$S_k = -4(k+1)(k+2)E_2,$$
$$k = 0, 1, 2, ..., m.$$

Theorem. *Lame's differential equation(4) has a polynomial solution if and only if there is a natural number m for which satisfies the conditions*

$$n = 2m,$$

$$\begin{vmatrix} \lambda & 2E_1 & -8E_2 & 0... & ...0 \\ -2m(2m+1) & \lambda & 12E_1 & -24E_2 & ...0 \\ 0 & 6-2m(2m+1) & \lambda & 30E_1 & -48E_2 \\ - & - & - & - & - \\ 0... & ...0 & -16m+12 & \lambda & 2m(2m-1)E_1 \\ 0... & 0 & ...0 & -8m+2 & \lambda \end{vmatrix} = 0.$$

Proof. The corresponding characteristic equation

$$4t^2 + 2t - n(n+1) = 0$$

has two roots

$$t_1 = \frac{n}{2}, t_2 = -\frac{n+1}{2}.$$

Thus for $n = 2m$ only one $t_1 = m$ is a natural number and the equation can have only one polynomial solution of degree m. So the first condition of existence is actually a natural number m for which $n = 2m$ is valid.

The second condition is the second condition of (5) for $n = 2m$ and is actually algebraic equation of degree $m+1$ in terms of the parameter λ.

The roots of this equation, real and different, are the values of the parameter λ, for which the Lame's equation has a polynomials solution of degree m. So in that case the equation (4) will have $m+1$ polynomial solutions of degree m appropriate for each of the $m+1$ roots of that equation, which is consistent with the classical results [1,3].

We now give some special cases.

For $n = 2$ in accordance with the theorem, the equation

$$4(x^3 + E_1x - E_2)y" + 2(3x^2 + E_1)y' - [6x - \lambda]y = 0,$$

has a polynomial solution of degree $m = 1$ if the parameter λ is one of the two roots of the equation

$$\Delta_1 = \begin{vmatrix} \lambda & 2E_1 \\ -6 & \lambda \end{vmatrix}$$
$$\equiv \lambda^2 + 12E_1 = 0.$$

So both polynomial solutions are

$$P_1^1 = x + \frac{1}{6}\lambda_1, P_1^2 = x + \frac{1}{6}\lambda_2$$

where λ_1 and λ_2 are the roots of the quadratic equation.

For $n = 4$ in accordance with the theorem, the equation

$$4(x^3 + E_1x - E_2)y" + 2(3x^2 + E_1)y' - [20x - \lambda]y = 0,$$

has a polynomial solution of degree $m = 2$ if the parameter λ is one of the three roots of equation

$$\Delta_2 = \begin{vmatrix} \lambda & 2E_1 & -8E_2 \\ -20 & \lambda & 12E_1 \\ 0 & -14 & \lambda \end{vmatrix}$$
$$\equiv \lambda^3 + (14 \cdot 12 + 40)E_1\lambda - 8 \cdot 14 \cdot 20E_2 = 0.$$

Three polynomial solutions are

$$P_2^1 = x^2 + \frac{1}{4}\lambda_1 + \frac{1}{20}[12E_1 + \frac{1}{4}\lambda_1^2],$$
$$P_2^2 = x^2 + \frac{1}{4}\lambda_2 + \frac{1}{20}[12E_1 + \frac{1}{4}\lambda_2^2],$$
$$P_2^3 = x^2 + \frac{1}{4}\lambda_3 + \frac{1}{20}[12E_1 + \frac{1}{4}\lambda_3^2],$$

where λ_1, λ_2 and λ_3 are the roots of algebraic equation of third degree.

These values of the parameter λ can be called their eigenvalues, which correspond to the appropriate polynomial solutions of Lame's equation, called eigenfunctions. Therefore this problem can be regarded as solving Sturm-Liouville-problem or a problem to find their eigenvalues and eigenfunctions for one special differential operator

$$l \equiv -\frac{d^2}{dx^2} + q(x).$$

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ЗА ПОЛИНОМИТЕ НА LAME

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Р е з и м е

Во овој труд се разгледува диференцијална равенка на Lame во алгебарски вид. За оваа класа диференцијални равенки се добиени потребни и доволни услови за егзистенција на полиномно решение. При тоа е добиена алгебарска равенка чии корени се вредностите на соодветен параметар за кои диференцијалната равенка има полиномно решение и кои можат да се интерпретираат и како сопствени вредности кај соодветна задача на Sturm-Liouville.

Клучни зборови: диференцијална равенка, полиномни решенија, сопствени вредности

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